PAGE
DATE

Probability Theory

chapter 4. Solution Toshinari Morimoto

1. $N : \Omega \to \mathbb{N} \cup \{\infty\}$.

(1) When $\{\omega \in \Omega \mid N(\omega) = n\} \in \mathcal{F}_n$ for all $n = 1, 2, 3 \cdots$

then N is called a stopping time

Here $\{\mathcal{F}_n\}_{n \geq 1}$ are sub-σ-algebras of $\mathcal{F}$ defined

($(\Omega, \mathcal{F}, P)$ .. Probability Space)

as $\mathcal{F}_n \overset{def}{=} \sigma[X_1, X_2 .. X_n] \left(= \sigma\left[\bigcup_{j=1}^{n} \sigma[X_j]\right] \right)$

$$\left(\sigma[X_j] = \{X_j^{-1}(B) \mid B \in \mathcal{B}(\mathbb{R}^1)\} \right)$$

(2) We define $N(\omega) = \inf\{n \mid S_n \in A\}$

then $N(\omega)$ is a stopping time.

(proof) $\{N(\omega) = n\} = \underbrace{\{S_1 \notin A\}}_{\sigma[X_1]} \cap \underbrace{\{S_2 \notin A\}}_{\in \sigma[X_1, X_2]} \cdots \cap \underbrace{\{S_n \in A\}}_{\in \sigma[X_1 .. X_n]}$

$$\in \sigma[X_1, X_2 .. X_n] = \mathcal{F}_n$$

So $N(\omega)$ is a stopping time

$\boxed{2}$ We examine if N_t is a stopping time or not.

$$\{N_t = n\} = \underbrace{\{S_1 \leq t\}}_{\sigma[X_1]} \cap \cdots \cap \underbrace{\{S_n \leq t\}}_{\sigma[X_1 \ldots X_n]} \cap \underbrace{\{S_{n+1} > t\}}_{\sigma[X_1 \ldots X_{n+1}]}$$

So $\{N_t + 1 = n\} = \{N_t = n-1\} \in \sigma[X_1 \ldots X_n]$

Thus $N_t + 1$ is a stopping time

PAGE

DATE

3 First we confirm the definition of $\mathcal{F}_N$ (N: stopping time)

$\mathcal{F}_N \overset{\text{def}}{=} \{A \in \mathcal{F} \mid \forall n = 1,2,3\cdots \quad A \cap \{N = n\} \in \mathcal{F}_n\}$.

Let k be a positive integer.

Let $A \in \mathcal{F}_N$.

Let $\{B_1, B_2 \dots B_k\}$ be Borel sets.

$$P\left(\bigcap_{j=1}^{k} \{B_j \in X_{N+j}\} \cap A \cap \{N<\infty\}\right) \qquad \{N<\infty\} = \sum_{n=1}^{\infty}\{N=n\}$$

$$= \sum_{n=1}^{\infty} P\left(\bigcap_{j=1}^{k} \{B_j \in X_{N+j}\} \cap \underbrace{A \cap \{N=n\}}\right)$$

Since $A \in \mathcal{F}_N$
hence $A \cap \{N=n\} \in \mathcal{F}_n$

$$= \sum_{n=1}^{\infty} P\left(\underbrace{\bigcap_{j=1}^{k} \{B_j \in X_{n+j}\}}_{\sigma[X_{n+1}, \cdots X_{n+k}]} \cap \underbrace{A \cap \{N=n\}}_{\in \mathcal{F}_n}\right)$$

These two events are independent

$$= \sum_{n=1}^{\infty} P\left(\bigcap_{j=1}^{k} \{B_j \in X_{n+j}\}\right) P(A \cap \{N=n\})$$

$$= \sum_{n=1}^{\infty} \left(\prod_{j=1}^{k} F(B_j)\right) P(A \cap \{N=n\}) = \prod_{j=1}^{k} F(B_j) \cdot P(A \cap \{N<\infty\})$$

($\because$ iid) $\qquad (X_1, X_2 \dots \overset{\text{iid}}{\sim} F)$

$$\therefore P\left(\bigcap_{j=1}^{K}\{B_j \in X_{N+j}\} \cap A \cap \{N<\infty\}\right)$$

$$= \prod_{j=1}^{K} \underbrace{P(B_j \in X_1)}_{=F(B_j)} \cdot P(A \cap \{N<\infty\})$$

(∵ iid)

This is what the statement means. ∎

$\boxed{4}$

$\boxed{\text{Step 1}}$ Consider $X_n \geq 0$ (non-negative)

Since $E[N] < \infty$, $N < \infty$ (a.s.).

$$E[S_N] = E[S_N \cdot \mathbb{I}\{N<\infty\}]$$

$$= \int_\Omega S_N \cdot \mathbb{I}\{N<\infty\}\, dP$$

$$= \int_\Omega \sum_{n=1}^{\infty} S_N \cdot \mathbb{I}\{N=n\}\, dP \qquad \downarrow \because N=n$$

$$= \int_\Omega \sum_{n=1}^{\infty} S_n \cdot \mathbb{I}\{N=n\}\, dP \qquad \downarrow \text{ Monotone Convergence Theorem}$$

$$= \sum_{n=1}^{\infty} \int_\Omega S_n \cdot \mathbb{I}\{N=n\}\, dP$$

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{n} \underbrace{\int_\Omega X_m\, \mathbb{I}\{N=n\}\, dP}_{\geq 0}$$

Since $\forall n, m$ $\int_\Omega X_m \cdot \mathbb{I}\{N=n\}\, dP$
we may swap $\sum_{n=1}$, $\sum_{m=1}$

$$= \sum_{m=1}^{\infty} \sum_{n=m}^{\infty} \int_\Omega X_m \cdot \mathbb{I}\{N=n\}\, dP$$

$$= \sum_{m=1}^{\infty} \int_\Omega X_m \cdot \mathbb{I}\{N \geq m\}\, dP$$

They are independent

$\{N \geq m\} = \{N < m\}^c$

PAGE

DATE

$$= \sum_{m=1}^{\infty} \int_{\Omega} X_m \cdot \mathbb{1}\{N \geq m\} \, dP \qquad (\because \{N<m\}^c = \{N \geq m\} \in \sigma(X_1 \cdots X_{m-1}))$$

$$= \sum_{m=1}^{\infty} E[X_m] \cdot P(N \geq m)$$

$$(= E[X_1] \; : \text{iid})$$

$$= E[X_1] \sum_{m=1}^{\infty} P(N \geq m)$$

$$= E[X_1] \, E[N] \quad (\text{See chapter 2, } \boxed{35})$$

Now the prof is complete. ∎

$\boxed{\text{Step 2}}$

$$S_N = X_1 + \cdots + X_N = (X_1^+ - X_1^-) + \cdots + (X_N^+ - X_N^-)$$

Let $S_N^{(P)} \stackrel{\text{def}}{=} X_1^+ + \cdots + X_N^+$

$S_N^{(N)} \stackrel{\text{def}}{=} X_1^- + \cdots + X_N^-$

Then $E[S_N^{(P)}] = E[X_1^+] \, E[N] < \infty \quad (S_N^{(P)} < \infty \text{ (a.s)})$

$E[S_N^{(N)}] = E[X_1^-] \, E[N] < \infty \quad (S_N^{(N)} < \infty \text{ (a.s)})$

$$\therefore E[S_N^{(P)} - S_N^{(N)}] = E[X_1] \, E[N]$$

This can be defined as.

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$$= E[S_N]$$

PAGE

DATE

$\boxed{5}$

① $\min\{S,T\}$ is a stopping time

$$\{\min\{S,T\}=n\} = \{S=n\}\cap \underbrace{\{T\geq n\}}_{\{T<n\}^c} \cup \{T=n\}\cap\underbrace{\{S\geq n\}}_{\{S<n\}^c}$$

$$\left(\in \sigma[X_1 \cdots X_{n-1}] \subseteq \sigma[X_1 \cdots X_n]\right)$$

$$\in \sigma[X_1, X_2 \cdots X_n]$$

Hence $\min\{S,T\}$ is a stopping time.

② $\max\{S,T\}$ is a stopping time

$$\{\max\{S,T\}=n\} = \{S=n\}\cap\{T\leq n\}\cup\{T=n\}\cap\{S\leq n\}$$

$$\in \sigma[X_1 \cdots X_n]$$

So it is a stopping time.

③. Let $T=n$. Then $\{T=k\} = \underset{(k\neq n)}{\phi}, \underset{(k=n)}{\Omega} \in \sigma[X_1 \cdots X_k]$ $(\because \sigma\text{-algebra})$

So T is a stopping time.

By ① and ② the proof is complete.

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$\boxed{6}$ The statement is true.

$$\{S+T=n\} = \sum_{k=1}^{n-1} \{S=k \cap T=n-k\}$$

$$= \sum_{k=1}^{n-1} \underbrace{\{S=k\}}_{\in\, \sigma[X_1 \cdots X_k]} \cap \underbrace{\{T=n-k\}}_{\in\, \sigma[X_1 \cdots X_{n-k}]}$$

$$\in \sigma[X_1 \cdots X_n]$$

So $S+T$ is a stopping time.

PAGE
DATE

$\boxed{7}$ $M \leq N$ is a stopping time then $\mathcal{F}_M \subseteq \mathcal{F}_N$

We show that $\forall A \in \mathcal{F}_M \Rightarrow A \in \mathcal{F}_N$

(proof) $A \in \mathcal{F}_M$

$$A \cap \{N=n\} = A \cap \{N=n\} \cap \{M \leq n\}$$

$$(\because M \leq N)$$

$$= \sum_{m=1}^{n} A \cap \{N=n\} \cap \{M=m\}$$

$$= \{N=n\} \cap \Big(\sum_{m=1}^{n} \underbrace{A \cap \{M=m\}}_{\in \mathcal{F}_M \ (\because A \in \mathcal{F}_M) \ = \sigma[X_1 \cdots X_m]} \Big)$$

$$\in \sigma[X_1 \cdots X_n]$$

$$A \cap \{N=n\} \in \sigma[X_1 \cdots X_n]$$

Now the proof is complete. ■

PAGE

DATE

$\boxed{8}$ $L \leq M$. $A \in \mathcal{F}_L$ (L, M: stopping time)

$$N \overset{\text{def}}{=} \begin{cases} L & (\omega \in A) \\ M & (\omega \notin A) \end{cases}$$

Then N is a stopping time

(proof)

$$\{N=n\} = \{N=n\} \cap A \cup \{N=n\} \cap A^c$$

$$= \underbrace{\{L=n\} \cap A}_{\in \mathcal{F}_n \ (\because A \in \mathcal{F}_L)} \cup \{M=n\} \cap A^c$$

$$\{M=n\} \cap A^c = \underbrace{\{M=n\}}_{\in \mathcal{F}_n} \setminus \underbrace{\{M=n\} \cap A}_{⊛}$$

⊛ $\{M=n\} \cap A = \{L \leq n\} \cap \{M=n\} \cap A$ $(\because L \leq M)$

$$= \sum_{l=1}^{n} \{L=l\} \cap \{M=n\} \cap A$$

$$= \underbrace{\{M=n\}}_{\in \mathcal{F}_n} \cap \underbrace{\sum_{l=1}^{n} \underbrace{\{L=l\} \cap A}_{\in \mathcal{F}_l}}_{\in \sigma[X_1 \cdots X_n]}$$

$\therefore$ ⊛ $\in \mathcal{F}_n$

Now the proof is complete. ■

9

(1) Definition of $E[X|\mathcal{F}]$.

$E[X|\mathcal{F}]$ is a $\mathcal{F}$-measurable function

such that $\forall A \in \mathcal{F}$ ($\mathcal{F} \subseteq \mathcal{F}_0$)

$\int_A X dP = \int_A E[X|\mathcal{F}] dP$

(In a sense of a.s. $E[X|\mathcal{F}]$ is unique)

(2) $\int_\Omega E[X|\mathcal{F}] dP = \int_\Omega X dP$

$\quad\quad \parallel$

$\int_\Omega Y dP$

$\therefore \left| \int_\Omega Y dP \right| = \left| \int_\Omega X dP \right| \leq \int_\Omega |X| dP < \infty$

Hence Y is integrable

PAGE
DATE

(3) We show that $Y \geq Y'$ (a.s) and
$Y \leq Y'$ (a.s)

① Let $A = \{Y < Y'\}$ ...

$A \in \mathcal{F}$ ($\because$ Y, Y' are $\mathcal{F}$-measurable)

$\int_A Y dP = \int_A X dP \in (-\infty, \infty)$ ($\because$ X: integrable)

$\int_A Y' dP = \int_A X dP \in (-\infty, \infty)$

Hence $\int_A Y dP - \int_A Y' dP = 0$

$\therefore \int_A \underbrace{(Y' - Y)}_{>0} dP = 0$

So $P(A) = 0$ $\therefore$ $Y \geq Y'$ (a.s)

② Let $A = \{Y' < Y\}$

In the same way, we have $Y' \geq Y$ (a.s)

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$\boxed{10}$ Let $B_1 = \{E[X_1|\mathcal{F}] > E[X_2|\mathcal{F}]\} \cap B. \in \mathcal{F}$

Then $\int_{B_1} X_1 dP = \int_{B_1} E[X_1|\mathcal{F}] dP \in (-\infty, \infty)$

$\int_{B_1} X_2 dP = \int_{B_1} E[X_2|\mathcal{F}] dP \in (-\infty, \infty)$

Since $X_1 = X_2$ (a.s) on B. (We suppose X_1, X_2 are integrable)

$\int_{B_1} E[X_1|\mathcal{F}] dP = \int_{B_1} E[X_2|\mathcal{F}] dP$

$\Rightarrow \int_{B_1} \underbrace{(E[X_1|\mathcal{F}] - E[X_2|\mathcal{F}])}_{>0} dP = 0$

$\therefore P(B_1) = 0$

$\Rightarrow E[X_1|\mathcal{F}] \leq E[X_2|\mathcal{F}]$ (a.s) on B.

, $B_2 = \{E[X_1|\mathcal{F}] < E[X_2|\mathcal{F}]\} \cap B$.

Similarly we have $E[X_1|\mathcal{F}] \geq E[X_2|\mathcal{F}]$ (a.s) on B.

$\boxed{11}$ $(\Omega, \mathcal{F}_0)$: measurable space

(1) $\forall A \in \mathcal{F}_0$ $\mu(A) = 0 \Rightarrow \nu(A) = 0$

Then $\nu \ll \mu$.

We call ν is absolutely continuous with regard to μ.

(2) Let μ, ν be σ-finite signed measures on $(\Omega, \mathcal{F}_0)$.

and suppose $\nu \ll \mu$.

Then there exists a $\mathcal{F}$-measurable function f

such that $\nu(A) = \int_A f d\mu$ for all $A \in \mathcal{F}$.

(3) Let $\nu(A) \overset{def}{=} \int_A X dP$. ($X$: integrable)

$(\nu : \mathcal{F}_0 \to [0, \infty])$

Then $\nu(A)$ is a signed-measure on $(\Omega, \mathcal{F}_0)$ and finite,

hence σ-finite. ($\because$ X is integrable)

And $\forall A \in \mathcal{F}_0$ $P(A)=0$, $\int_A X dP = 0$

$\Rightarrow \nu(A)=0$ $\therefore$ $\nu \ll P$. on $\mathcal{F}_0$.

$\Rightarrow \nu \ll P$ on $\mathcal{F}$. $(\mathcal{F} \subseteq \mathcal{F}_0)$

By R-N there exists a $\mathcal{F}$-measurable function f

s.t $\forall A \in \mathcal{F}$ $\nu(A) = \int_A f dP$. (hence $= \int_A X dP$)

Such a function f is the desired function.

We denote f as $E[X|\mathcal{F}]$

PAGE

DATE

12 Let $A \in \mathcal{F}$. By the definition,

$\int_A X dP = \int_A E[X|\mathcal{F}] dP$ for all $A \in \mathcal{F}$.

① $A = \{X > E[X|\mathcal{F}]\} \in \mathcal{F}$

$\int_A X dP = \int_A E[X|\mathcal{F}] dP \in (-\infty, \infty)$

$\Rightarrow \int_A (\underbrace{X - E[X|\mathcal{F}]}_{>0}) dP = 0$

$\Rightarrow P(A) = 0 \quad \therefore \quad X \leq E[X|\mathcal{F}]$ (a.s)

② $A = \{X < E[X|\mathcal{F}]\} \in \mathcal{F}$

Similarly we have $X \geq E[X|\mathcal{F}]$ (a.s)

$\therefore X = E[X|\mathcal{F}]$ (a.s)

$\boxed{13}$ Let $A \in \mathcal{F}$,

$$\int_A E[X|\mathcal{F}]\,dP = \int_A X\,dP$$

$$= \int_\Omega X \cdot \mathbb{I}_A\,dP$$

$$= E[X \cdot \mathbb{I}_A] = E[X]\,P(A)$$

$$= \int_A E[X]\,dP$$

$E[X]$ is a constant $\Rightarrow$ $\mathcal{F}$-measurable

And $\forall A \in \mathcal{F}$ $\int_A X\,dP = \int_A E(X)\,dP$

So $E[X|\mathcal{F}] = E(X)$ (as)

by the definition of conditional expectation

($\phi(\cdot,\cdot) : \mathbb{R}^2 \to \mathbb{R}$ is a Borel measurable function)

$\boxed{14}$ $E[\sim | X] \overset{def}{=} E[\sim | \sigma[X]]$

($\sigma(X) = \{ X^{-1}(B) \mid B \in \mathcal{B}(\mathbb{R}) \}$)

(proof)

Let $A \in \sigma[X]$. ($\therefore \exists B \in \mathcal{B}(\mathbb{R})$ s.t $A = X^{-1}(B)$)

$\int_A \phi(X,Y)\, dP = \int_A E[\phi(X,Y) | X]\, dP$

$\parallel$

$\int_\Omega \mathbb{I}_A \cdot \phi(X,Y)\, dP = \int_\Omega \mathbb{I}\{X(w) \in B\} \cdot \phi(X(w), Y(w))\, dP$

Now by change variable formula.

$\int_\Omega \mathbb{I}\{X \in B\}\, \phi(X,Y)\, dP$

$\equiv \int_{\mathbb{R}^2} \mathbb{I}_B(x)\, \phi(x,y)\, d\mu\, d\nu$

($X \sim \mu$, $Y \sim \nu$)

Since X, Y are independent

by Fubini's theorem

PAGE

DATE

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{I}_B(x)\, \phi(x,y)\, d\mu\, d\nu$$

$$= \int_B \underbrace{\int_{\mathbb{R}} \phi(x,y)\, d\nu}_{=E[\phi(x,Y)]}\, d\mu$$

↓ change variable formula

$$= \int_B E[\phi(x,Y)]\, d\mu$$

$$= \int_B g(x)\, d\mu = \int_{\mathbb{R}} \mathbb{I}_B(x)\, g(x)\, d\mu$$

↓ change variable formula

$$= \int_{\Omega} \mathbb{I}_A(\omega)\, g(X)\, dP$$

$$= \int_A g(X)\, dP.$$

So $\int_A \phi(X,Y)\, dP = \int_A g(X)\, dP$ for all $A \in \sigma(X)$

$\left(= \int_A E[\phi(X,Y) \mid X]\, dP\right)$

And $g(X) : \Omega \to \mathbb{R}$ is $\sigma(X)$-measurable. ...... ⊛

So $E[\phi(X,Y) \mid X] = g(X)$.

(⊛ ... $g(x) \overset{\text{def}}{=} \int_\Omega \phi(x,Y)\, dP = \int_{\mathbb{R}} \phi(x,y)\, \nu(dy)$

is a Borel-measurable function. See ch1. $\boxed{35}$

$g : \mathbb{R} \to \mathbb{R} \Rightarrow g(X)$ is $\sigma(X)$-measurable.

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$\boxed{15}$

(1) Let $a, b \in \mathbb{R}$ and let $A \in \mathcal{F}$.

$$\int_A (aX+bY)\,dP = \int_A E[aX+bY \mid \mathcal{F}]\,dP$$

(by definition)

Since X, Y are integrable,

$$\int_A (aX+bY)\,dP = a\int_A X\,dP + b\int_A Y\,dP$$

And by the definition of conditional expectation

$$= a\int_A E[X\mid\mathcal{F}]\,dP + b\int_A E[Y\mid\mathcal{F}]\,dP$$

$$= \int_A \left(aE[X\mid\mathcal{F}] + bE[Y\mid\mathcal{F}]\right)dP$$

$$\therefore \int_A E[aX+bY\mid\mathcal{F}]\,dP = \int_A \left(aE[X\mid\mathcal{F}] + bE[Y\mid\mathcal{F}]\right)dP$$

And $aE[X\mid\mathcal{F}] + bE[Y\mid\mathcal{F}]$ is also $\mathcal{F}$-measurable

Therefore the statement is true

(2) Suppose $X \leq Y$ and let $A \in \mathcal{F}$.

Then $\int_A X \, dP \leq \int_A Y \, dP$.

By definition we have

$$\int_A E[X|\mathcal{F}) \, dP \leq \int_A E[Y|\mathcal{F}) \, dP$$

for all $A \in \mathcal{F}$.

Hence we have $E[X|\mathcal{F}) \leq E[Y|\mathcal{F})$ (a.s)

(You may put $A = \{ E[Y|\mathcal{F}] < E[X|\mathcal{F}] \}$)

(3) ⊛ $\int_A X_n \, dP = \int_A E[X_n|\mathcal{F}] \, dP$ (by definition)

($A \in \mathcal{F}$)

By monotone convergence theorem ($\nearrow$)

$$\lim_{n\to\infty} \int_A X_n \, dP \underset{(MCT)}{\overset{\nearrow}{=}} \int_A X \, dP \underset{(\text{definition})}{=} \int_A E[X|\mathcal{F}) \, dP$$

⊛ ||

$$\lim_{n\to\infty} \int_A E[X_n|\mathcal{F}) \, dP \underset{(MCT)}{\overset{\nearrow}{=}} \int_A \lim_{n\to\infty} E[X_n|\mathcal{F}) \, dP$$

So we have $\int_A \lim_{n\to\infty} E[X_n|\mathcal{F}) \, dP = \int_A E[X|\mathcal{F}) \, dP$.

PAGE
DATE

By (2) $E[X_n|\mathcal{F}] \leq E[X_{n+1}|\mathcal{F})$ (a.s)

Thus $\lim_{n\to\infty} E(X_n|\mathcal{F})$ is defined almost surely.

($\because$ monotone-increasing)

$\lim_{n\to\infty} E[X_n|\mathcal{F})$ is also $\mathcal{F}$-measurable. (See ch1)

$(= \sup_{n\geq 1} E[X_n|\mathcal{F}])$

Therefore $\lim_{n\to\infty} E[X_n|\mathcal{F}) = E(X|\mathcal{F})$ (as)

CHA SHIN

PAGE

DATE

16

$$\left(\sup_{(a,b)\in S} \{ax+b\} = \phi(x) \right)$$

Let $S = \{ (a,b) \in \mathbb{Q}^2 \mid \phi(x) > ax+b \text{ for all } x \}$

$$E[\phi(X) \mid \mathcal{F}] \geq E[aX+b \mid \mathcal{F}] \quad ({}^{\forall}(a,b) \in S)$$

$$= aE[X \mid \mathcal{F}] + b \quad (a.s)$$

$$\sup_{(a,b)\in S} \{aE[X|\mathcal{F}] + b\} = \phi \circ E[X|\mathcal{F}]$$

Since S is a countable set.

if $\Omega^{(a,b)} \in \mathcal{F}$: $P(\Omega^{(a,b)}) = 1$

then $P\left(\bigcap_{(a,b)\in S} \Omega^{(a,b)} \right) = 1$ ...... $\circledast$

So $E[\phi(X) \mid \mathcal{F}] \geq \phi \circ E[X|\mathcal{F}]$ (a.s)

by $\circledast$, (a.s) still holds.

$\boxed{17}$ Let $\phi(x) = |x|^p$.

Then $\phi(x)$ is a convex function.

By $\boxed{16}$, $E[|X|^p \mid \mathcal{F}] \geq |E[X|\mathcal{F}]|^p$

So $E \circ E[|X|^p \mid \mathcal{F}] \geq E[|E[X|\mathcal{F}]|^p]$

$$\begin{array}{c} \| \\ E|X|^p \end{array}$$

$\therefore \{E|X|^p\}^{\frac{1}{p}} \geq E[|E[X|\mathcal{F}]|^p]^{\frac{1}{p}}$

$\|X\|_p \overset{\text{def}}{=} (E|X|^p)^{\frac{1}{p}}$

$T(\cdot) \overset{\text{def}}{=} E[\cdot \mid \mathcal{F}]$

Therefore $\|X\|_p \geq \|T(X)\|_p$

So this is the desired result.

$\boxed{18}$ Let $A \in \mathcal{F}$.

$\int_A X dP = \int_A E(X|\mathcal{F}) dp$ (by definition)

Since $A \in \mathcal{G}$ ($\because \mathcal{F} \subseteq \mathcal{G}$)

$\int_A X dp = \int_A E(X|\mathcal{G}) dp$.

So $\forall A \in \mathcal{F}$ $\int_A E(X|\mathcal{F}) dp$

$= \int_A E(X|\mathcal{G}) dp$

And $E(X|\mathcal{G})$ is also $\mathcal{F}$-measurable

Hence $E(X|\mathcal{F}) = E(X|\mathcal{G})$ (as) by uniqueness.

PAGE
DATE

$\boxed{19}$

(1) Let $X_1 = E[X|\mathcal{F}_1]$

X_1 is $\mathcal{F}_1$-measurable $\Rightarrow$ $\mathcal{F}_2$-measurable

By $\boxed{12}$, $E[X_1|\mathcal{F}_2] = X_1$ (a.s)

$\therefore E[E[X|\mathcal{F}_1]|\mathcal{F}_2] = E[X|\mathcal{F}_1]$

(2) Let $A \in \mathcal{F}_1$

Let $X_2 = E[X|\mathcal{F}_2]$

Then $\int_A X_2 dP = \int_A E[X_2|\mathcal{F}_1]\, dP$.

$LHS = \int_A E[X|\mathcal{F}_2]\, dP$ $(\because A \in \mathcal{F}_1 \Rightarrow A \in \mathcal{F}_2)$

$= \int_A X dP$

And $\int_A X dP = \int_A E[X|\mathcal{F}_1]\, dP$

$\therefore$ So for all $A \in \mathcal{F}_1$ $\int_A E[X_2|\mathcal{F}_1] dP = \int_A E[X|\mathcal{F}_1] dP$

CHA SHIN $\therefore E[E[X|\mathcal{F}_2]|\mathcal{F}_1] = E[X|\mathcal{F}_1]$ (a.s)

PAGE

DATE

$\boxed{20}$

$\boxed{\text{Step 1}}$ $X = \mathbb{1}_E(\omega)$ $(E \in \mathcal{F})$ and $A \in \mathcal{F}$.

$$\int_A XY\,dP = \int_A \mathbb{1}_{E}(\omega) \cdot Y\,dP$$

$$= \int_{A \cap E} Y\,dP \qquad (A \cap E \in \mathcal{F})$$

$$= \int_{A \cap E} E[Y|\mathcal{F})\,dP$$

$$= \int_A \mathbb{1}_E \cdot E[Y|\mathcal{F})\,dP$$

$$= \int_A X\,E[Y|\mathcal{F})\,dP$$

↳ This is also $\mathcal{F}$-measurable

Hence,

$$E[XY|\mathcal{F}) = X \cdot E[Y|\mathcal{F})$$

$\boxed{\text{Step 2}}$ $X = a_1 \mathbb{1}_{E_1}(\omega) + a_2 \mathbb{1}_{E_2}(\omega) + \dots + a_k \mathbb{1}_{E_k}(\omega)$

$\{a_i\}_{i=1}^{k} \subseteq [0, \infty)$, $\{E_i\}_{i=1}^{k} \subseteq \mathcal{F}$: disjoint

By $\boxed{15}$ (1) : Linearity and $\boxed{\text{Step 1}}$

We have $E[XY|\mathcal{F}) = X \cdot E[Y|\mathcal{F})$.

CHA SHIN

PAGE
DATE

$\boxed{\text{Step 3}}$ $X \geq 0$. $Y \geq 0$

We take $\{X_n\}_{n\geq 1}$: simple measurable functions

st $X_n \nearrow X$.

For each $n \geq 1$, $E[X_n Y | \mathcal{F}) = X_n E[Y | \mathcal{F})$

by $\boxed{\text{Step 2}}$.

By $\boxed{15}$ (3). $X_n Y \nearrow XY$

$\Rightarrow E[X_n Y | \mathcal{F}) \nearrow E[XY | \mathcal{F})$

$\therefore \lim_{n\to\infty} E(X_n Y | \mathcal{F}) = \lim_{n\to\infty} X_n E[Y | \mathcal{F})$

$= E[XY | \mathcal{F})$

$\therefore X E[Y | \mathcal{F}) = E[XY | \mathcal{F})$

$\boxed{\text{Step 4}}$ $Y \geq 0$.

By $\boxed{\text{Step 3}}$, $E[X^+ Y | \mathcal{F}) = X^+ E[Y | \mathcal{F})$ ... ①
$E[X^- Y | \mathcal{F}) = X^- E[Y | \mathcal{F})$ .. ②

And $E|XY| < \infty$, implies that

① - ② is defined (a.s)

So $E[XY | \mathcal{F}) = X E[Y | \mathcal{F})$ (a.s)

$\boxed{\text{Step 5}}$ General Case.

$E[XY^+ | \mathcal{F}) = X E[Y^+ | \mathcal{F})$ ... ③
$E(XY^- | \mathcal{F}) = X E(Y^- | \mathcal{F}]$. ④

(by $\boxed{\text{Step 4}}$)

By ③-④, we have the desired result.

PAGE
DATE

$\boxed{21}$ Let Z be a $\mathcal{F}$-measurable function

We consider $\underset{Z}{\operatorname{argmin}}\, E[(X-Z)^2]$.

$$E[(X-Z)^2] = E \circ E[(X-Z)^2 | \mathcal{F}]$$

$$= E[E[(X-E[X|\mathcal{F}]+E[X|\mathcal{F}]-Z)^2 | \mathcal{F}]]$$

$$= E\Big[\underbrace{E[(X-E[X|\mathcal{F}])^2 | \mathcal{F}]}_{①} + \underbrace{2E[(X-E[X|\mathcal{F}]) \cdot (E[X|\mathcal{F}]-Z) | \mathcal{F}]}_{②} + \underbrace{E[(E[X|\mathcal{F}]-Z)^2 | \mathcal{F}]}_{③}\Big]$$

$① = V[X|\mathcal{F}] \quad (\geq 0)$

$② = 2 \cdot (E[X|\mathcal{F}]-Z) \underbrace{E[(X-E[X|\mathcal{F}]) | \mathcal{F}]}_{=0}$

$= 0 \quad (\because \text{See } \boxed{12})$

CHA SHIN

PAGE
DATE

① is not related with Z.

So we just have to consider

$\underset{Z}{\text{argmin}}\, E[③]$

And $③ = (E[X|\mathcal{F}] - Z)^2$ $(\because \boxed{2})$

≥ 0

So $E[③] \geq 0$

Equality holds when $③ = 0$ (a.s)

$\therefore Z = E[X|\mathcal{F}]$ (a.s) minimizes it.

PAGE
DATE

$\boxed{22}$ $\mu(\omega, A)$: $(\omega \in \Omega,\ A \in \mathcal{S})$

- For almost everywhere $\omega \in \Omega$

  $\mu(\omega, \cdot) : \mathcal{S} \mapsto [0,1]$ is a measure

  (i.e. $\exists\, \tilde{\Omega} \in \mathcal{F}$ $P(\tilde{\Omega}) = 1$,

  $\forall \omega \in \tilde{\Omega}$: $\mu(\omega, \cdot) : \mathcal{S} \mapsto [0,1]$ is a measure)

- For all $A \in \mathcal{S}$

  $\mu(\cdot, A) : \Omega \to [0,1]$ is a conditional probability $= P(X \in A \mid \mathcal{G})$.

Then $\mu(\omega, A)$ is a regular conditional distribution for X given $\mathcal{G}$.

$\boxed{23}$

(1) When $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ and $\{\mathcal{F}_n\}_{n \geq 1}$: σ-algebras

then $\mathcal{F}_n$ is a filtration

(2) X_n is $\mathcal{F}_n$-measurable for all $n \geq 1$

(3) . $E|X_n| < \infty$

. X_n is $\mathcal{F}_n$-measurable (adapted to $\mathcal{F}_n$)

. $E[X_{n+1} | \mathcal{F}_n] = X_n$

(for all $n \geq 1$)

(4) . $E|X_n| < \infty$, X_n is $\mathcal{F}_n$-measurable

$E[X_{n+1} | \mathcal{F}_n] \leq X_n$

(5) $E|X_n| < \infty$ X_n is $\mathcal{F}_n$-measurable

$E[X_{n+1} | \mathcal{F}_n] \geq X_n$

$\boxed{24}$

① $E|S_n| \leq \sum_{j=1}^{n} E|\xi_j| \leq 2n E[\xi_j^+] < \infty$

$(\because E[\xi_j] = 0 \Rightarrow E[\xi_j^+] = E[\xi_j^-] < \infty)$

② $S_n = \xi_1 + \xi_2 + \cdots + \xi_n$

$= g(\xi_1, \xi_2, \cdots \xi_n)$

where $g(x_1, x_2, \cdots x_n) = x_1 + x_2 + \cdots + x_n$

Since $g : \mathbb{R}^n \to \mathbb{R}$ is continuous,

$g(\cdot)$ is Borel measurable

Let us recall ch2. $\boxed{9}$.

From its proof, we found that

$g(\xi_1, \xi_2, \cdots, \xi_n)$ is $\sigma\left[\bigcup_{j=1}^{n} \sigma[\xi_j]\right]$ - measurable

$$\begin{gathered} \sigma\left[\bigcup_{j=1}^{n} \sigma[\xi_j]\right] \\ \parallel \\ \sigma[\xi_1, \xi_2 \cdots \xi_n] \\ \parallel \\ \mathcal{F}_n \end{gathered}$$

PAGE
DATE

③ Finally, $E[S_{n+1} | \mathcal{F}_n]$

$= E[\xi_{n+1} + S_n | \mathcal{F}_n] = S_n + E[\xi_{n+1} | \mathcal{F}_n]$

$(\because \boxed{12})$ $(\because \boxed{13})$

$= S_n + E[\xi_{n+1}] = S_n$

(※ $\xi_1 \sim \xi_{n+1}$: independent
$\Rightarrow \sigma[\xi_1] \sim \sigma(\xi_{n+1})$: independent
$\Rightarrow \sigma[\xi_1 \cdots \xi_n], \sigma(\xi_{n+1})$ independent
$\because$ ch2, $\boxed{8}$)

$\therefore E[S_{n+1} | S_n] = S_n$

By ①~③, $\{S_n\}_{n\geq 1}$ is a martingale

PAGE
DATE

$\boxed{25}$

① $E[S_n^2] = n\sigma^2$

$\therefore E|S_n^2 - n\sigma^2| < 2n\sigma^2 < \infty$

② $S_n^2 - n\sigma^2 = g(\xi_1, \xi_2, \cdots \xi_n) - n\sigma^2$ where

$g(x_1, x_2 \cdots x_n) = (x_1 + \cdots + x_n)^2$ is Borel-Measurable

So $S_n^2 - n\sigma^2$ is $\underbrace{\sigma(\xi_1, \xi_2, \cdots, \xi_n)}_{= \mathcal{F}_n}$ -measurable

③ $E[S_{n+1}^2 - (n+1)\sigma^2 \mid \mathcal{F}_n]$

$= E[(S_n + \xi_{n+1})^2 - (n+1)\sigma^2 \mid \mathcal{F}_n]$

$= E[S_n^2 + 2S_n\xi_{n+1} + \xi_{n+1}^2 \mid \mathcal{F}_n] - (n+1)\sigma^2$ ↓ $\boxed{15}$.(1)

$= \underbrace{S_n^2}_{(\boxed{12})} + \underbrace{2S_n E[\xi_{n+1} \mid \mathcal{F}_n]}_{(\boxed{20}, \boxed{13})} + \underbrace{E[\xi_{n+1}^2]}_{(\boxed{13})} - (n+1)\sigma^2$

$E[\xi_{n+1} \mid \mathcal{F}_n] = 0$

$= S_n^2 + \sigma^2 - (n+1)\sigma^2 = S_n^2 - n\sigma^2$

So the proof is complete

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$\boxed{26}$

① $E|M_n| = \prod_{m=1}^{n} E|Y_m| < \infty$

$(\because E[Y_m] = 1 \Rightarrow E[Y_m^+], E[Y_m^-] < \infty)$

② $M_n = g(Y_1, Y_2 \cdots Y_n)$ where $g(x_1, \cdots, x_n) = x_1 x_2 \cdots x_n$

($=$ continuous $\Rightarrow$ Borel-measurable)

$\therefore M_n$ is $\sigma[Y_1, Y_2 \cdots Y_n]$- measurable. ($\mathcal{F}_n$- measurable)

③ $E[M_{n+1} | \mathcal{F}_n] = E[Y_{n+1} \cdot M_n | \mathcal{F}_n]$, $(\boxed{20})$

$= M_n E[Y_{n+1} | \mathcal{F}_n] = M_n \cdot E[Y_{n+1}] = M_n$

$(\boxed{13})$

$\therefore$ By ①~③, $\{M_n\}_{n \geq 1}$ is a martingale.

PAGE

DATE

$\boxed{27}$

Since $\{X_n\}_{n\geq 1}$ is a supermartingale,

$E[X_n \mid \mathcal{F}_{n-1}] \leq X_{n-1}$

By $\boxed{15}$ (2),

$$E\big[E[X_n \mid \mathcal{F}_{n-1}] \mid \mathcal{F}_{n-2}\big] \leq E[X_{n-1} \mid \mathcal{F}_{n-2}]$$

$$\leq X_{n-2} \quad \text{supermartingale}$$

By $\boxed{19}$ (2), LHS $= E[X_n \mid \mathcal{F}_{n-2}]$

So we have $E[X_n \mid \mathcal{F}_{n-2}] \leq X_{n-2}$

By repeating the same argument $n-m$ times

we have $E[X_n \mid \mathcal{F}_m] \leq X_m$.

PAGE
DATE

$\boxed{28}$ We only show that $E[\phi(X_{n+1}) \mid \mathcal{F}_n] \geq \phi(X_n)$.

By Jensen's inequality ($\boxed{16}$),

$$E[\phi(X_{n+1}) \mid \mathcal{F}_n] \geq \phi(E[X_{n+1} \mid \mathcal{F}_n]) = \phi(X_n)$$

($\because \{X_n\}_{n\geq 1}$: martingale)

Now the proof is complete.

(For example, $\phi(x) = |x|^p$)

PAGE

DATE

$\boxed{29}$

By Jensen's inequality

$E[\phi(X_{n+1}) | \mathcal{F}_n] \geq \phi \circ E[X_{n+1} | \mathcal{F}_n]$

Since $\phi(\cdot)$ is an increasing function,

$E[X_{n+1} | \mathcal{F}_n] \geq X_n$ ($\because$ sub-martingale)

implies $\phi \circ E[X_{n+1} | \mathcal{F}_n] \geq \phi(X_n)$

So $E[\phi(X_{n+1}) | \mathcal{F}_n] \geq \phi(X_n)$

$\therefore$ $\phi(X_n)$ is a sub-martingale.

$\boxed{30}$

(1) $\{H_n\}_{n\geq 1}$ is a predictable sequence means that H_n is $\mathcal{F}_{n-1}$-measurable.

(2) $(H \cdot X)_n \overset{\text{def}}{=} \sum_{m=1}^{n} H_m (X_m - X_{m-1})$

3. • X_n … supermartingale

• $H_n \geq 0$ and predictable, and bounded

① $E|(H\cdot X)_n| \leq \sum_{m=1}^{n} E|H_m||X_m - X_{m-1}|$

$\leq M \sum_{m=1}^{n} E[|X_m| + |X_{m-1}|]$

$(|H_n| \leq M < \infty$ … ∵ Bounded$)$

$\leq 2M(E|X_0| + \dots + E|X_n|) < \infty$

$(\because \{X_n\}_{n\geq 1}$, supermartingale $\Rightarrow E|X_n| < \infty)$

② $\sum_{m=1}^{n} \underbrace{H_m}_{\mathcal{F}_{m-1}\text{-measurable}} \cdot \underbrace{(X_m - X_{m-1})}_{\mathcal{F}_m\text{-measurable}}$ … $\mathcal{F}_n$- measurable

③ $E[(H\cdot X)_{n+1} | \mathcal{F}_n]$

$= E[(H\cdot X)_n + H_{n+1}(X_{n+1} - X_n) | \mathcal{F}_n]$

$= (H\cdot X)_n + E[H_{n+1}(X_{n+1}-X_n) \mid \mathcal{F}_n]$ $\boxed{20}$

$= (H\cdot X)_n + \underbrace{H_{n+1}}_{\geq 0} \cdot \underbrace{E[X_{n+1}-X_n \mid \mathcal{F}_n]}_{= E[X_{n+1} \mid \mathcal{F}_n] - X_n \geq 0}$ (H_{n+1} .. $\mathcal{F}_n$ – measurable)

(by assumption) ($\because$ $\{X_n\}_{n\geq 1}$ .. superma)

$\geq (H\cdot X)_n$

So $\{(H\cdot X)_n\}_{n\geq 1}$ is also a super-martingale ∎

$\boxed{\text{NOTE}}$. If $\{X_n\}_{n\geq 1}$ – submartingale.

Then $E[(H\cdot X)_{n+1} \mid \mathcal{F}_n] = (H\cdot X)_n + \underbrace{H_{n+1}}_{\geq 0} \cdot \underbrace{E[X_{n+1}-X_n \mid \mathcal{F}_n]}_{\leq 0}$

$\leq (H\cdot X)_n$

So $\{(H\cdot X)_n\}_{n\geq 1}$ is a submartingale.

$\boxed{\text{NOTE}}$ If $\{X_n\}_{n\geq 1}$ – martingale. Then $\{(H\cdot X)_n\}_{n\geq 1}$ is a martingale (H_n does not have to be positive.)

PAGE
DATE

$\boxed{32}$

$\begin{cases} \cdot\ \{X_n\}_{n\geq 1} \cdots \text{ supermartingale} \\ \cdot\ N \cdots \text{ stopping time} \end{cases}$

Let $H_n \overset{\text{def}}{=} \mathbb{I}_{\{N \geq n\}}$.

Then $\{H_n\}_{n\geq 1}$ is a predictable sequence

because $\{N \geq n\}^c = \{N < n\} \in \mathcal{F}_{n-1}$

$\Rightarrow \{N \geq n\} \in \mathcal{F}_{n-1}$. $\therefore H_n \cdots \mathcal{F}_{n-1}$-measurable

Now we consider

$(H \cdot X)_n = \sum_{m=1}^{n} H_m (X_m - X_{m-1})$

$\boxed{\text{case 1}}$ $N \geq n$

$\Rightarrow N \geq m$ for all $m = 1 \sim n$

$\Rightarrow H_m = 1$ for all $m = 1 \sim n$

PAGE
DATE

$\therefore (H\cdot X)_n = X_n - X_0$

$\boxed{\text{case}}$ $N < n$

$\Rightarrow H_m = I\{N \geq m\} = 0$ for all $m = N+1 \sim n$

$\Rightarrow (H\cdot X)_n = \sum_{m=1}^{n} H_m (X_m - X_{m-1}) = \sum_{m=1}^{N} (\sim) + \sum_{m=N+1}^{n} (\sim)$

$= \sum_{m=1}^{N} 1 \cdot (X_m - X_{m-1}) = X_N - X_0$

Hence $(H\cdot X)_n = X_{n\wedge N} - X_0$ $\left(n \wedge N \overset{\text{def}}{=} \min\{n, N\} \right)$

Since $\{H_n\}_{n\geq 1}$ is positive, bounded

so $(H\cdot X)_n = X_{n\wedge N} - X_0$ is also a supermartingale.

It's easy to verify that $X_{n\wedge N}$ is a supermartingale

(⊛ $\{X_n\}, \{Y_n\}$ are supermartingale $\Rightarrow \{X_n + Y_n\}$: supermartingale

Let $Y_n \overset{\text{def}}{=} X_0$. $\Rightarrow \{Y_n\}_{n\geq 1}$: martingale $\Rightarrow$ supermartingale)

$\boxed{33}$ Here we prove the upcrossing inequality.

(1) For proof, we introduce $\{N_k\}_{k\geq 0}$ stopping times

- $N_0 \stackrel{def}{=} -1$.
- $N_{2k-1} \stackrel{def}{=} \inf\{n \mid n > N_{2k-2},\ X_n \leq a\}$
- $N_{2k} \stackrel{def}{=} \inf\{n \mid n > N_{2k-1}\ \ X_n \geq b\}$

⊛ [We can imagine that $\{X_n\}_{n\geq 1}$ is going back and forth around $[a,b]$.

N_{2k-1}, N_{2k} are the index number of $\{X_n\}_{n\geq 1}$ which goes above "b" or below "a".]

Now we prove that $\{N_k\}_{k\geq 1}$ are stopping times by induction.

PAGE
DATE

Suppose $N_1, N_2 \ldots N_{2k-2}$ are stopping times.

Then $\{N_{2k-1} = n\} = \bigcup_{m=1}^{n-1} \{N_{2k-2} = m\} \cap \{N_{2k-1} = n\}$

$$= \bigcup_{m=1}^{n-1} \{N_{2k-2} = m\} \cap \{X_{m+1} > a\} \cap \ldots\ldots$$

$$\cap \{X_{n-1} > a\} \cap \{X_n \leq a\}$$

$$\in \mathcal{F}_n$$

$(\because$ By assumption $\{N_{2k-2} = m\} \in \mathcal{F}_m \subseteq \mathcal{F}_n$

and $\{X_{m+1} > a\} \ldots \{X_n \leq a\} \in \mathcal{F}_n.)$

So N_{2k-1} is also a stopping time

Similarly N_{2k} is also a stopping time

Hence the proof is complete.

PAGE
DATE

(2) $H_m \overset{\text{def}}{=} \mathbb{I}_{\bigcup_{k=1}^{\infty}\{N_{2k-1} < m \leq N_{2k}\}}$

$= \mathbb{I}_{\bigcup_{k=1}^{\infty}\{N_{2k-1} < m\}} \cdot \mathbb{I}_{\{m \leq N_{2k}\}}$

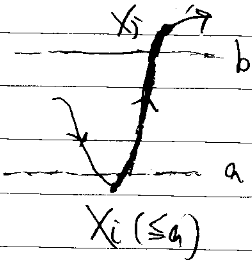
$= \mathbb{I}_{\bigcup_{k=1}^{\infty}\underbrace{\{N_{2k-1} < m\}}_{\in \mathcal{F}_{m-1}}} \cdot \mathbb{I}_{\underbrace{\{N_{2k} < m\}^c}_{\in \mathcal{F}_{m-1}}}$

Hence, H_m is $\mathcal{F}_{m-1}$ - measurable.

$\therefore$ It's a predictable sequence.

(3) Now we define $U_n = \sup\{k \mid N_{2k} \leq n\}$

U_n is the number of upcrossings seen between $\{X_1, X_2 \cdots X_n\}$.



$\cdots$ an upcrossing

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(4) If $\{X_n\}_{n\geq 1}$ is a submartingale, then

$$(b-a)E[U_n] \leq E[(X_n-a)^+] - E[(X_0-a)^+]$$

(proof)

Let $\phi(x) = a + (x-a)^+$. $\left((x-a)^+ = \max\{x-a, 0\} \right)$

Then $\phi(x)$ is convex and monotone increasing.

$Y_n \overset{def}{=} \phi(X_n)$.

By $\boxed{29}$, Y_n is also a submartingale.

By $\boxed{31}$, $(H\cdot Y)_n$ is also a submartingale

(See $\boxed{31}$. $\boxed{\text{NOTE}}$ the statement still holds for submartingale)

(H_n is defined as (2))

Now $(H\cdot Y)_n = \sum_{m=1}^{n} H_m (Y_m - Y_{m-1})$

$$= \sum_{m=1}^{n} \underbrace{\mathbb{I}\bigcup_{k=1}^{\infty}\{N_{2k-1} < m \leq N_{2k}\}}_{} \cdot (Y_m - Y_{m-1})$$

This part is 1 when X_n is going up from "a" to "b"

otherwise it is 0. (See the definition of N_{2k-1} and N_{2k})

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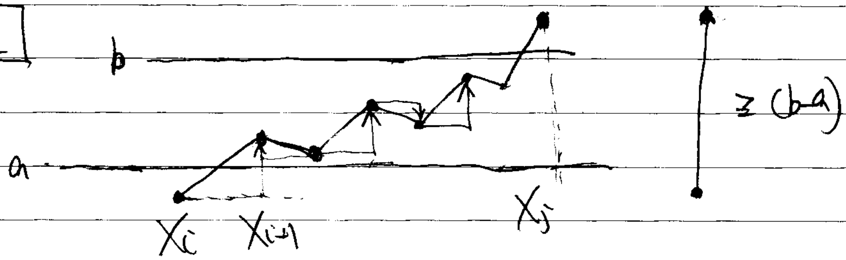
So if we sum up $(X_m - X_{m-1})$ for such m,

then it will exceed $(b-a)$ ($\times$ number of upcrossings)

Fig



$$\Rightarrow (X_{i+1} - X_i) + (X_{i+2} - X_{i+1}) + \dots + (X_j - X_{j-1}) = X_j - X_i \geq (b-a)$$

It's not difficult to see that we may change X_m to Y_m. ($Y_i \to 0$, $Y_{i+1} = X_{i+1}, \dots$)

So $(H \cdot Y)_n \geq (b-a) \cdot U_n$

number of upcrossings

Now let $K_m = 1 - H_m$

Then $(K\cdot Y)_n + \underbrace{(H\cdot Y)_n}_{\geq (b-a)U_n} = Y_n - Y_0$

$\therefore Y_n - Y_0 \geq (K\cdot Y)_n + (b-a)\cdot U_n \quad \cdots \; (*_1)$

We should notice that $K_m \geq 0$ and K_m is also a predictable sequence, hence $(K\cdot Y)_n$ is a submartingale.

$\therefore E[(K\cdot Y)_n \mid \mathcal{F}_{n-1}] \geq (K\cdot Y)_{n-1}$

$\Rightarrow E[(K\cdot Y)_n] \geq E[(K\cdot Y)_{n-1}]$

$\therefore \Rightarrow E[(K\cdot Y)_n] \geq E[(K\cdot Y)_1]$

$(K\cdot Y)_1 = \mathbb{I}_{\bigcap_{k=1}^{\infty}\{N_{2k-1} < m \leq N_{2k}\}^c}(Y_1 - Y_0) \quad (= Y_1 - Y_0 \text{ or } 0)$

$\geq \min\{0, Y_1 - Y_0\}$

$\therefore E[(K\cdot Y)_1] \geq \min\{E[0], E[Y_1 - Y_0]\} \geq 0$

($\because \{Y_n\}_{n\geq 1}$ submartingale)

$\therefore E[(K\cdot Y)_n] \geq 0 \quad \cdots \; (*_2)$

Finally $E[Y_n - Y_0] \geq (b-a)E[U_n]$ (by $(*_1)$, $(*_2)$)

PAGE

DATE

$\boxed{34}$ We show the martingale convergence theory

- $\{X_n\}_{n\geq 1}$ ... submartingale
- $\sup_{n\geq 1} E[X_n^+] < \infty$

Recall the upcrossing inequality

$$(b-a)E[U_n] \leq E[(X_n-a)^+] - E[(X_0-a)^+]$$

By definition of $U_n = \sup_{k\geq 1}\{k \mid N_{2k} \leq n\}$

$U_n \leq U_{n+1}$. $\therefore U_n \uparrow U$.

So monotone convergence theorem implies that

$$\lim_{n\to\infty} (b-a)E[U_n] = (b-a)E[U]$$

$$(\sim) \leq \liminf_{n\to\infty} \left(E[(X_n-a)^+] - E[(X_0-a)^+] \right)$$

Since $(X_n-a)^+ \leq X_n^+ + |a|$

We have $(b-a)E[U] \leq \sup_{n\geq 1} E[X_n^+] + |a| - E[(X_0-a)^+] < \infty$.

PAGE
DATE

So $E[U] < \infty \Rightarrow$ "$U < \infty$" (a.s.)

This means that the number of upcrossings in $\{X_n\}_{n \in \mathbb{N}}$ is finite almost surely.

Hence the number of $\{X_n\}_{n \geq 1}$ which is larger than "b" is finite (a.s). Similarly the number of $\{X_n\}_{n \geq 1}$ which is smaller than "a" is finite (a.s)

So $P(\{\liminf X_n < a < b < \limsup X_n\}) = 0$

($\because$ $\limsup X_n \leq b$ (a.s) ; $\liminf X_n \geq a$ (a.s))

(for all a, b)

$\therefore P\left(\bigcup_{(a,b) \in \mathbb{Q}^2} \{\liminf X_n < a < b < \limsup X_n\}\right) = 0$

(Q: rational numbers)

$\Rightarrow P(\liminf X_n = \limsup X_n) = 1$ (Q: dense set)

$\therefore \lim_{n \to \infty} X_n$ exists almost surely.

Finally we show that $E|X| < \infty$

- By Fatou's lemma

$$E[\liminf X_n^+] = E[X^+] \leq \liminf E[X_n^+] \leq \sup_{n \geq 1} E[X_n^+] < \infty$$

$\therefore E[X^+] < \infty$

- Again by Fatou's lemma.

$$E[\liminf X_n^-] = E[X^-] \leq \liminf E[X_n^-] \quad \cdots \circledast$$

And $E[X_n^-] = E[X_n^+] - E[X_n]$.

Since $\{X_n\}_{n \geq 1}$ is a submartingale,

$$E[X_n] \geq E[X_{n-1}] \geq \cdots \geq E[X_0]$$

So $E[X_n^-] \leq E[X_n^+] - E[X_n] \leq E[X_n^+] - E[X_0]$

$\therefore \circledast \leq \sup_{n \geq 1} E[X_n^+] - E[X_0] < \infty$

$\therefore E[X^-] < \infty$

PAGE
DATE

$\boxed{35}$ $X_n \geq 0$, X_n: supermartingale.

$\Rightarrow$ $-X_n \leq 0$, $-X_n$: submartingale

$\Rightarrow$ $(-X_n)^+ = 0$ $\therefore$ $\sup_{n\geq 1} E[(-X_n)^+] = 0 < \infty$

By $\boxed{34}$, $-X_n$ converges (a.s)

$\Leftrightarrow$ X_n converges (a.s)

$\therefore$ $X_n \to X$ (a.s)

Since $\{X_n\}_{n\geq 1}$ is a supermartingale

$E[X_n | \mathcal{F}_{n-1}] \leq X_{n-1}$ $\Rightarrow$ $E[X_n] \leq E[X_{n-1}] \leq E[X_0]$

By Fatou's lemma, $\underset{\displaystyle E[X]}{\underset{\shortparallel}{E[\liminf X_n]}} \leq \liminf E[X_n] \leq E[X_0]$

So the proof is complete.

PAGE
DATE

$\boxed{36}$

Let $S_0 = 1$, $S_1 = 1 + \xi_1$, $S_2 = 1 + \xi_1 + \xi_2$, ...

where $\xi_1, \xi_2 \ldots \xi_n \ldots$ are i.i.d random variables

$P(\xi_i = 1) = P(\xi_i = -1) = \frac{1}{2}$.

Now let $\mathcal{F}_n = \sigma[\xi_1, \xi_2, \ldots \xi_n]$.

then $E[S_{n+1} | \mathcal{F}_n] = E[\xi_{n+1} + S_n | \mathcal{F}_n]$

$= S_n + E[\xi_{n+1} | \mathcal{F}_n] = S_n + \underset{\overset{\shortparallel}{0}}{E[\xi_{n+1}]} = S_n$

$\boxed{12}$ $\boxed{13}$

. $E[S_n] = 0 \in (-\infty, \infty)$

- S_n : $\mathcal{F}_n$-measurable

→ So $\{S_n\}$ is a martingale

Let $N = \inf\{n \mid S_n = 0\}$.

Then N is a stopping time.

By $\boxed{32}$, $S_{n \wedge N}$ is a martingale

($\because$ It's easy to verify $\boxed{32}$ holds for martingales / submartingales)

Hence $S_{n \wedge N}$ is a supermartingale

And $S_{n \wedge N} \geq 0$ ($\because N \overset{def}{=} \inf\{n \mid S_n = 0\}$
and $S_0 = 1 > 0$)

By $\boxed{35}$ $S_{n \wedge N}$ converges almost surely. $(S_{n \wedge N} \xrightarrow{a.s} S^*)$

And $\nexists k \geq 1$, $P(S^* = k) > 0$.

because if $S_{n \wedge N} = k$. then $S_{(n+1) \wedge N} = k+1$ or $k-1$

Therefore $P(S^* = 0) = 1$ $\therefore S^* = 0$ (a.s)

Now $E|S_{n \wedge N} - S^*| = E|S_{n \wedge N}| = E[S_{n \wedge N}] \geq 1 > 0$ $(\forall n \geq 1)$

$\Rightarrow$ does not converge in L^1

PAGE

DATE

$\boxed{37}$ Consider $\{X_k\}_{k\geq 1}$: which is defined as

- $X_k \mid X_{k-1} = 0 \quad \begin{cases} 1 & \left(\frac{1}{2k}\right) \\ -1 & \left(\frac{1}{2k}\right) \\ 0 & \left(1-\frac{1}{k}\right) \end{cases}$

- $X_k \mid X_{k-1} \neq 0 \quad \begin{cases} kX_{k-1} & \left(\frac{1}{k}\right) \\ 0 & \left(1-\frac{1}{k}\right) \end{cases}$

- $X_0 = 0$

$\mathcal{F}_k = \sigma[X_1, X_2 \ldots X_k]$

Then $E[X_k \mid \mathcal{F}_{k-1}] = E[X_k \mid X_{k-1}] = X_{k-1}$

because $\cdot$ $E[X_k \mid X_{k-1} = 0] = \frac{1}{2k} \cdot (1) + \frac{1}{2k}(-1) = 0$

$\cdot$ $\underset{(\neq 0)}{E[X_k \mid X_{k-1} = x_{k-1}]} = (kx_{k-1}) \cdot \frac{1}{k} = x_{k-1}$.

$\therefore E[X_k] = E[X_{k-1}] = E[X_0] = 0$

So $\{X_k\}_{k\geq 1}$ is a martingale

PAGE

DATE

$P(X_k=0) = E[\mathbb{I}\{X_k=0\}] = E\cdot E[\mathbb{I}\{X_k=0\} \mid X_{k-1}]$

$= E\cdot P(X_k=0 \mid X_{k-1}) = E[1-\frac{1}{k}] = 1-\frac{1}{k}$

$\therefore \lim_{k\to\infty} P(X_k=0)=1 \quad \therefore X_k \xrightarrow{P} 0.$

Next we show that $X_k \not\to 0$ (a.s)

Consider the probability $P(X_k=0 \text{ for all } k\geq K)$

$= \underbrace{P(X_K=0)}_{\leq 1}\cdot \underbrace{P(X_{K+1}=0 \mid X_K=0)}_{=1-\frac{1}{K+1}}\cdot \underbrace{P(X_{K+2}=0 \mid X_{K+1}=0)}_{=1-\frac{1}{K+2}}\cdots$

$\therefore P(X_k=0 \text{ for all } k\geq K) \leq \prod_{k\geq K+1}^{\infty}(1-\frac{1}{k}) = 0 \ \cdots ⊛$

(⊛ $\forall x\in\mathbb{R},\ e^{x}\geq 1+x \Rightarrow e^{-x}\geq 1-x \Rightarrow -x \geq \ln(1-x)$

$\Rightarrow \ln(1-\frac{1}{k}) \leq -\frac{1}{k} \Rightarrow \sum_{k\geq K+1}^{\infty} \ln(1-\frac{1}{k}) = -\infty \Rightarrow \prod_{k\geq K+1}^{\infty}(1-\frac{1}{k})=0$)

And X_k is integer-valued.

So $X_k\to 0 \Leftrightarrow \exists K$ s.t $X_k=0$ for all $k\geq K$.

$\therefore P(X_k\to 0) = P\left(\bigcup_{K=1}^{\infty}\{X_k=0 \text{ for all } k\geq K\}\right) \leq \sum_{K=1}^{\infty} P(\sim) = 0$

PAGE

DATE

$\boxed{38}$ $Z_n = X_n \vee Y_n = \max\{X_n, Y_n\}$

is also a submartingale

① $|Z_n| \leq |X_n| + |Y_n|$

$E|Z_n| \leq E|X_n| + E|Y_n| < \infty$

② Z_n is $\mathcal{F}_n$-measurable (X_n, Y_n are $\mathcal{F}_n$-measurable)

③ $Z_{n+1} = X_{n+1} \vee Y_{n+1} \geq X_{n+1}$

$(\geq Y_{n+1})$

$\therefore E[Z_{n+1} | \mathcal{F}_n] \geq E[X_{n+1} | \mathcal{F}_n] \geq X_n$

$\because$ X_n is a submartingale

Similarly $E[Z_{n+1} | \mathcal{F}_n] \geq Y_n$

$\therefore E[Z_{n+1} | \mathcal{F}_n] \geq \max\{X_n, Y_n\}$

$\qquad\qquad\qquad\qquad\qquad\qquad\ \ \|$

$\qquad\qquad\qquad\qquad\qquad\qquad\ Z_n$

$\therefore \{Z_n\}_{n \geq 1}$ is a sub-martingale

PAGE

DATE

$\boxed{39}$

(1) We use the supermartingale convergence theorem $\boxed{35}$.

$X_n \geq 0$ and X_n is a martingale ($\Rightarrow$ supermartingale)

$\therefore X_n \to X$ (as)

Next, $X_n \to X$ (as) $\Rightarrow X_n \xrightarrow{P} X$.

It's enough to show that $X_n \xrightarrow{P} 0$

$P(|Y_m - 1| > 0) = P(Y_m \neq 1) > 0$

$\Rightarrow$ We may find small $\varepsilon > 0$ st $P(|Y_m - 1| \geq \varepsilon) > 0$

Let $\delta > 0$ be an arbitrary positive number

$P(|X_{n+1} - X_n| \geq \delta\varepsilon) = P(|Y_{n+1} - 1||X_n| \geq \delta\varepsilon)$

$\geq P(|Y_{n+1} - 1| \geq \varepsilon \cap |X_n| \geq \delta)$

$= \underbrace{P(|Y_{n+1} - 1| \geq \varepsilon)}_{>0} P(|X_n| \geq \delta)$

And LHS : $P(|X_{n+1} - X_n| \geq \delta\varepsilon) \leq P(|X_{n+1} - X| \geq \frac{\delta\varepsilon}{2})$

$+ P(|X_n - X| \geq \frac{\delta\varepsilon}{2})$

$\to 0$ as $n \to \infty$. ($\because X_n \xrightarrow{P} X$)

PAGE

DATE

Hence RHS $\to 0$ as $n \to \infty$

So $P(|X_n| \geq \delta) \to 0$ as $n \to \infty$

Since $\delta > 0$ is arbitrary, so $X_n \xrightarrow{P} 0$.

Now the proof is complete.

(2) $\frac{1}{n} \log \prod_{m=1}^{n} Y_m = \frac{\log Y_1 + \dots + \log Y_n}{n}$

By SLLN. $\xrightarrow{as} E[\log Y_1] = c$

It's enough to show that $E[\log Y_1] < 0$

By Jensen's inequality. $E[\log Y_1] \leq \log \underbrace{E[Y_1]}_{1} = 0$,

Its equality holds when $Y_1 =$ constant (as)

But by assumption. $P(Y_1 = 1) < 1$

So equality does not hold

$\therefore$ $c = E[\log Y_1] < 0$

$\boxed{40}$ Let $W_n \overset{\text{def}}{=} \dfrac{X_n}{\prod_{m=1}^{n-1}(1+Y_m)}$.

Then $E[W_{n+1} \mid \mathcal{F}_n] = E\left[\dfrac{X_{n+1}}{\prod_{m=1}^{n}(1+Y_m)} \,\middle|\, \mathcal{F}_n\right]$

$\boxed{20}$ $\left(\prod_{m=1}^{n}(1+Y_m) : \mathcal{F}_n\text{-measurable}\right)$

$= \dfrac{1}{\prod_{m=1}^{n}(1+Y_m)} E[X_{n+1} \mid \mathcal{F}_n]$

by assumption

$\leq \dfrac{1}{\prod_{m=1}^{n}(1+Y_m)}(1+Y_n)X_n = \dfrac{X_n}{\prod_{m=1}^{n-1}(1+Y_m)} = W_n$

So $\{W_n\}_{n\geq 1}$ is a supermartingale.

Since $\{X_n\}_{n\geq 1}$, $\{Y_n\}_{n\geq 1}$ are positive, $W_n \geq 0$

We may apply $\boxed{35}$ to $\{W_n\}_{n\geq 1}$

$\therefore W_n \to W$ (as)

PAGE
DATE

Next. We consider convergence of $\prod_{m=1}^{n-1}(1+Y_m)$.

- $\ln \prod_{m=1}^{n-1}(1+Y_m) = \sum_{m=1}^{n-1} \ln(1+Y_m)$
- $1+x \leq e^x \Rightarrow \ln(1+x) \leq x \Rightarrow \sum_{m=1}^{n-1} \ln(1+Y_m) \leq \sum_{m=1}^{n-1} Y_m$

$\therefore 0 \leq \sum_{m=1}^{n-1} \ln(1+Y_m) \leq \sum_{m=1}^{n-1} Y_m$

(Y_n: positive, so $\ln(1+Y_m) \geq 0$)

Since $\sum_{n=1}^{\infty} Y_n < \infty$ (a.s), $\sum_{n=1}^{\infty} \ln(1+Y_n)$ also converges

to a finite limit (a.s).

Hence $\prod_{m=1}^{n-1}(1+Y_m)$ converges (a.s)

$\therefore \dfrac{X_n}{\prod_{m=1}^{n-1}(1+Y_m)} \cdot \prod_{m=1}^{n-1}(1+Y_m) = X_n$ converges (a.s)

$\boxed{4}$

$\boxed{\text{Step 1}}$

Let $N_k \overset{\text{def}}{=} \inf\{n \mid X_n \leq -k\}$

Since $D = \{\limsup X_n = \infty \cap \liminf X_n = -\infty\}$

If $w \in D^c$, then $\underbrace{\liminf X_n(w) > -\infty}_{①}$ or $\underbrace{\limsup X_n(w) < \infty}_{②}$

Now suppose ① $\liminf_{n\to\infty} X_n(w) > -\infty$.

We may find a positive number $\tilde{K}(w) < \infty$

such that $\lim_{n\to\infty} \inf_{m \geq n} X_m(w) > -\tilde{K}(w)$

For sufficiently large m (say $m > M(w)$), $X_m(w) > -\tilde{K}(w)$

Let $-K(w) = \min\{X_1(w), \ldots, X_{M(w)}(w), -\tilde{K}(w)\}$

Then $X_m(w) > -K(w)$ for all $m \geq 1$.

$\therefore \{\liminf X_n(w) > -\infty\} \subseteq \bigcup_{k=1}^{\infty} \{X_m(w) > -k \text{ for all } m \geq 1\}$

$\subseteq \bigcup_{k=1}^{\infty} \{N_k = \infty\}$

(Similar argument is valid for ②)

$\boxed{\text{Step2}}$

- Again $N_k \overset{def}{=} \inf\{n \mid X_n \le -K\}$
  where K is a constant and is not related with ω.

- Since N_k is a stopping time,
  $X_{n \wedge N_k}$ is also a martingale by $\boxed{32}$.

- $X_{n \wedge N_k} + K + M \ge 0$
  $(\because X_{n \wedge N_k - 1} > -K$ and $|X_m - X_{m-1}| \le M < \infty)$

- We apply $\boxed{35}$ to $\{X_{n \wedge N_k} + K + M\}_{n \ge 1}$
  $\lim_{n \to \infty} X_{n \wedge N_k} + K + M$ exists and finite (a.s)

$(0 \le E[\lim_{n \to \infty} X_{n \wedge N_k} + K + M] \le E[X_1 + K + M] < \infty$

$\therefore \lim_{n \to \infty} X_{n \wedge N_k} + K + M < \infty$ (a.s) $)$ (We suppose $X_0 = 0$)

- Hence $\lim_{n \to \infty} X_{n \wedge N_k}$ exists and finite (a.s)

- If $\omega \in \{N_k = \infty\}$, Then $X_{n \wedge N_k} = X_n$
  So $\lim_{n \to \infty} X_n$ exists and finite

PAGE
DATE

$\boxed{\text{step3}}$ Now recall $\boxed{\text{Step 1}}$

$\omega \in D^c \overset{*}{\Rightarrow} \omega \in \bigcup_{k=1}^{\infty} \{N_k = \infty\}$

$\Rightarrow \exists K \in \mathbb{N}$ st $N_K = \infty$ $\downarrow$ $\boxed{\text{step2}}$

$\Rightarrow \lim_{n\to\infty} X_n$ exists and finite (as)

$\Rightarrow \omega \in C$ (as)

(Except ω in a measure zero set)

$\therefore P(C \cup D) = 1$ ■

(* We should both discuss $\omega \in \{\liminf X_n > -\infty\}$
and $\omega \in \{\limsup X_n < \infty\}$

but they are quite similar

PAGE
DATE

$\boxed{42}$ (Doob's decomposition)

$A_n \overset{\text{def}}{=} \sum_{m=1}^{n} E[X_m - X_{m-1} \mid \mathcal{F}_{m-1}]$

$M_n \overset{\text{def}}{=} X_n - A_n$

Then $X_n = M_n + A_n$

① A_n is a predictable and increasing sequence

- A_n is $\mathcal{F}_{n-1}$-measurable by its definition.
- $E[X_m - X_{m-1} \mid \mathcal{F}_{m-1}] = \underbrace{E[X_m \mid \mathcal{F}_{m-1}]}_{\geq X_{m-1}} - X_{m-1} \geq 0$

So A_n is increasing.

② M_n is a martingale

- $E[A_{n+1} \mid \mathcal{F}_n] = A_{n+1}$ ($\because$ A_{n+1} is $\mathcal{F}_n$-measurable and by $\boxed{12}$)
- $E[M_{n+1} \mid \mathcal{F}_n) = E[X_{n+1} - A_{n+1} \mid \mathcal{F}_n)$

$= E[\underbrace{X_{n+1}}_{(X_{n+1} - X_n + X_n.)} \mid \mathcal{F}_n] - A_{n+1}$

CHA SHIN

PAGE
DATE

$$= E[X_{n+1} - X_n \mid \mathcal{F}_n] + E[X_n \mid \mathcal{F}_n] - A_{n+1}$$

$$\left(\hookrightarrow \sum_{m=1}^{n-1} E[X_m - X_{m-1} \mid \mathcal{F}_{m-1}] \right)$$

$$= E[X_n \mid \mathcal{F}_n] - \sum_{m=1}^{n} E[X_m - X_{m-1} \mid \mathcal{F}_{m-1}]$$

$$= X_n - \sum_{m=1}^{n} E[X_m - X_{m-1} \mid \mathcal{F}_{m-1}]$$

$$= X_n - A_n = M_n$$

$\therefore$ M_n is a martingale

PAGE

DATE

$\mathcal{F}_0 = \{\phi, \Omega\}$, $A_n \in \mathcal{F}_n$

$\boxed{43}$ The statement in the textbook may be wrong

Here we prove that $P\left(\{A_n \text{ i.o.}\} \triangle \left\{\sum_{n=1}^{\infty} P(A_n \mid \mathcal{F}_{n-1}) = \infty\right\}\right) = 0$

(proof) $X_n \stackrel{def}{=} \sum_{m=1}^{n} \mathbb{I}_{A_m}$

$$E[X_{n+1} \mid \mathcal{F}_n] = X_n + \underbrace{E[\mathbb{I}_{A_{n+1}} \mid \mathcal{F}_n]}_{\geq 0} \geq X_n$$

$\therefore \{X_n\}_{n\geq 1}$ is a submartingale

By $\boxed{42}$, $X_n = M_n + K_n$ (Doob's decomposition)

where M_n: martingale, K_n: increasing predictable sequence

$$M_n = X_n - \sum_{m=1}^{n} E[X_m - X_{m-1} \mid \mathcal{F}_{m-1}]$$

$$= \sum_{m=1}^{n} \mathbb{I}_{A_m} - \sum_{m=1}^{n} E[\mathbb{I}_{A_m} \mid \mathcal{F}_{m-1}]$$

$$= \sum_{m=1}^{n} \left(\mathbb{I}_{A_m} - P(A_m \mid \mathcal{F}_{m-1})\right)$$

$$|M_n - M_{n-1}| = |\mathbb{I}_{A_n} - P(A_n \mid \mathcal{F}_{n-1})| \leq 2$$

So we may apply $\boxed{41}$.

PAGE

DATE

$C = \{ \lim_{n\to\infty} M_n$ exists and is finite $\}$
$D = \{ \limsup_{n\to\infty} M_n = \infty$ and $\liminf_{n\to\infty} M_n = -\infty \}$

$P(C \cup D) = 1$

① $\omega \in C$

If $\sum_{n=1}^{\infty} \mathbb{I}_{A_n}(\omega) = \infty$, then $\sum_{n=1}^{\infty} P(A_n | \mathcal{F}_{n-1}) = \infty$

($\because M_n = \sum_{m=1}^{n} \{ \mathbb{I}_{A_m} - P(A_m | \mathcal{F}_{m-1}) \}$
and $\lim_{n\to\infty} M_n$ exists and finite)

Similarly, $\sum_{n=1}^{\infty} P(A_n | \mathcal{F}_{n-1}) = \infty \Rightarrow \sum_{n=1}^{\infty} \mathbb{I}_{A_n}(\omega) = \infty$

$\therefore$ " $\sum_{n=1}^{\infty} \mathbb{I}_{A_n}(\omega) = \infty \Leftrightarrow \sum_{n=1}^{\infty} P(A_n | \mathcal{F}_{n-1}) = \infty$ "

② $\omega \in D$

$\sum_{n=1}^{\infty} \mathbb{I}_{A_n}(\omega) = \infty$ and $\sum_{n=1}^{\infty} P(A_n | \mathcal{F}_{n-1}) = \infty$

So if $\omega \in C \cup D \Rightarrow$ " $\sum_{n=1}^{\infty} \mathbb{I}_{A_n}(\omega) = \infty \Leftrightarrow \sum_{n=1}^{\infty} P(A_n | \mathcal{F}_{n-1}) = \infty$ "

$\therefore \underset{(\{A_n \text{ i.o.}\})}{\{ \sum_{n=1}^{\infty} \mathbb{I}_{A_n}(\omega) = \infty \}} \triangle \{ \sum_{n=1}^{\infty} P(A_n | \mathcal{F}_{n-1}) = \infty \} \subseteq (C \cup D)^c$

$P((C \cup D)^c) = 0$. The statement is true. ∎

CHA SHIN

PAGE

DATE

$\boxed{44}$ $\{\xi_i^{(n)}\}_{\substack{i\geq 1\\ n\geq 1}}$ … iid, non-negative integer valued.

$$Z_{n+1} \overset{\text{def}}{=} \begin{cases} \xi_1^{(n+1)} + \cdots + \xi_{Z_n}^{(n+1)} & (\text{if } Z_n \geq 1) \\ 0 & (\text{if } Z_n = 0) \end{cases}$$

Then $\{Z_n\}_{n\geq 1}$ is called a Galton-Watson process

CHA SHIN

45 Z_n is a Galton-Watson process

where $E[\xi_i^{(n)}] = \mu > 0$.

Let $W_n = \frac{Z_n}{\mu^n}$.

① W_n is $\mathcal{F}_n$-measurable

- $Z_1 = \xi_1^{(1)} \cdots \mathcal{F}_1$-measurable

- Suppose Z_k is $\mathcal{F}_k$-measurable

Then $Z_{k+1} = \underbrace{\mathbb{I}\{Z_k > 0\}}_{\in \mathcal{F}_k \subseteq \mathcal{F}_{k+1}} \cdot \underbrace{\left(\sum_{j=1}^{Z_k} \xi_j^{(k+1)} \right)}_{\in \sigma[\mathcal{F}_k \cup \{\xi_i^{(k+1)}\}_{i \geq 1}] \subseteq \mathcal{F}_{k+1}}$

By induction Z_k is $\mathcal{F}_k$-measurable (So is W_k)

② $E[W_{n+1} | \mathcal{F}_n] = E\left[\frac{Z_{n+1}}{\mu^{n+1}} \middle| \mathcal{F}_n\right] = E\left[\frac{\xi_1^{(n+1)} + \cdots + \xi_{Z_n}^{(n+1)}}{\mu^{n+1}} \middle| \mathcal{F}_n\right]$

$= \frac{\mu \cdot Z_n}{\mu^{n+1}} = \frac{Z_n}{\mu^n} = W_n$ ($\xi_i^{(n+1)} \sim \xi_{Z_n}^{(n+1)}$ are independent of $\mathcal{F}_n$)

③ $E[W_{n+1}] = E[W_n]$ (by ②)

$E[W_1] = E\left[\frac{Z_1}{\mu}\right] = E\left[\frac{\xi_1}{\mu}\right] = 1$

$\therefore |E[W_n]| < \infty$

$\boxed{46}$ We show that if $\mu<1$, then $Z_n=0$ for sufficiently large n.

$W_n \overset{def}{=} \frac{Z_n}{\mu^n}$ (by $\boxed{45}$ $E[W_n]=1 \Rightarrow E[Z_n]=\mu^n$)

Since W_n is a martingale and non-negative

by $\boxed{35}$, $W_n \to W$ (as) $E[W] \leq E[W_1]=1$

Next we show that $W_n \xrightarrow{P} 0$.

$P(|W_n|>\varepsilon) \leq P(W_n>0) = P(Z_n>0) \leq E[Z_n \cdot \mathbb{I}\{Z_n>0\}]$

$\leq E[Z_n] = \mu^n$ $\quad$ ($E[\mathbb{I}\{Z_n>0\}] \leq E[Z_n \cdot \mathbb{I}\{Z_n>0\}]$

$\because Z_n>0 \Rightarrow Z_n \geq 1$)

Hence $n \nearrow \infty$ $\mu^n \downarrow 0$. $\therefore W_n \xrightarrow{P} 0$. $\therefore W=0$

$\frac{Z_n}{\mu^n} \xrightarrow{as} 0$. Since Z_n is non-negative integer-valued

and $\mu^n \ll 1$ $Z_n=0$ for sufficiently large n.

PAGE

DATE

Otherwise, there exist infinitely many n such that $Z_n \geq 1$. Then $\limsup \frac{Z_n}{\mu^n} = +\infty$.

So the statement holds ■

CHA SHIN

PAGE

DATE

$\boxed{47}$ $\boxed{\text{step 1}}$

Let $k \geq 1$ (integer) and $N \geq 1$ (integer)

Consider $P(Z_n = k \text{ for all } n \geq N)$

$= P(Z_N = k, Z_{N+1} = k \dots)$

$\leq P(Z_N = k) \cdot P(Z_{N+1} = k \mid Z_N = k) \cdot P(Z_{N+2} = k \mid$

$Z_{N+1} = k, Z_N = k) \cdot \dots\dots$

$= P(Z_N = k) \cdot \underbrace{P(Z_{N+1} = k \mid Z_N = k)}_{= \rho} \cdot \underbrace{P(Z_{N+2} = k \mid Z_{N+1} = k)}_{= \rho} \dots\dots$

$\leq P(Z_N = k) \cdot \rho^m \quad (m \geq 1)$

$\rho < 1$ because $\rho = 1$ only when $k = 1$ and

$P(\xi_i^{(n)} = 1) = 1$.

But $P(\xi_i^{(m)} = 1) < 1$ so $\rho < 1$

By taking $m \to \infty$ we have R.H.S $\to 0$.

CHA SHIN

So L.H.S $= 0$.

PAGE

DATE

$\therefore P(\{Z_n=k \text{ for all } n \geq N\})=0$

$\Rightarrow P\left(\bigcup_{N=1}^{\infty}\{Z_n=k \text{ for all } n \geq N\}\right)=0$

$= P(\{Z_n=k \text{ for sufficiently large } n\})=0$

for all $k \geq 1$.

$\boxed{\text{Step2}}$

$W_n = \frac{Z_n}{\mu^n} = Z_n$ is a martingale and non-negative

By $\boxed{35}$, $Z_n \to Z$ (as) and $Z < \infty$ (as)

Similar to $\boxed{46}$, $Z_n = Z$ for sufficiently large $n \geq 1$

because Z_n is integer-valued (so is Z).

Otherwise there exists infinitely many $n \geq 1$

$Z_n \neq Z \Rightarrow \limsup |Z_n - Z| \geq 1$.

$\boxed{\text{Step3}}$ Finally by $\boxed{\text{Step1}}$ and $\boxed{\text{Step2}}$,

we have $P(\{Z_n=0 \text{ for sufficiently large } n\})=1$

CHA SHIN

So $Z=0$ (as)

$\boxed{48}$ $\{X_n\}_{n\geq 1}$: submartingale
N ... stopping time with $P(N \leq k) = 1$

We show that $E[X_0] \leq E[X_N] \leq E[X_k]$.

(proof)

By $\boxed{32}$, $X_{n \wedge N}$ is also a submartingale, hence

$E[X_{m \wedge N} \mid \mathcal{F}_{m-1}] \geq X_{m-1 \wedge N}$

$\Rightarrow E[X_{m \wedge N}] \geq E[X_{m-1 \wedge N}]$ $\quad (\because E \circ E[X_{n \wedge N} \mid \mathcal{F}_m] = E[X_{n \wedge N}])$

$\therefore E[X_{0 \wedge N}] \leq \cdots \leq E[X_{k \wedge N}]$

$\Rightarrow E[X_0] \leq E[X_N] \quad \cdots ①$

Let $K_m \overset{def}{=} \mathbb{I}\{N \leq m-1\}$ ... this is a predictable sequence

By $\boxed{31}$, $(K \cdot X)_n$ is a submartingale.

$(K \cdot X)_n = \sum_{m=1}^{n} \underset{\sim}{K_m} (X_m - X_{m-1}) =$

$\begin{pmatrix} \cdot\ N \leq m-1 \Rightarrow 1 \\ \cdot\ N \geq m \Rightarrow 0 \end{pmatrix} \Rightarrow$

$\therefore$ ($\cdot$ If $N \geq n$, $K_m = 0$ for $m = 1 \sim n$
$\cdot$ If $N \leq n-1$, $K_m = 1$ for $m = N+1 \sim n$.
$K_m = 0$ for $m = 1 \sim N$.

$$= \begin{cases} \cdot\ 0 & (\text{if } N \geq n) \\ \cdot\ \sum_{m=N+1}^{n} 1 \cdot (X_m - X_{m-1}) = X_n - X_N & (\text{if } N \leq n-1) \end{cases}$$

$$= X_n - X_{n \wedge N} \qquad (\wedge \cdots \min\{\cdot,\cdot\})$$

So $\{X_n - X_{n\wedge N}\}_{n \in \mathbb{Z}_+}$ is a submartingale

$$\therefore E[X_k - X_{k\wedge N}] \geq E[X_0 - X_{0\wedge N}] = 0$$

$$\Rightarrow E[X_k] \geq E[X_{k\wedge N}] = E[X_N] \quad \cdots ②$$

① and ② $\Rightarrow$ $E[X_0] \leq E[X_N] \leq E[X_k]$

$\boxed{49}$ (Doob's inequality)

- $\{X_n\}_{n\geq 1}$ … submartingale
- $\overline{X_n} \overset{\text{def}}{=} \max_{0\leq m\leq n}\{X_m^+\}$
- $A \overset{\text{def}}{=} \{\overline{X_n} \geq \lambda\}$ $(\lambda>0)$

We show that $\lambda P(A) \leq E[X_n \cdot I_A] \leq E[X_n^+]$

(proof)

- $N \overset{\text{def}}{=} \inf\{m \mid X_m \geq \lambda\} \wedge n$ $(a\wedge b = \min\{a, b\})$
- When $\omega \in A$, $\max_{0\leq m\leq n}\{X_m^+\} \geq \lambda \Rightarrow$

There exists $m_0^{(\omega)}$ $(0\leq m_0 \leq n)$ s.t. $X_{m_0}^+ \geq \lambda$ (>0)

$\Rightarrow X_{m_0} \geq \lambda \Rightarrow X_N \geq \lambda$ ($\because$ Definition of N)

$\therefore A \subseteq \{X_N \geq \lambda\}$

$\therefore E[\lambda \cdot I_A] \leq E[X_N \cdot I_A]$,

$\Rightarrow \lambda P(A) \leq E[X_N \cdot I_A] \cdots ①$

Next since N is a stopping time and $P(N \leq n) = 1$,

by $\boxed{4.8}$ $E[X_N] \leq E[X_n] \cdots ②$

Furthermore. $\omega \in A^c \Rightarrow \max_{0 \leq m \leq n} \{X_m^+\} < \lambda \Rightarrow X_0 \sim X_n < \lambda$

$\Rightarrow \inf\{m \mid X_m \geq \lambda\} > n \Rightarrow N = n \Rightarrow X_N = X_n$

Thus $E[X_N \cdot I_{A^c}] = E[X_n \cdot I_{A^c}] \cdots ③$

② ③ $\cdots$ $E[X_N \cdot I_A] \leq E[X_n \cdot I_A] \cdots ④$

① & ④ $\Rightarrow$ $\lambda P(A) \leq E[X_n \cdot I_A]$

And $X_n \cdot I_A \leq X_n^+ \cdot I_A \leq X_n^+$ $\therefore$ $E[X_n I_A] \leq E[X_n^+]$

PAGE
DATE

$\boxed{50}$ $S_n = \xi_1 + \xi_2 + \cdots + \xi_n$ ($\xi_1 \sim \xi_n$ .. independent)

$E[\xi_m] = 0$. $E[\xi_m^2] = \sigma_m^2 < \infty$

$\mathcal{F}_n = \sigma[\xi_1, \xi_2, \dots \xi_n]$

- $\{S_n^2\}_{n \geq 1}$ is a submartingale

$E[S_{n+1}^2 | \mathcal{F}_n] = E[S_n^2 + 2S_n \xi_{n+1} + \xi_{n+1}^2 | \mathcal{F}_n]$

$= S_n^2 + 2S_n \underbrace{E[\xi_{n+1} | \mathcal{F}_n]}_{0} + \underbrace{E[\xi_{n+1}^2 | \mathcal{F}_n]}_{\sigma_{n+1}^2}$

$\geq S_n^2$

- $A \overset{def}{=} \{ \max_{1 \leq m \leq n} (S_m^2)^+ \geq x^2 \}$

$= \{ \max_{1 \leq m \leq n} S_m^2 \geq x^2 \}$

$= \{ \max_{1 \leq m \leq n} |S_m| \geq x \}$

By $\boxed{49}$ $x^2 \cdot P(A) \leq E[(S_n^2)^+] = E[S_n^2] = V[S_n]$

$\therefore x^2 \cdot P(\max_{1 \leq m \leq n} |S_m| \geq x) \leq V[S_n]$

CHA SHIN

PAGE
DATE

$\boxed{5}$ $\{X_n\}_{n\geq 1}$: submartingale

then $E[(\bar{X}_n)^p] \leq \left(\frac{p}{p-1}\right)^p E[(X_n^+)^p]$ $(p\in(1,\infty))$

(proof)

$\bar{X}_n = \max_{1\leq m\leq n} \{X_m^+\}$.

Let $M \in (0,\infty)$

- $E[(\bar{X}_n\wedge M)^p] = \int_0^\infty p\lambda^{p-1}\cdot P(\bar{X}_n\wedge M > \lambda)\, d\lambda$

($\because$ $\boxed{ch2}$: $X\geq 0 \Rightarrow E[X^p] = \int_\Omega p x^{p-1}\cdot P(X>\lambda)\,dx$)

Since the number of points of discontinuity is at most countable and a countable set has measure zero (Lebesgue measure),

$\int_0^\infty p\lambda^{p-1} P(\bar{X}_n\wedge M>\lambda)\,d\lambda = \int_0^\infty p\lambda^{p-1} P(\bar{X}_n\wedge M\geq\lambda)\,d\lambda$

Let $A = \{\bar{X}_n \wedge M \ge \lambda\}$.

$\bar{X}_n \wedge M = \max_{1 \le m \le n} \{X_m^+\} \wedge M = \max_{1 \le m \le n} \{X_m^+ \wedge M\}$

$= \max_{1 \le m \le n} \{(X_m \wedge M)^+\}$

Let $Y_n = (X_n \wedge M)$

Then $A = \{\bar{X}_n \wedge M \ge \lambda\} = \{\bar{Y}_n \ge \lambda\}$

By $\boxed{49}$, $\lambda P(A) \le E[Y_n \cdot \mathbb{1}_A]$

$= E[(X_n \wedge M) \cdot \mathbb{1}_A]$

$\le E[X_n^+ \mathbb{1}_A]$

Hence, $\int_0^\infty p\lambda^{p-1} P(\bar{X}_n \wedge M \ge \lambda)\, d\lambda = \int_0^\infty p\lambda^{p-2} \cdot \underline{\lambda P(A)}\, d\lambda$

$\le \int_0^\infty p\lambda^{p-2} \cdot E[X_n^+ \cdot \mathbb{1}_A]\, d\lambda$

$$= \int_0^\infty p\lambda^{p-2} \cdot \int_\Omega X_n^+ \cdot \mathbb{I}_A \, dP \, d\lambda$$

$$= \int_\Omega \int_0^\infty X_n^+ \cdot \mathbb{I}_A \cdot p\lambda^{p-2} \, d\lambda \, dP$$

Tonelli's Theorem

$$= \int_\Omega \int_0^{X_n \wedge M} X_n^+ \, p\lambda^{p-2} \, d\lambda \, dP \qquad (A = \{X_n \wedge M \geq \lambda\})$$

$$= \int_\Omega X_n^+ \cdot \left(\frac{p}{p-1}\right) \cdot (X_n \wedge M)^{p-1} \, dP$$

$$= \left(\frac{p}{p-1}\right) \int_\Omega X_n^+ \cdot (X_n \wedge M)^{p-1} \, dP$$

(By Hölder's inequality $\|fg\| \leq \|f\|_p \|g\|_q$ $\left(\frac{1}{p}+\frac{1}{q}=1\right)$

$f = X_n^+ \quad g = (X_n \wedge M)^{p-1}$

$$\leq \frac{p}{p-1} \, E\left[(X_n^+)^p\right]^{\frac{1}{p}} \cdot E\left[(X_n \wedge M)^{pq-q}\right]^{\frac{1}{q}}$$

$$= \left(\frac{p}{p-1}\right) \cdot E\left[(X_n^+)^p\right]^{\frac{1}{p}} \cdot E\left[(X_n \wedge M)^p\right]^{\frac{1}{q}}$$

$(\because \; pq = p+q \Rightarrow pq - q = p)$

So $LHS = E\left[(X_n \wedge M)^p\right]$

$$\leq RHS = \left(\frac{p}{p-1}\right) E\left[(X_n^+)^p\right]^{\frac{1}{p}} E\left[(X_n \wedge M)^p\right]^{\frac{1}{q}}$$

$$\therefore \; E\left[(X_n \wedge M)^p\right]^{1-\frac{1}{q}} \leq \left(\frac{p}{p-1}\right) E\left[(X_n^+)^p\right]^{\frac{1}{p}}$$

$$\therefore \; E\left[(X_n^+ \wedge M)^p\right]^{p-\frac{p}{q}} \leq \left(\frac{p}{p-1}\right)^p \cdot E\left[(X_n^+)^p\right]$$

$$\left(\frac{1}{p}+\frac{1}{q}=1 \Rightarrow 1+\frac{p}{q}=p \Rightarrow p-\frac{p}{q}=1\right)$$

PAGE

DATE

$\therefore$ LHS $= E[(\overline{X_n} \wedge M)^p]$

$\leq$ RHS $= \left(\frac{p}{p-1}\right)^p \cdot E[(X_n^+)^p]$ (for all $M>0$)

By taking $M \nearrow \infty$, and by monotone convergence theorem. LHS $\nearrow E[(\overline{X_n})^p]$

So we have $E[(\overline{X_n})^p] \leq \left(\frac{p}{p-1}\right)^p E[(X_n^+)^p]$ ∎

CHA SHIN

$\boxed{52}$ (L^p- convergence theorem)

$\{X_n\}_{n\geq 1}$ ... martingale $\quad \sup_{n\geq 1} E|X_n|^p < \infty \quad (p>1)$

Then $X_n \to X$ (as) and $X_n \xrightarrow{L^p} X$.

(proof)

$\boxed{\text{Step 1}}$. $\sup_{n\geq 1} E[X_n^+] < \infty$

because $\left(E[X_n^+]\right)^p \leq \left(E|X_n|\right)^p \leq E\left[|X_n|^p\right]$

(Jensen's inequality)

$\leq \sup_{n\geq 1} E|X_n|^p < \infty$

So we have $\sup_{n\geq 1} E[X_n^+] < \infty$.

By martingale convergence theorem, ($\boxed{34}$)

$X_n \to X$ (as) $\quad$ ($\because$ $\sup_{n\geq 1} E[X_n^+] < \infty$

$\{X_n\}_{n\geq 1}$: martingale)

$\Rightarrow$ submartingale

PAGE
DATE

$\boxed{\text{Step2}}$ Use L^p-maximal inequality ($\boxed{51}$)

$$Y_n \overset{\text{def}}{=} |X_n|$$

$$E[(\overline{Y_n})^p] \leq \left(\frac{p}{p-1}\right)^p E[(Y_n^+)^p]$$

$$\|$$

$$E\left[\left(\max_{1 \leq m \leq n}\{|X_m|\}\right)^p\right] \leq \left(\frac{p}{p-1}\right)^p E[|X_n|^p]$$

$$\leq \left(\frac{p}{p-1}\right)^p \sup_{n \geq 1} E[|X_n|^p] < \infty$$

By taking $n \nearrow \infty$.

$$LHS = E\left[\left(\sup_{n \geq 1} |X_n|\right)^p\right] < \infty$$

So $\sup_{n \geq 1} |X_n|$ is L^p-integrable

PAGE

DATE

$\boxed{\text{Step 7}}$ L.D.C.T

$$|X_n - X|^p \le \left(2 \sup_{n\ge1} |X_n|\right)^p = 2^p \cdot \left(\sup_{n\ge1} |X_n|\right)^p \cdots \text{integrable}$$

By Lebesgue's Dominated Convergence Theorem

$$\lim_{n\to\infty} E[|X_n - X|^p] = E[\lim_{n\to\infty} |X_n - X|^p] = 0$$

$\therefore X_n \xrightarrow{L^p} X$.   Now the proof is complete ∎

CHA SHIN

PAGE

DATE

$\boxed{53}$

(1) Let $K_m \overset{\text{def}}{=} \mathbb{I}\{M<m\leq N\} \geq 0$

K_m is a predictable sequence

and K_m is bounded.

Hence, by $\boxed{31}$, $(K\cdot X)_n$ is also a submartingale.

$$(K\cdot X)_n = \sum_{m=1}^{n} K_m \cdot (X_m - X_{m-1})$$

$$= \begin{cases} 0 & (n\leq M) \\ X_n - X_M & (M<n\leq N) \\ X_N - X_M & (n>N) \end{cases}$$

$$= X_{n\wedge N} - X_{n\wedge M}$$

So $E[(K\cdot X)_n \mid \mathcal{F}_{n-1}] \geq (K\cdot X)_{n-1}$ (submartingale)

$\Rightarrow E[(K\cdot X)_n] \geq E[(K\cdot X)_{n-1}] \geq \cdots E[(K\cdot X)_1]$

≥ 0 or $E[X_1 - X_0]$.

≥ 0 ($\because \{X_n\}_{n\geq 1}$ submartingale)

$\therefore E[X_{n\wedge N}] \geq E[X_{n\wedge M}]$

PAGE

DATE

$n \overset{\text{put}}{=} k$ ... We have $E[X_N] \geq E[X_M]$

(12) We use the fact $\boxed{8}$:

$$\left(\begin{array}{l} M \leq N \text{ and } A \in \mathcal{F}_M \\ \text{then } L \overset{\text{def}}{=} \begin{cases} M & (\omega \in A) \\ N & (\omega \notin A) \end{cases} \text{ is a stopping time} \end{array} \right.$$

- By (1), $L \leq N$. $\Rightarrow$ $E[X_L] \leq E[X_N]$ ... ①
- $\omega \in A \Rightarrow L = M \Rightarrow E[X_L \cdot \mathbb{I}_A] = E[X_M \cdot \mathbb{I}_A]$ ... ②
- $\omega \in A^c \Rightarrow L = N \Rightarrow E[X_L \mathbb{I}_{A^c}] = E[X_N \cdot \mathbb{I}_{A^c}]$ ... ③

① - ③ $\Rightarrow$ $E[X_L \cdot \mathbb{I}_A] \leq E[X_N \cdot \mathbb{I}_A]$ ... ④

② & ④ $\Rightarrow$ $E[X_M \mathbb{I}_A] \leq E[X_N \cdot \mathbb{I}_A]$ ... ⑤

⑤ $= \int_A X_M dP \leq \int_A X_N dP$ for all $\forall A \in \mathcal{F}_M$

$$\| $$

$\int_A E[X_N | \mathcal{F}_M] dP$ (By definition)

We should this is $\mathcal{F}_M$-measurable

CHA SHIN

PAGE
DATE

Finally we show that X_M is a $\mathcal{F}_M$-measurable function.

Let $c \in \mathbb{R}$. We show that $\{X_M > c\} \in \mathcal{F}_M$

$\{X_M > c\} \cap \{M = m\}$

$= \underbrace{\{X_m > c\}}_{\in \sigma[X_m] \subseteq \mathcal{F}_m} \cap \underbrace{\{M = m\}}_{\in \mathcal{F}_m} \in \mathcal{F}_m$

So for $\forall m$ $\{X_M > c\} \cap \{M = m\} \in \mathcal{F}_m$

$\Rightarrow \{X_M > c\} \in \mathcal{F}_M$. $\therefore$ X_M is $\mathcal{F}_M$-measurable

$\forall A \in \mathcal{F}_M$ $\int_A X_M \, dP \leq \int_A E[X_N | \mathcal{F}_M] \, dP$

$\Rightarrow$ $X_M \leq E[X_N | \mathcal{F}_M]$ (a.s.) $\blacksquare$

$\boxed{54}$

$A \overset{\text{def}}{=} \{\omega \in \Omega \mid \max_{1 \le m \le n} |S_m| > x\}$

$N \overset{\text{def}}{=} \inf\{m \mid |S_m| > x\} \wedge n$ (This is a stopping time)

Since $P(N \le n) = 1$, we may apply $\boxed{48}$,

$E[S_1^2 - s_1^2] \le E[S_N^2 - s_N^2]$

$\quad\; \| $

$\quad\; 0 \qquad\qquad \Downarrow$

$\therefore 0 \le E[(S_N^2 - s_N^2) \cdot \mathbb{I}_A + (S_N^2 - s_N^2) \cdot \mathbb{I}_{A^c}]$

- $\omega \in A \Rightarrow \exists\, m = N$ such that $|S_m| > x$

$\therefore |S_N| > x$ but $|S_{N-1}| \le x$ $\quad (* S_0 \overset{\text{def}}{=} 0)$

So we have $|S_N| \le (x + K)$ $\quad (\because |X_i| \le K)$

- $\omega \in A^c \Rightarrow N = n$ (see definition of N)

And $|S_n| \le x$

$$(0 \leq)$$

Therefore $E[(S_N^2 - S_N^2) \cdot I_A + (S_N^2 - S_N^2) I_{A^c}]$

$\leq E[S_N^2 \cdot I_A + (S_N^2 - S_N^2) I_{A^c}]$

$\leq E[(x+K)^2 \cdot I_A + (x^2 - \underbrace{s_n^2}_{= V[S_n]}) \cdot I_{A^c}]$

$\therefore 0 \leq \underbrace{P(A) \cdot (x+K)^2}_{= (x+K)^2 - (x+K)^2 (1-P(A))} + (x^2 - V[S_n]) \cdot (1-P(A))$

$\therefore 0 \leq (x+K)^2 + \{x^2 - V[S_n] - (x+K)^2\} (1-P(A))$

$\Rightarrow \underbrace{(1-P(A)) \{(x+K)^2 - x^2 + V[S_n]\}}_{\geq (1-P(A)) \cdot V[S_n]} \leq (x+K)^2$

$\therefore \underbrace{(1-P(A))}_{= P(A^c) = P(\max_{1 \leq m \leq n} |S_m| \leq x)} \leq \dfrac{(x+K)^2}{V[S_n]}$

Now we have the desired result. ∎

PAGE
DATE

$\boxed{55}$

Consider $E[Y_{n-1}(X_n - X_{n-1}) | \mathcal{F}_{n-1}]$

$= Y_{n-1} \cdot \{ \underbrace{E[X_n | \mathcal{F}_{n-1}]}_{= X_{n-1}} - X_{n-1} \} = 0$

※ By Cauchy Shwartz's Inequality

$E[X_n Y_{n-1}]^2 \leq E[X_n^2] E[Y_{n-1}]^2 < \infty$

So it's finite

$\therefore E \circ E[Y_{n-1}(X_n - X_{n-1}) | \mathcal{F}_{n-1}] = 0$

$\therefore E[Y_{n-1}(X_n - X_{n-1})] = 0 \Rightarrow E[X_n Y_{n-1}] = E[X_{n-1} Y_{n-1}] \cdots ①$

Similarly we have $E[X_{n-1} Y_n] = E[X_{n-1} Y_{n-1}] \cdots ②$

$E[(X_n - X_{n-1})(Y_n - Y_{n-1})] = E[X_n Y_n - \underbrace{X_{n-1} Y_n}_{= E[X_{n-1} Y_{n-1}]} - \underbrace{X_n Y_{n-1}}_{= E[X_{n-1} Y_{n-1}]} + X_{n-1} Y_{n-1}]$

$= E[X_n Y_n] - E[X_{n-1} Y_{n-1}]$

So $\sum_{m=1}^{n} E[(X_m - X_{m-1}) \cdot (Y_m - Y_{m-1})] = \sum_{m=1}^{n} (E[X_m Y_m] - E[X_{m-1} Y_{m-1}])$

And we have the desired result.

CHA SHIN

PAGE

DATE

$\boxed{56}$

(1) $\{X_i\}_{i\in I}$ … uniformly integrable

$$\lim_{M\to\infty} \sup_{i\in I} E[|X_i|\cdot \mathbb{I}\{|X_i|>M\}] = 0$$

(2) We show that

$$① \quad \lim_{M\to\infty} \sup_{i\in I} E[|X_i|\cdot \mathbb{I}\{|X_i|>M\}] = 0$$

$\Leftrightarrow$ ② $\cdot \ \sup_{i\in I} E|X_i| < \infty$ (Absolutely Continuous)

$$\cdot \ \lim_{\substack{P(A)\to 0 \\ A\in\mathcal{F}}} \sup_{i\in I} E[|X_i|\cdot \mathbb{I}_A] = 0$$

(i.e. $\forall \varepsilon>0 \ \exists \delta_\varepsilon>0$ s.t. $\forall A\in\mathcal{F}$ $P(A)<\delta$

$\sup_{i\in I} E[|X_i|\cdot \mathbb{I}_A] < \varepsilon$.)

$\boxed{\text{Step 1}}$ ① $\Rightarrow$ ②

Let $A\in\mathcal{F}$.

$$E|X_i|\cdot\mathbb{I}_A = E|X_i|\cdot \mathbb{I}\{\omega\in A \mid |X_i|>M\} + E|X_i|\cdot\mathbb{I}\{\omega\in A \mid |X_i|\le M\}$$

PAGE

DATE

$\leq E|X_i| \cdot \mathbb{I}\{|X_i|>M\} + M \cdot P(A)$

$\leq \sup_{i\in I} E|X_i| \cdot \mathbb{I}\{|X_i|>M\} + MP(A)$

By taking large $M(\varepsilon)>0$, we have $\sup_{i\in I} E|X_i| \cdot \mathbb{I}\{|X_i|>M\} < \frac{\varepsilon}{2}$.

And then $\forall A \in \mathcal{F}$ $P(A) < \frac{\varepsilon}{2M}$ ... $\sup_{i\in I} E|X_i| \cdot \mathbb{I}_A < \varepsilon$.

If $A=\Omega$, $\sup_{i\in I} E|X_i| \leq \sup_{i\in I} E|X_i| \mathbb{I}\{|X_i|>M\} + M < \infty$

So $\{X_i\}_{i\in I}$ is L^1-bounded.

$\boxed{\text{Step2}}$ ② ⇒ ①

$\forall A \in \mathcal{F}$: $P(A) \leq \delta_\varepsilon$ $\sup_{i\in I} E[|X_i| \cdot \mathbb{I}_A] < \varepsilon$

Let $M_\varepsilon = \frac{1}{\delta_\varepsilon} \sup_{i\in I} E|X_i| < \infty$. If $M > M_\varepsilon$,

Then $P(|X_i|>M) \leq P(|X_i|>M_\varepsilon) \leq \frac{1}{M_\varepsilon} E|X_i|$

$\leq \frac{1}{M_\varepsilon} \sup_{i\in I} E|X_i| = \delta_\varepsilon$ (for all $i\in I$)

Hence $\sup_{i\in I} E[|X_i| \cdot \mathbb{I}\{|X_i|>M\}] < \varepsilon$

This implies $\lim_{M\to\infty} \sup_{i\in I} E[|X_i| \cdot \mathbb{I}\{|X_i|>M\}] = 0$

$\boxed{57}$

$\boxed{\text{Step 1}}$ We show "$^\forall \varepsilon > 0 \ ^\exists \delta_\varepsilon > 0$ s.t. $^\forall A \in \mathcal{F}$

$P(A) < \delta_\varepsilon \Rightarrow \int_A |X| dp < \varepsilon$"

(proof)

$$\int_A |X| dp = \int_{\{\omega \in A | \, |X| > M\}} |X| dp + \int_{\{\omega \in A | \, |X| \le M\}} |X| dp$$

$$\le \underbrace{E[|X| \cdot \mathbb{I}\{|X| > M\}]} + M P(A)$$

By L.D.C.T. $M \uparrow \infty$.

We have $< \frac{\varepsilon}{2}$

Then if $P(A) < \frac{\varepsilon}{2M} \Rightarrow \int_A |X| dp < \varepsilon$.

$\boxed{\text{Step2}}$

$$E\left[\,|E[X|\mathcal{F}]|\cdot I\{|E[X|\mathcal{F}]|>M\}\,\right]$$ Jensen's inequality

$$\leq E\left[\,E[|X|\,|\mathcal{F}]\cdot I\{E[|X|\,|\mathcal{F}]>M\}\,\right]$$

$$= E\left[\,E[|X|\,|\mathcal{F}]\cdot I_A\,\right] \quad (A \overset{\text{def}}{=} \{E[|X|\,|\mathcal{F}]>M\})$$

$$= \int_A E[|X|\,|\mathcal{F}]\,dP \quad (A\in\mathcal{F})$$ By the definition of Conditional expectation.

$$= \int_A E|X|\,dP$$

Be careful of the fact that $P(A)$

$= P(E[|X|\,|\mathcal{F}]>M) \leq \frac{1}{M}E|X| < \delta_\varepsilon \quad (\text{if } M > \frac{1}{\delta_\varepsilon}E|X|)$

for all $A(\mathcal{F})$

Hence by $\boxed{\text{Step1}}$, $\int_A E|X|\,dP < \varepsilon$.

So $\sup_{\mathcal{F}} E\left[\,|E[X|\mathcal{F}]|\cdot I\{|E[X|\mathcal{F}]|>M\}\,\right] < \varepsilon$

Now the proof is complete ■

$\boxed{58}$

$$E\left[|X_i| \cdot \mathbb{I}\{|X_i| > M\} \right]$$

$$= E\left[\frac{|X_i|}{\phi(|X_i|)} \cdot \mathbb{I}\{|X_i| > M\} \cdot \phi(|X_i|) \right]$$

⊛ By assumption, $\forall \varepsilon > 0 \ \exists M_\varepsilon$ s.t. $\forall M > M_\varepsilon$

$$\frac{|X_i|}{\phi(|X_i|)} \mathbb{I}\{|X_i| > M\} < \frac{\varepsilon}{C}$$

$$\left(\because \ \frac{\phi(x)}{x} \uparrow \infty \text{ as } x \uparrow \infty\right)$$

$$< E\left[\frac{\varepsilon}{C} \phi(|X_i|) \right]$$

$$\leq \frac{\varepsilon}{C} \cdot \sup_{i \in I} E[\phi(|X_i|)]$$

$$\leq \frac{\varepsilon}{C} \cdot C = \varepsilon$$

$$\therefore \lim_{M \uparrow \infty} \sup_{i \in I} E\left[|X_i| \cdot \mathbb{I}\{|X_i| > M\} \right] = 0$$

$\boxed{59}$ Suppose $\{X_n\}_{n\geq 1}$ $\quad X_n \xrightarrow{p} X$,

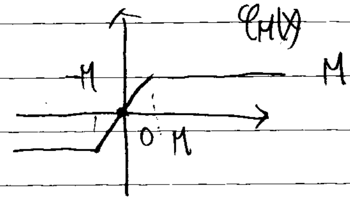
Now the following statements are equivalent.

① $\{X_n\}_{n\geq 1}$ Uniformly Integrable
② $X_n \xrightarrow{L^1} X$
③ $E|X_n| \to E|X| < \infty$ $\quad$ (⊛ We should assume $E|X_n| < \infty$ for all $n \geq 1$)

(proof) ① ⇒ ②

Consider a bounded continuous function $\varphi_M(x)$.

$$\varphi_M(x) = \begin{cases} M & x \geq M \\ x & |x| \leq M \\ -M & x \leq -M \end{cases}$$



By triangle inequality, we have

$$|X_n - X| = |X_n - \varphi(X_n) + \varphi(X_n) - \varphi(X) + \varphi(X) - X|$$

$$\leq \underbrace{|X_n - \varphi(X_n)|}_{(i)} + \underbrace{|\varphi(X_n) - \varphi(X)|}_{(ii)} + \underbrace{|\varphi(X) - X|}_{(iii)}$$

PAGE
DATE

(i) and (iii) :

$$|x - \varphi(X)| = \begin{cases} 0 \cdots & |X| \leq M \\ \big||X| - M\big| \cdots & |X| > M \end{cases}$$

$$\leq \begin{cases} 0 \cdots & |X| \leq M \\ |X| \cdots & |X| > M \end{cases}$$

$$\leq |X| \cdot \mathbb{I}\{|X| > M\}$$

Therefore (i) $\leq |X_n| \cdot \mathbb{I}\{|X_n| > M\}$

(iii) $\leq |X| \cdot \mathbb{I}\{|X| > M\}$

$$E[|X_n - X|] \leq E|X_n| \cdot \mathbb{I}\{|X_n| > M\} + E|X| \cdot \mathbb{I}\{|X| > M\}$$

$$+ E|\varphi(X_n) - \varphi(X)|$$

And $\leq \sup_{n \geq 1} E|X_n| \mathbb{I}\{|X_n| > M\} + E|X| \cdot \mathbb{I}\{|X| > M\}$

$$+ E[|\varphi(X_n) - \varphi(X)|]$$

- The first term $\to 0$ as $M \to \infty$ because $\{X_n\}_{n \geq 1}$ is uniformly integrable

$(\because\ |X|\cdot I_{\{|X|>M\}} \leq |X| \in L^1)$

- The second term $\downarrow 0$ as $M \uparrow \infty$ by L.D.C.T

because $E|X| < \infty$. (ch4 $\boxed{56}$ (2))

(*) $\{X_n\}_{n\geq 1}$: Uniformly integrable $\Rightarrow$ $\sup_{n\geq 1} E[|X_n|] < \infty$

$X_n \xrightarrow{P} X \Rightarrow |X_n| \xrightarrow{P} |X| \Rightarrow$ By Fatou's lemma (ch2 $\boxed{49}$)

$E[|X|] \leq \liminf E[|X_n|] \leq \sup_{n\geq 1} E[|X_n|] < \infty$

Hence $E[|X|] < \infty$

- Finally the third term $\downarrow 0$ as $n \to \infty$

by bounded convergence theorem $(\because\ \phi(X_n) \xrightarrow{P} \phi(X)$ (ϕ : continuous))

So $\lim E[|X_n - X|] = 0$ $\therefore X_n \xrightarrow{L^1} X$

(proof) ② ⇒ ③ ⊛

$$|E[X_n] - E[X]| = |E[X_n - X]| \le E[|X_n - X|] \to 0 \quad (\text{as } n\to\infty)$$

So $E[X_n] \to E[X]$

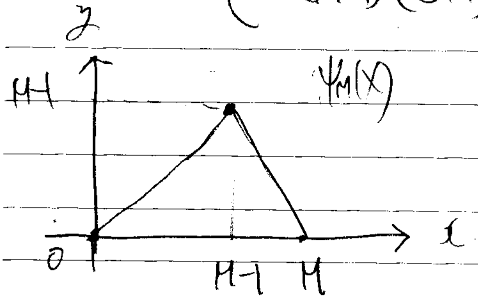
⊛ "$X_n \xrightarrow{L^1} X$" : We suppose this implies $E|X_n|, E|X| < \infty$

Hence $|E[X]| < \infty$ ($\Leftarrow E|X| < \infty$) and ⊛ is valid.

(proof) ③ ⇒ ①

We consider a continuous and bounded function $\psi_M(x)$.

$$\psi_M(x) = \begin{cases} x & \cdots \quad 0 \le x \le M-1 \\ 0 & \cdots \quad M \le x < \infty \\ -(M-1)(x-M) & \cdots \quad M-1 \le x \le M \end{cases}$$



$\boxed{\text{Step1}}$

$x \geq 0 \Rightarrow \psi_M(x) \leq x$ and $\psi_M(x) \nearrow x$ (as $M \nearrow \infty$).

So $\psi_M(|X|) \nearrow |X|$ as $M \nearrow \infty$

Hence by Monotone Convergence Theorem,

$\lim_{M \nearrow \infty} E[\psi_M(|X|)] = E[|X|] < \infty$

$\forall \varepsilon > 0 \ \exists M(\varepsilon) > 0$ s.t. $E[|X|] - E[\psi_M(|X|)] < \frac{\varepsilon}{2}$

$\boxed{\text{Step2}}$ (Here $M = M(\varepsilon)$ given in $\boxed{\text{Step1}}$.)

Next, $0 \leq \psi_M(|X_n|) \leq (M-1)$ .. bounded

and $\psi_M(\cdot)$ is continuous $X_n \xrightarrow{P} X \Rightarrow |X_n| \xrightarrow{P} |X| \Rightarrow$

$\Rightarrow \psi_M(|X_n|) \xrightarrow{P} \psi_M(|X|) \Rightarrow E[\psi_M(|X_n|)] \to E[\psi_M(|X|)]$

($\because$ Bounded Convergence Theorem in $\boxed{\text{ch2}}$. $\boxed{41}$)

$\boxed{\text{Step3}}$

Finally we consider $E[|X_n| \cdot I\{|X_n| > M\}]$.

$= E[|X_n|] - E[|X_n| \cdot I\{|X_n| \leq M\}]$

$\leq E[|X_n|] - E[\psi_M(|X_n|)]$

$\begin{cases} \cdot\ 0 \leq x \leq M \ \cdots \\ \quad x \cdot I\{x \leq M\} \geq \psi_M(x) \\ \cdot\ x > M \cdots \\ \quad x \cdot I\{x \leq M\} = \psi_M(x) = 0 \end{cases}$

$$E[|X_n|] - E[\psi_M(|X_n|)]$$

$$\leq E[|X_n|] - E[\psi_M(|X|)] + \frac{\varepsilon}{4}$$

$\left(\because \text{ By } \boxed{\text{Step2}} \quad {}^{\forall}\varepsilon>0 \quad {}^{\exists}N_1(M,\varepsilon)>0, \text{ s.t } {}^{\forall}n>N_1 \quad |E[\psi_M(|X_n|)] - E[\psi_M(|X|)]| < \frac{\varepsilon}{4}\right)$

(※, $N_1(M,\varepsilon)$ is determined by M, ε. But M is also determined by $\varepsilon>0$.
N_1 actually only depends on $\varepsilon>0$)

$$\leq E[|X|] - E[\psi_M(|X|)] + \frac{\varepsilon}{4} + \frac{\varepsilon}{4}$$

$\left(\because E|X_n| \to E|X| \Rightarrow {}^{\exists}N_2(\varepsilon)>0 \text{ such that } {}^{\forall}n>N_2(\varepsilon) \quad \big|E|X| - E|X_n|\big| < \frac{\varepsilon}{4}\right)$

So we take $n > N \overset{\text{def}}{=} \max\{N_1(\varepsilon), N_2(\varepsilon)\}$)

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \quad (\text{by } \boxed{\text{Step1}})$$

Hence $\sup_{n>N(\varepsilon)} \{E[|X_n| \cdot \mathbb{I}\{|X_n|>L\}]\} < \varepsilon \quad ({}^{\forall}L > M(\varepsilon))$

PAGE
DATE

Moreover, for $i = 1 \sim N(\varepsilon)$,

By taking large $K_1(\varepsilon), K_2(\varepsilon), \dots, K_N(\varepsilon)$,

We have $\begin{cases} \cdot\ E[|X_1| \cdot \mathbb{I}\{|X_1| > K_1\}] < \varepsilon \\ \quad \vdots \\ \cdot\ E[|X_{N(\varepsilon)}| \cdot \mathbb{I}\{|X_N| > K_N\}] < \varepsilon \end{cases}$ (So we should assume $E|X_n| < \infty$ for all $n \geq 1$)

($\because$ L.D.C.T)

Let $K = \max\{K_1(\varepsilon), \dots, K_{N(\varepsilon)}\}$

then $\forall L > \max\{K(\varepsilon), M(\varepsilon)\}$

$\sup_{n \geq 1} E[|X_n| \cdot \mathbb{I}\{|X_n| > L\}] < \varepsilon$

So $\{|X_n|\}_{n \geq 1}$ is uniformly integrable. ∎

$\boxed{60}$ Let $\{X_n\}_{n\geq 1}$ – submartingale

① ⇒ ②

(∵ $\boxed{56}$-(2))

$\{X_n\}_{n\geq 1}$: Uniformly Integrable $\Rightarrow \sup_{n\geq 1} E|X_n| < \infty$

$\Rightarrow \sup_{n\geq 1} E[X_n^+] < \infty \Rightarrow X_n \to X$ (a.s) (∵ $\boxed{54}$)

$E|X| < \infty$.

Next, $X_n \to X$ (a.s) $\Rightarrow X_n \xrightarrow{P} X$

By $\boxed{59}$, $X_n \xrightarrow{P} X$ and $\{X_n\}_{n\geq 1}$: Uniformly Integrable

$\Rightarrow X_n \xrightarrow{L^1} X$.

② ⇒ ③

This is trivial

③ ⇒ ①

$X_n \xrightarrow{L^1} X \Rightarrow X_n \xrightarrow{P} X$ and By $\boxed{59}$, $\{X_n\}_{n\geq 1}$

is uniformly integrable

PAGE

DATE

$\boxed{6}$

$$\left| E[X_n \cdot \mathbb{I}_A] - E[X \cdot \mathbb{I}_A] \right| \leq E\left[|X_n - X| \cdot \mathbb{I}_A \right]$$

$$\leq E|X_n - X| \to 0 \quad (\text{as } n \to \infty)$$

$$(\because |X_n - X| \cdot \mathbb{I}_A \leq |X_n - X|)$$

PAGE

DATE

$\boxed{62}$. Let $A \in \mathcal{F}_n$ (an arbitrary $\mathcal{F}_n$-measurable set)

- $E[X_{n+1} | \mathcal{F}_n] = X_n$ ($\because$ martingale)
- $E[X_{n+2} | \mathcal{F}_n] = E[\underbrace{E[X_{n+2} | \mathcal{F}_{n+1}]}_{X_{n+1}\ (\because \text{ martingale})} | \mathcal{F}_n] = E[X_{n+1} | \mathcal{F}_n] = X_n$

(By $\boxed{19}$)

So similarly $E[X_m | \mathcal{F}_n] = X_n$ (for all $m \geq n+1$)

$\Rightarrow \int_A E[X_m | \mathcal{F}_n] dP = \int_A X_n dP$ ... ①
$(m \geq n+1)$

Moreover,

$\int_A E[X_m | \mathcal{F}_n] dP = \int_A X_m dP$ ($A \in \mathcal{F}_n$; By the definition of conditional expectation)

... ②

By ① and ② $\int_A X_m dP = \int_A X_n dP$ ($\forall m \geq n+1$) ~ ③

And $\boxed{61}$ implies $\int_A X_n dP \to \int_A X dP$

But ③ implies $\int_A X_n dP = \int_A X dP$

In conclusion, $\forall A \in \mathcal{F}_n$ $\int_A X dP = \int_A E[X | \mathcal{F}_n] dP = \int_A \overset{\mathcal{F}_n\text{-measurable}}{X_n} dP$.

So $E[X | \mathcal{F}_n] = X_n$ by uniqueness of conditional expectation ($\boxed{9}$ (3))

PAGE
DATE

$\boxed{63}$ $\{X_n\}_{n\geq 1}$ ... martingale Then ①~④ are equivalent

①⇒②

$\{X_n\}_{n\geq1}$ .. martingale → submartingale

⇒ By $\boxed{60}$, this immediately holds

②⇒③

Similarly, by $\boxed{60}$, this also holds

③⇒④

By $\boxed{62}$, $X_n \xrightarrow{L^1} X \Rightarrow X_n = E[X|\mathcal{F}_n]$

(We assume $X_n \xrightarrow{L^1} X$ implies X is integrable.)

So ④ holds

④ ⇒ ①

$X_n = E[X|\mathcal{F}_n]$ (X: integrable)

$\{X_n\}_{n \geq 1} = \{E[X|\mathcal{F}_n]\}_{n \geq 1}$ ... By $\boxed{57}$

this is uniformly integrable

So ① holds ∎

$\boxed{64}$ $\mathcal{F}_n \uparrow \mathcal{F}_\infty \left(= \sigma\left[\bigcup_{n=1}^{\infty} \mathcal{F}_n\right]\right)$ (and $E|X| < \infty$)

We show that $E[X|\mathcal{F}_n] \xrightarrow{as} E[X|\mathcal{F}]$ and
$E[X|\mathcal{F}_n] \xrightarrow{L^1} E[X|\mathcal{F}]$.

(proof)

Let $Y_n = E[X|\mathcal{F}_n]$

By $\boxed{19}$ (ch.4), $E[E[X|\mathcal{F}_{n+1}]|\mathcal{F}_n] = E[X|\mathcal{F}_n]$

$\Rightarrow E[Y_{n+1}|\mathcal{F}_n] = Y_n$

Hence $\{Y_n\}_{n\geq 1}$ is a martingale

And also by $\boxed{57}$, $\{Y_n\}_{n\geq 1}$ is uniformly integrable

By $\boxed{63}$, $\{Y_n\}_{n\geq 1}$ is a uniformly integrable martingale

$\Rightarrow Y_n \to Y$ (as) and (L^1)

By $\boxed{62}$: $\{Y_n\}_{n\geq 1}$ is a martingale and $Y_n \xrightarrow{L^1} Y$

$\Rightarrow Y_n = E[Y|\mathcal{F}_n]$ ... $\circledast$

PAGE

DATE

Now consider a family of set $\mathcal{D}$.

$$\mathcal{D} \overset{\text{def}}{=} \{ A \in \mathcal{F}_\infty \mid \int_A X dP = \int_A Y dP \}$$

- $\underline{\mathcal{F}_n \subseteq \mathcal{D}}$

(proof)

$$A \in \mathcal{F}_n \Rightarrow \int_A X dP = \int_A E[X|\mathcal{F}_n] dP$$

(By definition of conditional expectation)

$$= \int_A Y_n dP = \int_A E[Y|\mathcal{F}_n] dP = \int_A Y dP$$

($Y_n = E[X|\mathcal{F}_n]$) (By ⊛) (By definition of conditional expectation)

So $\int_A X dP = \int_A Y dP$ for all $A \in \mathcal{F}_n$

Therefore $\mathcal{F}_n \subset \mathcal{D}$

PAGE
DATE

$\therefore$ • $\mathcal{D}$ is a Dynkin-system

(proof)

• $\Omega \in \mathcal{D}$ ($\because$ $\mathcal{F}_n \subseteq \mathcal{D}$ and $\Omega \in \mathcal{F}_n$

$\rightarrow \Omega \in \mathcal{D}$)

• $A, \hat{A} \in \mathcal{D}$ $(A \subseteq \hat{A})$

$\int_A X dP = \int_A Y dP$ (finite) ($\because$ X : integrable)

$\int_{\hat{A}} X dP = \int_{\hat{A}} Y dP$ (finite)

$\therefore \int_{\hat{A} \setminus A} X dP = \int_{\hat{A} \setminus A} Y dP$

$\therefore \hat{A} \setminus A \in \mathcal{D}$

• $\{A_n\}_{n \geq 1} \subseteq \mathcal{D}$ $A_n \nearrow A$

$\int_{A_n} X dP = \int_\Omega X \cdot \mathbb{I}_{A_n} dP = \int_{A_n} Y dP = \int_\Omega Y \cdot \mathbb{I}_{A_n} dP$

Since $|X \cdot \mathbb{I}_{A_n}| \leq |X|$: integrable
$|Y \cdot \mathbb{I}_{A_n}| \leq |Y|$: integrable

$X \cdot \mathbb{I}_{A_n} \nearrow X \cdot \mathbb{I}_A$, $Y \cdot \mathbb{I}_{A_n} \nearrow Y \cdot \mathbb{I}_A$

By LDCT $\lim_{n \to \infty} \int_\Omega X \cdot \mathbb{I}_{A_n} dP = \int_\Omega X \cdot \mathbb{I}_A dP = \int_A X dP$

SHIN

PAGE

DATE

$= \lim_{n\to\infty} \int_\Omega Y \cdot \mathbb{I}_{A_n} dP = \int_\Omega Y \cdot \mathbb{I}_A dP = \int_A Y dP$

$\therefore \int_A X dP = \int_A Y dP \Rightarrow A \in \mathcal{D}$

$\therefore \mathcal{D}$ is a Dynkin-system

$\underline{\mathcal{F}_\infty \subseteq \mathcal{D}}$

(proof)

$\mathcal{F}_n \subseteq \mathcal{D}$ (for all $n \geq 1$)

$\Rightarrow \bigcup_{n=1}^{\infty} \mathcal{F}_n \subseteq \mathcal{D}$

$\Rightarrow \delta\left[\bigcup_{n=1}^{\infty} \mathcal{F}_n\right] \subseteq \mathcal{D}$

($\delta(\cdot)$: the smallest Dynkin system containing $\cdot$)

And $\bigcup_{n=1}^{\infty} \mathcal{F}_n$ is a π-system.

($\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$ $A, B \in \bigcup_{n=1}^{\infty} \mathcal{F}_n \Rightarrow A \in \mathcal{F}_{m_1}, B \in \mathcal{F}_{m_2}$

$m_1 < m_2 \Rightarrow A, B \in \mathcal{F}_{m_2} \Rightarrow A \cap B \in \mathcal{F}_{m_2}$

$\Rightarrow A \cap B \in \bigcup_{n=1}^{\infty} \mathcal{F}_n$)

PAGE

DATE

$\therefore \mathcal{D}[\bigcup_{n=1}^{\infty} \mathcal{F}_n] = \sigma[\bigcup_{n=1}^{\infty} \mathcal{F}_n] \quad (= \mathcal{F}_\infty)$

(By π-λ theorem)

$\Rightarrow \quad \mathcal{F}_\infty \subset \mathcal{D}$

Finally, $\quad \int_A X dP = \int_A Y dP \quad (\forall A \in \mathcal{F}_\infty)$

$$\int_A Y dP = \int_A E[X \mid \mathcal{F}_\infty] \, dP$$

($\because$ by the definition of conditional expectation)

Y is also $\mathcal{F}_\infty$-measurable

($\because$ Y_n is $\mathcal{F}_n$-measurable $\Rightarrow$ $\mathcal{F}_\infty$-measurable
So. $Y_n \xrightarrow[(as)]{} Y \Rightarrow Y$ is also $\mathcal{F}_\infty$-measurable)

So by uniqueness of conditional expectation

$Y = E[X \mid \mathcal{F}_\infty] \quad (as)$ ∎

PAGE
DATE

$\boxed{65}$ (Levy's 0-1 law)

This is just a special version of $\boxed{64}$

Let $X = \mathbb{I}_A$ and $A \in \mathcal{F}_\infty$ where

$\mathcal{F}_n \nearrow \mathcal{F}_\infty$ ($\mathcal{F}_\infty = \sigma\left[\bigcup_{n=1}^{\infty} \mathcal{F}_n\right]$)

By $\boxed{64}$, $E[\mathbb{I}_A \mid \mathcal{F}_n] \xrightarrow{a.s / L^1} E[\mathbb{I}_A \mid \mathcal{F}_\infty]$

$$E[\mathbb{I}_A \mid \mathcal{F}_\infty] = \mathbb{I}_A$$

[66] (Dominated Convergence Theorem for conditional expectation)

$Y_n \to Y$ (as) and $|Y_n| \leq Z$ for all $n \geq 1$

where $E|Z| < \infty$. And suppose $\mathcal{F}_n \uparrow \mathcal{F}_\infty = \sigma[\bigcup_{n=1}^{\infty} \mathcal{F}_n]$.

We show that "$E[Y_n | \mathcal{F}_n] \to E[Y | \mathcal{F}_\infty]$ (a.s)"

(proof) (Our first goal is to show that $\limsup_{n\to\infty} E[|Y_n - Y| \,|\, \mathcal{F}_n] = 0$ (as))

Let $W_N \overset{\text{def.}}{=} \sup_{n,m \geq N} \{|Y_n - Y_m|\}$.

$W_N \leq 2Z$ Hence $E[W_N] < \infty$

Now fix N, and suppose $n \geq N$.

$|Y_n - Y| = \lim_{\ell\to\infty} |Y_n - Y_\ell| \leq \sup_{k,\ell \geq N} |Y_k - Y_\ell| = W_N$

Thus $E[|Y_n - Y| \,|\, \mathcal{F}_n] \leq E[W_N | \mathcal{F}_n]$

By taking limit, we have:

$$\limsup_{n\to\infty} E[|Y_n - Y| \,|\, \mathcal{F}_n] \leq \lim_{n\to\infty} E[W_N | \mathcal{F}_n]$$

By [64], (W_N is integrable), RHS $= E[W_N | \mathcal{F}_\infty]$.

PAGE

DATE

Now taking $N \uparrow \infty$, $W_N \downarrow 0$ (a.s).

because $\{Y_n(w)\}_{n\geq 1}$ converges almost surely

$\Rightarrow$ "$\{Y_n(w)\}_{n\geq 1}$ is a Cauchy sequence almost surely"

$\Rightarrow$ "$N \uparrow \infty \Rightarrow W_N \downarrow 0$ (a.s)"

Furthermore, this implies that

$N \uparrow \infty$ $E[W_N | \mathcal{F}_\infty] \downarrow 0$ (a.s) $\cdots$ ⊛

• To prove ⊛ we recall that $\boxed{15}$ (3).

Consider $X_n \geq 0$ $X_n \downarrow 0$ $E[X_1] < \infty$

Let $Y_n \overset{def}{=} X_1 - X_n$. Then $Y_n \geq 0$ $Y_n \uparrow X_1$.

By $\boxed{15}$ (3) $\lim_{n\to\infty} E[Y_n | \mathcal{F}] = E[X_1 | \mathcal{F}]$ (a.s)

$$\begin{aligned} &\| \\ &\lim_{n\to\infty} E[X_1 - X_n | \mathcal{F}] \\ &\| \\ &\lim_{n\to\infty} (E[X_1 | \mathcal{F}] - E[X_n | \mathcal{F}]) \end{aligned}$$

So $\lim_{n\to\infty} E[X_n | \mathcal{F}] = 0$ (a.s)

CHA SHIN

PAGE
DATE

$W_N \geq 0$, $W_N \downarrow 0$ (as) and $W_1 \leq 2Z \Rightarrow E[W_1] < \infty$

So $E[W_N | \mathcal{F}_\infty] \downarrow 0$ (as).

Therefore $\limsup_{n\to\infty} E[|Y_n - Y| \,|\, \mathcal{F}_n] = 0$ (as)

Finally, $\limsup_{n\to\infty} |E[Y|\mathcal{F}_n] - E[Y|\mathcal{F}_\infty]|$

$$= \limsup_{n\to\infty} |E[Y|\mathcal{F}_n] - E[Y_n|\mathcal{F}_n] + E[Y_n|\mathcal{F}_n] - E[Y|\mathcal{F}_\infty]|$$

$$\leq \underbrace{\limsup_{n\to\infty} E[|Y - Y_n| \,|\, \mathcal{F}_n]}_{\to 0 \text{ (as)}} + \underbrace{\limsup_{n\to\infty} |E[Y_n|\mathcal{F}_n] - E[Y|\mathcal{F}_\infty]|}_{\text{By } \boxed{64} \to 0 \text{ (as)}}$$

Now the proof is complete ■

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[67] $X_n: \Omega \to [0, \infty)$ (r.v)

$D = \{X_n = 0 \text{ for some } n \geq 1\}$

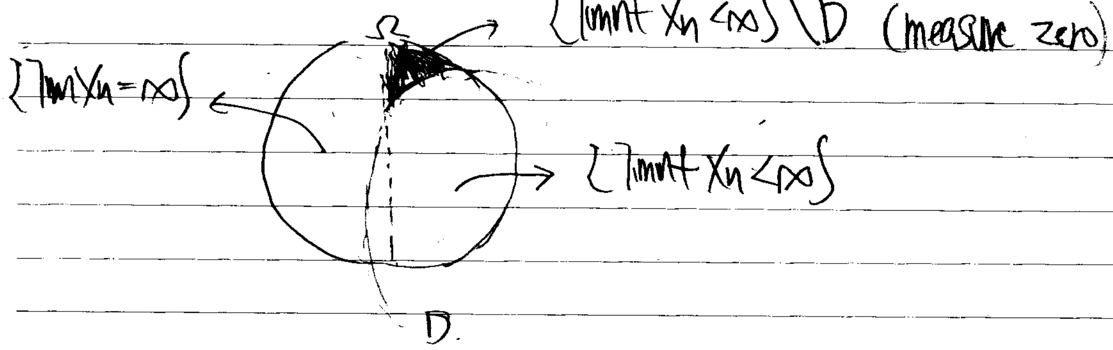
$P(D \mid X_1 \cdots X_n) \geq \delta(x) > 0$ (a.s) on $\{X_n \leq x\}$

We show that $P(D \cup \{\lim_{n\to\infty} X_n = \infty\}) = 1$.

(proof)

We try to show that If $\omega \in \{\liminf X_n < \infty\}$

then $\omega \in D$ (a.s).



This will prove the statement of [67]

$\boxed{\text{step 1}}$

Let $\mathcal{F}_n = \sigma[X_1, X_2 \dots X_n]$ and $\mathcal{F}_\infty = \sigma[\bigcup_{n=1}^{\infty} \mathcal{F}_n]$.

$D = \{X_n = 0 \text{ for some } n \geq 1\} \in \bigcup_{n=1}^{\infty} \mathcal{F}_n \subseteq \sigma[\bigcup_{n=1}^{\infty} \mathcal{F}_n] = \mathcal{F}_\infty$

So $D \in \mathcal{F}_\infty$.

By Levy's 0-1 theorem. $\boxed{65}$

$\mathcal{F}_n \uparrow \mathcal{F}_\infty$ and $D \in \mathcal{F}_\infty \Rightarrow E[\mathbb{I}_D | \mathcal{F}_n] \to \mathbb{I}_D$ (as)

$= P(D | X_1 \dots X_n) \to \mathbb{I}_D$ (as)

$\boxed{\text{Step 2}}$

Now let $w \in \{\liminf X_n \leq M\}$ $(M < \infty)$

$\Rightarrow w \in \{X_n \leq M \text{ i.o}\}$

$(\because w \in \{X_n \leq M \text{ i.o}\}^c \Rightarrow w \in \{X_n \leq M \text{ only for finite } n\}$

$\subseteq \{X_n > M \text{ for sufficiently large } n\} \subseteq \{\liminf X_n > M\}$

$= \{\liminf X_n \leq M\}^c$)

And $w \in \{X_n \leq M \text{ i.o}\}$

$\Rightarrow \{P(D | X_1 \dots X_n) \geq \delta(M) \text{ i.o}\}$ (as)

(Because $P(D | X_1 \dots X_n) \geq \delta(x)$ (as) on $\{X_n \leq x\}$)

PAGE

DATE

$= \{ E[\mathbb{I}_D | \mathcal{F}_n] \geq \delta(M) \text{ i.o} \}$

$\subseteq \{ \limsup E[\mathbb{I}_D | \mathcal{F}_n] \geq \delta(M) \}$

By the conclusion of $\boxed{\text{step 1}}$

$\lim_{n\to\infty} P(D | X_1 \cdots X_n) = \lim_{n\to\infty} E[\mathbb{I}_D | \mathcal{F}_n] = \mathbb{I}_D$ (a.s)

$\therefore \omega \in \{ \liminf X_n \leq M \} \Rightarrow \omega \in \{ \mathbb{I}_D(\omega) \geq \delta(M) \}$ (a.s)

Since $\delta(M) > 0$, $\omega \in \{ \mathbb{I}_D(\omega) \geq \delta(M) \}$

$= \{ \mathbb{I}_D(\omega) = 1 \}$

$= D$

$\forall M < \infty$ $\omega \in \{ \liminf X_n \leq M \} \Rightarrow \omega \in D$ (a.s)

Therefore $\omega \in \underbrace{\bigcup_{M=1}^{\infty} \{ \liminf X_n \leq M \}}_{\{ \liminf X_n < \infty \}} \Rightarrow \omega \in D$ (a.s)

So the proof is complete ∎

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$\boxed{68}$ Show that $\mathcal{F}_n \uparrow \mathcal{F}_\infty$ ($\mathcal{F}_n, \mathcal{F}_\infty$: σ-algebras)

and $Y_n \xrightarrow{L^1} Y \Rightarrow E[Y_n | \mathcal{F}_n] \xrightarrow{L^1} E[Y | \mathcal{F}_\infty]$

(proof)

Our goal is to show $\limsup_{n\to\infty} E\left| E[Y_n|\mathcal{F}_n] - E[Y|\mathcal{F}_\infty] \right|$

By triangle inequality,

$\limsup E\left| E[Y_n|\mathcal{F}_n] - E[Y|\mathcal{F}_n] + E[Y|\mathcal{F}_n] - E[Y|\mathcal{F}_\infty] \right|$

$\leq \limsup E\left| E[Y_n|\mathcal{F}_n] - E[Y|\mathcal{F}_n] \right| \cdots ①$

$+ \limsup E\left| E[Y|\mathcal{F}_n] - E[Y|\mathcal{F}_\infty] \right| \cdots ②$

$① \cdots = \limsup E|Y_n - Y| = 0$ ($\because Y_n \xrightarrow{L^1} Y$)

$② \cdots$ By $\boxed{64}$, $E[Y|\mathcal{F}_n] \xrightarrow{L^1} E[Y|\mathcal{F}_\infty]$

$\therefore ② \to 0.$

Now the proof is complete. ■

PAGE

DATE

$\boxed{69}$

(1) Consider $\{X_n\}_{n\le 0}$ a sequence of r.v

- X_n is $\mathcal{F}_n$-measurable. (adapted to $\mathcal{F}_n$) ($\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$)
- $E|X_n| < \infty$ ($n \le 0$)
- $E[X_{n+1} | \mathcal{F}_n] = X_n$ for all $n \le -1$

Then $\{X_n\}_{n\le 0}$ is called a backward martingale.

(2) Let $\{X_n\}_{n\le 0}$ be a backward martingale

We show that $\lim_{n\to-\infty} X_n$ exists

($X_{-\infty} \overset{\text{def}}{=} \lim_{n\to-\infty} X_n$) and $X_n \xrightarrow{L^1} X_{-\infty}$ (as $n \downarrow -\infty$)

$\boxed{\text{Step 1}}$

Consider upcrossing of $[a,b]$ by $\{X_{-n}, X_{-n+1}, \dots X_1, X_0\}$

As we have shown in $\boxed{33}$,

$(b-a)E[U_n] \le E[(X_0-a)^+] - E[(X_{-n}-a)^+]$

$\le E[(X_0-a)^+] \le E[X_0^+] + |a| < \infty$

CHA SHIN

PAGE

DATE

$\rightarrow$ (M.C.T)

Similar to $\boxed{33}$, $\lim_{n\to\infty}(b-a)E[U_n] = (b-a)E[U] < \infty$

(U: number of upcrossings of $[a,b]$ by $\{X_n\}_{n\le\infty}$)

$\therefore U < \infty$ (as) $\Rightarrow$

$\omega \in \{X_n \le a$ only for finite times$\} \cup \{X_n \ge b$ only for finite times$\}$

$\subseteq \{X_n > a$ for sufficiently large $n\} \cup \{X_n < b$ for sufficiently large $n\}$

$\subseteq \{\liminf X_n > a\} \cup \{\limsup X_n < b\}$

$\subseteq \{\liminf X_n \ge a\} \cup \{\limsup X_n \le b\}$ (as)

So $P\left(\left(\{\liminf X_n \ge a\} \cup \{\limsup X_n \le b\}\right)^c\right) = 0$

$\therefore P(\liminf X_n < a < b < \limsup X_n) = 0$

$\therefore P\left(\bigcup_{\substack{a<b \\ a,b\in\mathbb{Q}}} \{\liminf X_n < a < b < \limsup X_n\}\right) = 0$

$\therefore P(\lim X_n$ does not exist$) = 0$

$\therefore \lim X_n$ exists (as)

PAGE

DATE

$\boxed{\text{Step2}}$

We prove $X_n = E[X_0 | \mathcal{F}_n]$. for all $n \leq 0$.

Since $E[X_{n+1} | \mathcal{F}_n] = X_n$, by putting $n=-1$

we have $E[X_0 | \mathcal{F}_{-1}] = X_{-1}$

Next $E[X_0 | \mathcal{F}_{-2}] = E[E[X_0 | \mathcal{F}_{-1}] | \mathcal{F}_{-2}]$

by $\boxed{19}$ and $\mathcal{F}_{-2} \subseteq \mathcal{F}_{-1}$.

So $E[X_0 | \mathcal{F}_{-2}] = E[X_{-1} | \mathcal{F}_{-2}] = X_{-2}$ $(n=-2)$

By induction, we have $E[X_0 | \mathcal{F}_n] = X_n$

$\boxed{\text{Step3}}$

By $\boxed{\text{Step2}}$ and $\boxed{57}$, $\{X_n\}_{n \geq 1}$ is uniformly integrable.

By $\boxed{59}$ $X_n \to X_{-\infty}$ (as) $\Rightarrow$ $X_n \xrightarrow{P} X_{-\infty}$

and $\{X_n\}_{n \geq 1}$ is uniformly integrable

CHA SHIN

$\Rightarrow$ $X_n \xrightarrow{L^1} X_{-\infty}$

$\boxed{70}$ Suppose $\{X_n\}_{n \leq 0}$ is a backward martingale.

$X_{-\infty} = \lim_{n \to -\infty} X_n$ and $\mathcal{F}_{-\infty} = \bigcap_{n \leq 0} \mathcal{F}_{n}$ ($\because$ this is σ-algebra)

We show that $X_{-\infty} = E[X_0 \mid \mathcal{F}_{-\infty}]$

$\boxed{\text{Step 1}}$ $X_{-\infty}$ is $\mathcal{F}_{-\infty}$-measurable

Consider $\{X_{-\infty} \geq a\}$.

$= \{X_{-\infty} \geq a\} = \{\limsup_{n \to -\infty} X_n \geq a\}$

$= \{\lim_{n \to -\infty} \sup_{m \leq n} X_m \geq a\}$

$= \bigcap_{n \leq -\infty} \underbrace{\{\sup_{m \leq n} X_m \geq a\}}_{\mathcal{F}_n}$

Hence $\{X_{-\infty} \geq a\} \in \mathcal{F}_n$ for all $n \leq -\infty$

$\Rightarrow \{X_{-\infty} \geq a\} \in \bigcap_{n \leq 0}^{\infty} \mathcal{F}_n$

PAGE
DATE

$\boxed{\text{step2}}$

$X_n = E[X_0 | \mathcal{F}_n]$ (see $\boxed{69}$)

Let $A \in \mathcal{F}_{-\infty}$.

$\int_A X_n \, dP = \int_A E[X_0 | \mathcal{F}_n] \, dP = \int_A X_0 \, dP$ for all $n \leq 0$

(• $A \in \mathcal{F}_{-\infty} \subseteq \mathcal{F}_n$ for all $n \geq 1$
• By the definition of conditional expectation)

$\int_A X_n \, dP = E[X_n \cdot I_A]$

Since $X_n \xrightarrow{L^1} X_{-\infty}$ (by $\boxed{69}$) and

By $\boxed{61}$, $E[X_n \cdot I_A] \to E[X_{-\infty} \cdot I_A]$

But $\int_A X_n \, dP = E[X_n \cdot I_A] = E[X_0 \cdot I_A]$ (for all $n \leq 0$)

So we find that $E[X_0 \cdot I_A] = E[X_{-\infty} I_A]$.

PAGE

DATE

Finally for $\forall A \in \mathcal{F}_{-\infty}$ we have

$$\int_A X_0 \, dP = \int_A X_{-\infty} \, dP$$

$$\| \qquad\qquad \Downarrow \text{ (by the definition of conditional expectation)}$$

$$\int_A E[X_0 | \mathcal{F}_{-\infty}] \, dP$$

Therefore, $X_{-\infty} = E[X_0 | \mathcal{F}_{-\infty}]$ (a.s.)

$\boxed{71}$ Suppose $\mathcal{F}_{n-1} \subseteq \mathcal{F}_n$ $(n \le 0)$

$$\mathcal{F}_{-\infty} = \bigcap_{n \le 0} \mathcal{F}_n.$$

We show that $E[Y|\mathcal{F}_n] \xrightarrow{as/L^1} E[Y|\mathcal{F}_{-\infty}]$

(proof)

(Here we suppose $E|Y| < \infty$)

Let $X_n \overset{def}{=} E[Y|\mathcal{F}_n]$.

$$E[X_{n+1}|\mathcal{F}_n] = E[E[Y|\mathcal{F}_{n+1}]|\mathcal{F}_n] \quad (\mathcal{F}_n \subseteq \mathcal{F}_{n+1})$$

$$= E[Y|\mathcal{F}_n] = X_n \quad (\text{by } \boxed{19})$$

So $\{X_n\}_{n \le 0}$ is a backward martingale

By $\boxed{69}$.(2) $X_n \xrightarrow{as} X_{-\infty}$

$X_n \xrightarrow{L^1} X_{-\infty}$

And by $\boxed{70}$

$$X_{-\infty} = E[X_0|\mathcal{F}_{-\infty}]$$

$$= E[E[Y|\mathcal{F}_0]|\mathcal{F}_{-\infty}]$$

$$= E[Y|\mathcal{F}_{-\infty}] \quad \text{by } \boxed{19}$$

So the proof is complete ■

$\boxed{72}$ $\{X_n\}_{n\geq 1}$ - a uniformly integrable submartingale.

We show that $\{X_{n\wedge N}\}_{n\geq 1}$ is uniformly integrable

where N is a stopping time.

$\boxed{\text{Step 1}}$ $\sup_{n\geq 1} E[X_{N\wedge n}^+] < \infty$

- $\phi(x) = \max\{0, x\}$ is a convex and monotone increasing function $\therefore$ $X_n^+ = \phi(X_n)$ is also a submartingale (By $\boxed{29}$)
- $N\wedge n$ is also a stopping time

and $P(N\wedge n \leq n) = 1$

Applying $\boxed{48}$ to $\{X_n^+\}$ and $N\wedge n$

we have $E[X_{N\wedge n}^+] \leq E[X_n^+]$

- $\{X_n\}_{n\geq 1}$: uniformly integrable $\Rightarrow$ $\sup_{n\geq 1} E|X_n| < \infty$

$\Rightarrow \sup_{n\geq 1} E[X_n^+] < \infty$ (See $\boxed{56}$ (2))

$\therefore \sup_{n\geq 1} E[X_{N\wedge n}^+] < \infty$

$\boxed{\text{Step 2}}$ By martingale convergence theorem ($\boxed{32}$, $\boxed{34}$)

$X_{N\wedge n} \to X_N$ (as) and $E|X_N| < \infty$.

(This is a submartingale)

Finally consider $\lim_{k\to\infty} \sup_{n\geq 1} E[|X_{N\wedge n}| \cdot \mathbb{I}\{|X_{N\wedge n}| > k\}]$

$$E[|X_{N\wedge n}| \cdot \mathbb{I}\{|X_{N\wedge n}| > k\}]$$

$$\|$$

$$E[|X_{N\wedge n}| \cdot \mathbb{I}\{|X_{N\wedge n}| > k,\ N > n\}] +$$
$$E[|X_{N\wedge n}| \cdot \mathbb{I}\{|X_{N\wedge n}| > k,\ N \leq n\}]$$

$$\|$$

$$E[|X_n| \cdot \mathbb{I}\{|X_n| > k,\ N > n\}] +$$
$$E[|X_N| \cdot \mathbb{I}\{|X_N| > k,\ N \leq n\}]$$

PAGE

DATE

$$\leq \sup_{n \geq 1} E\left[|X_n| \cdot \mathbb{I}\{|X_n| > K\}\right]$$

$$+ \underbrace{E\left[|X_N| \cdot \mathbb{I}\{|X_N| > K\}\right]}$$ not related to n

$$\therefore \sup_{n \geq 1} E\left[|X_{N \wedge n}| \cdot \mathbb{I}\{|X_{N \wedge n}| > K\}\right]$$

$$\leq \sup_{n \geq 1} E\left[|X_n| \cdot \mathbb{I}\{|X_n| > K\}\right] + E\left[|X_N| \cdot \mathbb{I}\{|X_N| > K\}\right]$$

$$(E|X_N| < \infty)$$

By taking large $K > 0$. we have $< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

PAGE
DATE

$\boxed{13}$

(1) $\{X_n\}_{n\geq 1}$ ... a uniformly integrable submartingale

N ... a stopping time

We show that $E[X_0] \leq E[X_N] \leq E[X_\infty]$

$(X_\infty = \lim X_n)$

(proof)

(Similar to $\boxed{12}$)

Apply $\boxed{4.8}$ to $\{X_n\}_{n\geq 1}$ and $N \wedge n$

(This is also a stopping time and $P(N\wedge n \leq n) = 1$)

Then we have $E[X_0] \leq E[X_{N\wedge n}] \leq E[X_n]$ ... (*)

CHA SHIN

PAGE
DATE

Recall $\boxed{60}$ — ($\{Y_n\}_{n\geq 1}$ submartingale.

· Uniformly integrable $\Leftrightarrow$ converge in L^1)

Since $\{X_{N\wedge n}\}$, $\{X_n\}$ are both submartingales ($\boxed{32}$)

and are uniformly integrable ($\because \boxed{72}$),

$\{X_{N\wedge n}\}$, $\{X_n\}$ converges in L^1.

This implies $E[X_{N\wedge n}] \to E[X_N]$ ($\because X_{N\wedge n} \xrightarrow{a.s/L^1} X_N$ See $\boxed{72}$)

$E[X_n] \to E[X_\infty]$

By taking $n \nearrow \infty$ to $\otimes$,

we have $E[X_0] \leq E[X_N] \leq E[X_\infty]$

PAGE

DATE

(2) If $E|X_N| < \infty$ and $X_n \cdot \mathbb{I}\{N > n\}$ is uniformly integrable.

Then $X_{N \wedge n}$ is also uniformly integrable and hence $E[X_0] \leq E[X_N]$

(proof)

$E[|X_{N\wedge n}| \cdot \mathbb{I}\{|X_{N\wedge n}| > M\}]$

$= E[|X_{N\wedge n}| \cdot \mathbb{I}\{|X_{N\wedge n}| > M, N > n\}]$

$+ E[|X_{N\wedge n}| \cdot \mathbb{I}\{|X_{N\wedge n}| > M, N \leq n\}]$

$= E[|X_n| \cdot \mathbb{I}\{|X_n| > M, N > n\}]$ -- ①

$+ E[|X_N| \cdot \mathbb{I}\{|X_N| > M, N \leq n\}]$ -- ②

① $= E[|X_n| \cdot \mathbb{I}\{N > n\} \cdot \mathbb{I}\{|X_n| \mathbb{I}\{N > n\} > M\}]$

$\leq \sup_{n \geq 1} E[\cdot]$

② $\leq E[|X_N| \cdot \mathbb{I}\{|X_N| > M\}]$

CHA SHIN

So by taking large $M > 0$ we have $< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$

PAGE
DATE

(If X_n is a submartingale)

$X_{n \wedge N}$ is a uniformly integrable submartingale

By (1) $E[X_{0 \wedge N}] \leq E[X_{N \wedge N}]$

So $E[X_0] \leq E[X_N]$.

(3) $\{X_n\}_{n\ge 1}$ .. a non-negative supermartingale

$N(\le\infty)$ .. a stopping time

(proof) $N\wedge n$ is a stopping time and $P(N\wedge n\le n)=1$.

$-X_n$ is a submartingale

By $\boxed{4.2}$, $E[-X_0]\le E[-X_{N\wedge n}]$

$\Rightarrow E[X_{N\wedge n}]\le E[X_0]$ for all $n\ge 1$

By Fatou's lemma

$E[\liminf X_{N\wedge n}]\le \liminf E[X_{N\wedge n}]\le E[X_0]$

$\therefore E[X_N]\le E[X_0]$

$\boxed{74}$ $\{X_n\}_{n\ge 1}$ .. submartingale, $E[\,|X_{n+1}-X_n|\,|\,\mathcal{F}_n] \le B$ (a.s)

N .. a stopping time $E[N] < \infty$

We show that $X_{N\wedge n}$ is uniformly integrable and hence $E[X_0] \le E[X_N]$

$\boxed{\text{Step 1}}$ $|X_{N\wedge n}| \le |X_0| + \sum_{m=0}^{\infty} |X_{m+1}-X_m| \cdot \mathbb{I}\{N>m\}$ ... ⊛

because $X_{N\wedge n} = X_0 + \sum_{m=0}^{N\wedge n-1} (X_{m+1}-X_m)$ and

so $|X_{N\wedge n}| \le |X_0| + \sum_{m=0}^{N\wedge n-1} |X_{m+1}-X_m|$

$= |X_0| + \sum_{m=0}^{(N-1)\wedge n-1} |X_{m+1}-X_m|$

$\le |X_0| + \sum_{m=0}^{n-1} |X_{m+1}-X_m| \cdot \mathbb{I}\{N>m\}$

$\le |X_0| + \sum_{m=0}^{\infty} |X_{m+1}-X_m| \cdot \mathbb{I}\{N>m\}$

$\boxed{\text{Step 2}}$ $E[\,|X_{m+1}-X_m| \cdot \mathbb{I}\{N>m\}]$

$= E[\,E[\,|X_{m+1}-X_m| \cdot \mathbb{I}\{N>m\} \,|\, \mathcal{F}_m]\,]$, $\{N>m\} \in \mathcal{F}_m$

$= E[\,E[\,|X_{m+1}-X_m| \,|\, \mathcal{F}_m] \cdot \mathbb{I}\{N>m\}]$ by $\boxed{20}$ and

$\le E[B \cdot \mathbb{I}\{N>m\}] = B \cdot P(N>m)$

PAGE

DATE

Therefore $\boxed{\text{Step}}$ ⊛ :

$$E[\boxed{R.H.S}] = E[|X_0|] + \sum_{m\ge 0}^{\infty} B \cdot P(N>m)$$

$$= E|X_0| + B\,E[N] \quad \text{ch2 } \boxed{35}$$

$$< \infty$$

$$|X_{N\wedge n}| \le \underbrace{|X_0| + \sum_{m\ge 0}^{\infty} |X_{m+1} - X_m| \cdot \mathbb{I}\{N>m\}}_{} = \underset{\sim}{Y}$$

This is an integrable r.v not related to n.

$$\text{So } \lim_{M\to\infty} \sup_{n\ge 1} E[|X_{N\wedge n}| \cdot \mathbb{I}\{|X_{N\wedge n}|>M\}]$$

$$\le \lim_{M\to\infty} E[Y \cdot \mathbb{I}\{Y>M\}] = 0 \quad (\because Y: \text{integrable})$$

$\therefore \{X_{N\wedge n}\}_{n\ge 1}$ is uniformly integrable