chapter 3. Solution Manual by Toshinari Morimoto

$\boxed{1}$ $F(\cdot)$ is a right continuous function

(proof) $F(x) \overset{def}{=} \inf_{z > x} \{ F_D(z) \}$

By this definition, $\forall \varepsilon > 0$, $\forall x \in \mathbb{R}$ $\exists y(\varepsilon, x) \in D$, $y > x$

such that $F(x) \leq F_D(y) < F(x) + \varepsilon$ ... (*)

Let $x^* \in (x, y)$ then $\underset{①}{F(x) \leq} F(x^*) \underset{②}{\leq} F_D(y) \underset{③}{<} F(x) + \varepsilon$

Hence

$F(\cdot)$ is right-continuous. $\blacksquare$

($\because$) ① ... $F(x)$ is monotone-increasing.

$\because$ $x_1 < x_2 \Rightarrow (x_1, \infty) \supseteq (x_2, \infty)$

$\Rightarrow \{F_D(z)\}_{z \in (x_1, \infty)} \supseteq \{F_D(z)\}_{z \in (x_2, \infty)}$

$\Rightarrow \inf \{F_D(z)\}_{z \in (x_1, \infty)} \leq \inf \{F_D(z)\}_{z \in (x_2, \infty)}$

② ... $F(x^*) = \inf \{F_D(z)\}_{z \in (x^*, \infty)}$ and $F_D(y) \in \{F_D(z)\}_{z \in (x^*, \infty)}$

$\therefore F(x^*) \leq F_D(y)$ $\quad (\because y \in (x^*, \infty))$

③ .. by (*)

Also $F(\cdot)$ is a probability distribution function (cdf)

(I) $\lim\limits_{x\to\infty} F(x)=1$.

$\forall \varepsilon>0,\ \exists y_\varepsilon \in D$ s.t. $\forall \underset{z\in D}{z>y_\varepsilon}$ $F_D(z)>1-\varepsilon$.

$\left(\because \lim\limits_{\substack{x\in D\\ x\to\infty}} F_D(x)=1\right)$

Hence $\inf\limits_{\substack{z>y_\varepsilon\\ z\in D}}\{F_D(z)\}\geq 1-\varepsilon$

$\therefore F(y)\geq 1-\varepsilon$ $\therefore$ $\forall x>y_{(\varepsilon)}$ $F(x)\geq 1-\varepsilon$ ■

(II) $\lim\limits_{x\to-\infty} F(x)=0$

$\forall \varepsilon>0\ \exists y\in D$ s.t. $\forall \tilde{y}\in D$ $(\tilde{y}<y)$

$F_D(\tilde{y})<\varepsilon$ $\left(\because \lim\limits_{\substack{x\to-\infty\\ x\in D}} F_D(x)=0\right)$

We take $\tilde{y}_1<\tilde{y}_2<y$ $\tilde{y}_1, \tilde{y}_2\in D$

$F(\tilde{y}_1)=\inf\limits_{\substack{z\in D\\ z>\tilde{y}_1}} F_D(z)\leq F_D(\tilde{y}_2)<\varepsilon$

This implies $\forall \tilde{y}<y$ $F(\tilde{y})<\varepsilon$

Hence $\lim\limits_{x\to-\infty} F(x)=0$

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$\boxed{2}$ The definition of "vague convergence" or "weak convergence" is somewhat different from Durrett's. Here we state the definition by Resnick.

- vague convergence $\forall (a,b]$ $a,b \in C(F)$ $\quad$ (i.e. $F(\{a\}) = F(\{b\}) = 0$)

$$\lim_{n\to\infty} F_n((a,b]) = F((a,b]) \qquad (\{F_n\}_{n\ge1} \text{ sequence of distribution function})$$
$$\Vert$$
$$(:= F_n(b) - F_n(a))$$

- proper convergence $\forall (a,b] \in C(F)$

$$\lim_{n\to\infty} F_n((a,b]) = F((a,b]) \qquad (\{F_n\}_{n\ge1} : \text{distribution function})$$

and $F(\cdot)$ is a distribution function.

- weak convergence $\forall b \in C(F)$

$$\lim_{n\to\infty} F_n(b) = F(b) \qquad (\{F_n\}_{n\ge1} : \text{distribution function})$$

- complete convergence $\forall b \in C(F)$

$\lim_{n\to\infty} F_n(b) = F(b)$ and $F(\cdot)$ is also a distribution function.

($\{F_n\}$: distribution function)

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$\boxed{3}$ Let $F(\cdot)$ be a distribution function

$F_n(x) \overset{def}{=} F(x+(-1)^n \cdot n)$

$\{F_n\}_{n\geq 1}$ is also a sequence of distribution

$\cdot\ F_{2n+1}(x) = F(x-(2n+1))$

$\cdot\ F_{2n}(x) = F(x+2n)$

$\lim_{n\to\infty} F_{2n+1}(x) = 0 \quad (\text{for all } x\in\mathbb{R})$

$\lim_{n\to\infty} F_{2n}(x) = 1 \quad (\text{for all } x\in\mathbb{R})$

So $F_n(\cdot)$ does not converge at any x.

However $F_n((a,b]) = F_n(b+(-1)^n\cdot n) - F_n(a+(-1)^n\cdot n)$

$\lim_{n\to\infty} F_n((a,b]) = 0 \quad (\because\ n:$ odd.. $0-0=0$, even.. $1-1=0)$

Let $F(x)=0$

Then $F_n((a,b]) \to 0 = F((a,b])$

So F_n vaguely converges to F. $(F=0)$

(F is "not" a distribution function.

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So this is not weak convergence)

$\boxed{4}$ (Equivalance of four convergence)

"F is proper" means "F is a distribution function"

(i.e $\lim_{x\to\infty} F(x) = 1$, $\lim_{x\to-\infty} F(x) = 0$

and $F(\cdot)$ is right-continuous)

① vague convergence
② proper convergence
③ weak convergence
④ complete convergence

If F is proper, then obviously ① ⇔ ② and ③ ⇔ ④.

So we prove that ② ⇒ ④ and ④ ⇒ ②

(I) ④ ⇒ ②

It's easy. $F_n((a.b]) = F_n(b) - F_n(a)$ $(a.b \in C(F))$

When $n \to \infty$, $F_n(b) \to F(b)$ $F_n(a) \to F(a)$

Hence $\lim_{n\to\infty} F_n((a.b]) = F(b) - F(a)$ (F: distribution function)

And $F(b) - F(a) = F((a.b])$

So $\lim_{n\to\infty} F_n((a.b]) = F((a.b])$

(II) ② → ①

${}^\forall a,b \in C(F)$ $\lim_{n\to\infty} F_n((a,b]) = F((a,b])$

Now we prove that ${}^\forall b \in C(F)$ $\lim_{n\to\infty} F_n(b) = F(b)$

(i) "$\liminf F_n(b) \geq F(b)$"

$$\lim F_n((a,b]) = \lim (F_n(b) - F_n(a)) \leq \liminf F_n(b)$$
$$\| $$
$$F(b) - F(a)$$

$a \downarrow -\infty$ ($a \in C(F)$) $F(a) \downarrow 0$.

So we have $\liminf F_n(b) \geq F(b)$

$\boxed{\text{NOTE}}$ $F(\cdot)$ has at least countable points of discontinuity

So we may take infinitely small $a \in C(F)$

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(ii) "$\limsup F_n(b) \leq F(b)$"

pick b_1, b_2 : $b_1 < b < b_2$ $b_1, b_2 \in C(F)$

Since $F_n((b_1, b_2]) \to F((b_1, b_2])$

$\therefore$ $F_n((b_1, b_2]^c) \to F((b_1, b_2]^c)$

And we may pick b_1, b_2 so that

$F((b_1, b_2]^c) < \varepsilon$ ($\because$ F: distribution function)

So for sufficiently large $n \geq 1$

$F_n((b_1 b_2]^c) < 2\varepsilon$

Now
$$\begin{aligned} F_n(b) &= F_n(b) - F_n(b_1) + F_n(b_1) \\ &= F_n((b_1, b]) + F_n(b_1) \\ &\leq F_n((b_1, b]) + F_n((b_1, b_2]^c) \quad \text{(for sufficiently large } n \geq 1) \\ &\leq F_n((b_1, b]) + 2\varepsilon \end{aligned}$$

Hence
$$\begin{aligned} \limsup_{n \to \infty} F_n(b) &\leq F((b_1, b]) + 2\varepsilon \\ &\leq F(b) + 2\varepsilon \end{aligned}$$

$\varepsilon \downarrow 0$, we have the desired result

$\boxed{5}$ $X_n = (-1)^n \cdot X$ where $X \sim N(0,1)$

Then $X_n \sim N(0,1)$ (for all n)

$\therefore F_n(x) = P(X_n \le x) = F(x)$ $\left(F(x) = P(X \le x) \right)$

$\Rightarrow \lim_{n\to\infty} F_n(x) = F(x)$ $(\forall x \in \mathbb{R})$

Hence $X_n \xrightarrow{d} X$.

However $\limsup_{n\to\infty} P(|X_n - X| > \varepsilon) > 0$

So $X_n \not\xrightarrow{p} X$. $\left(\therefore X_n \not\xrightarrow{as} X \right)$

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$\boxed{6}$ The following conditions are stronger than weak convergence

① $^{\forall}B \in \mathcal{B}(\mathbb{R})$ (Borel-Sets)

$$\lim_{n\to\infty} F_n(B) = F(B)$$

$$\left(F_n(B) \overset{\text{def}}{=} P(X_n \in B) \quad X_n \sim F_n \right)$$

② $\sup_{B \in \mathcal{B}(\mathbb{R})} |F_n(B) - F(B)| = 0.$

[7] Consider a sequence of distribution functions

$\{F_n\}_{n\geq 1}$ ⋯ ($X_n \sim F_n$ $P(X_n = \frac{k}{n}) = \frac{1}{n}$ $k=1,2,3 \cdots n$)

$$F_n(x) = P(X_n \leq x) = \begin{cases} \frac{[nx]}{n} & (0 \leq x \leq 1) \\ 1 & (x > 1) \end{cases}$$

So $\forall x \in [0,1)$ $\lim_{n\to\infty} F_n(x) = x$

$\therefore$ $X_n \xrightarrow{d}$ Uniform $(0,1)$

However it does not satisfy $\boxed{6}$.

Let $B = \mathbb{Q} \cap [0,1]$ ($\mathbb{Q}$: set of rational numbers)

Then $F_n(B) = P(X_n \in B) = 1$.

$(\because \{\frac{1}{n}, \frac{2}{n}, \ldots \frac{n}{n}\} \subseteq B)$

However $F(B) = P(X \in B) = 0$

($X \sim$ Uniform$(0,1)$)

because X is a continuous random variable

but B is countable. hence $P(X \in B) = 0$.

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$\boxed{8}$

(1) $F_n(B) - F(B) = \int_B f_n dx - \int_B f dx$ ... ①

$\int_{\mathbb{R}} f_n dx = \int_{\mathbb{R}} f dx = 1$

$\Rightarrow \int_{\mathbb{R}} f_n dx - \int_{\mathbb{R}} f dx = 0$ ... ②

① − ② $= -\{\int_{B^c} f_n dx - \int_{B^c} f dx\} = F_n(B) - F(B)$ ... ①'

$|①| = |\int_B (f_n - f) dx| \leq \int_B |f_n - f| dx$

$|①'| = |-\int_{B^c} (f_n - f) dx| \leq \int_{B^c} |f_n - f| dx$

$\therefore \underbrace{|①| + |①'|}_{= 2|F_n(B) - F(B)|} \leq \int_{\mathbb{R}} |f_n - f| dx$

$\therefore \sup_{B \in \mathcal{B}(\mathbb{R})} |F_n(B) - F(B)| \leq \frac{1}{2} \int_{\mathbb{R}} |f_n - f| dx$

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Next Let $B = \{f_n - f \geq 0\}$

- $F_n(B) - F(B) = \int_B \underbrace{(f_n - f)}_{\geq 0} dx = |F_n(B) - F(B)| \cdots ③$

$$= \int_B |f_n - f| \, dx$$

- $F_n(B) - F(B) = \int_{B^c} \underbrace{(f - f_n)}_{\geq 0} dx = |F_n(B) - F(B)| \cdots ④$

$$= \int_{B^c} |f_n - f| \, dx$$

↳ (See the previous page)

$\therefore ③ + ④ = 2|F_n(B) - F(B)| = \int_{\mathbb{R}} |f_n - f|$

$\therefore \exists B \in \mathcal{B}(\mathbb{R}) \quad |F_n(B) - F(B)| = \frac{1}{2} \int_{\mathbb{R}} |f_n - f| \, dx$

$\Rightarrow \sup_{B \in \mathcal{B}(\mathbb{R})} |F_n(B) - F(B)| \geq \frac{1}{2} \int_{\mathbb{R}} |f_n - f| \, dx$

Hence the proof is complete ■

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(2) $\int_R |f_n - f| dx = \int_R |f - f_n|$

$= \int_R (f-f_n)^+ dx + \int_R (f-f_n)^- dx$

And

$\int_R (f-f_n)^+ - \int_R (f-f_n)^- = 0$

$(\because = \int_R (f-f_n) dx = \int_R f dx - \int_R f_n dx = 1 - 1 = 0)$

So $\int_R |f_n - f| dx = 2 \int_R (f-f_n)^+ dx$

$0 \leq (f-f_n)^+ \leq f \in L^1$ (bounded by an integrable function)

Hence by LDCT $\lim_{n\to\infty} 2 \int_R (f-f_n)^+ dx = 2 \int_R \lim_{n\to\infty} (f-f_n)^+ dx$

$= 0$

[9] We show that $\sum_{j=1}^{n} \log(1+C_{jn}) \to \lambda$

$\therefore \left| \sum_{j=1}^{n} \log(1+C_{jn}) - \lambda \right| \leq \left| \sum_{j=1}^{n} \log(1+C_{jn}) - \sum_{j=1}^{n} C_{jn} \right| \cdots ①$

$+ \left| \sum_{j=1}^{n} C_{jn} - \lambda \right| \cdots ②$

$② \to 0$ (by assumption)

$① \leq \sum_{j=1}^{n} \left| \log(1+C_{jn}) - C_{jn} \right|$ $\quad \left| \log(1+x) - x \right| \leq \frac{x^2}{2}$

$\leq \sum_{j=1}^{n} \frac{1}{2} |C_{jn}|^2$

$\leq \sum_{j=1}^{n} \frac{1}{2} |C_{jn}| \max_{1 \leq j \leq n} |C_{jn}|$

$= \max_{1 \leq j \leq n} |C_{jn}| \cdot \sum_{j=1}^{n} \frac{1}{2} |C_{jn}|$

$\leq \max_{1 \leq j \leq n} |C_{jn}| \cdot \underbrace{\sup_{n \geq 1} \sum_{j=1}^{n} \frac{1}{2} |C_{jn}|}_{< \infty}$

$\therefore n \to \infty \quad \to 0$

$\boxed{10}$ Let $(\Omega, \mathcal{F}, P)$ be a probability space.

$\Omega = (0,1)$ $\mathcal{F} \overset{def}{=} \mathcal{B}((0,1)) \overset{def}{=} \{B \cap (0,1) \mid B \in \mathcal{B}(\mathbb{R})\}$

$P(\cdot) =$ Lebesgue Measure $(P((a,b]) = b-a)$

And $Y_n(\omega) \overset{def}{=} \inf\{y \mid F_n(y) \geq \omega\}$
$Y(\omega) \overset{def}{=} \inf\{y \mid F(y) \geq \omega\}$

By chapter 1. $\boxed{3}$ $Y_n \sim F_n$ and $Y \sim F$.

Now we prove $Y_n(\omega) \to Y(\omega)$ (a.s)

① $\forall \varepsilon > 0$, $\forall \omega \in \Omega$.

Consider an open interval $(Y(\omega) - \varepsilon, Y(\omega))$.

There are uncountable points in $(Y(\omega) - \varepsilon, Y(\omega))$

However there are only countable points of discontinuity for $F(\cdot)$ on $\mathbb{R}$.

So we may find $y \in (Y(\omega) - \varepsilon, Y(\omega))$

s.t $\lim_{n\to\infty} F_n(y) = F(y)$.

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Since $y < Y(w) = \inf\{x \mid F(x) \geq w\}$. so

$y \notin \{x \mid F(x) \geq w\}$.

$\Rightarrow F(y) < w$,

For sufficiently large n, $F_n(y) < w$.

$(\because F_n(y) \to F(y))$

So $y \notin \{x \mid F_n(y) \geq w\}$

$\therefore y < \inf\{x \mid F_n(y) \geq w\} = Y_n(w)$

And $Y(w) - \varepsilon < y < Y_n(w)$. (for sufficiently large n)

$\therefore Y(w) \leq \liminf Y_n(w)$

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② $\forall \varepsilon > 0 \quad \forall \hat{w} > w$

Similarly we may pick $y \in (Y(\hat{w}), Y(\hat{w})+\varepsilon)$

Where $F(y)$ is continuous at y.

$Y(\hat{w}) < y \Rightarrow \inf\{x \mid F(x) \geq \hat{w}\} < y$

$\Rightarrow y \in \{x \mid F(x) \geq \hat{w}\} \Rightarrow F(y) \geq \hat{w}$

$\Rightarrow F(y) > w \quad (\because \hat{w} > w)$

And for suffiently large n, $F_n(y) > w$

$\Rightarrow y \in \{x \mid F_n(x) \geq w\}$

$\Rightarrow y \geq \inf\{x \mid F_n(x) \geq w\} = Y_n(w)$

$\therefore Y_n(w) \leq y < Y(\hat{w}) + \varepsilon$ (for suffiently large n)

$\therefore \limsup Y_n(w) \leq Y(\hat{w})$

$Y(w)$ is an increasing function. So there are only cuntable points of discontinuity ($\Rightarrow$ a.e continuous)

So by $\hat{w} \downarrow w$. we have $\limsup Y_n(w) \leq Y(w)$

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$\boxed{11}$ 「$X_n \xrightarrow{d} X$」 $\Leftrightarrow$ 「$\forall g$: bounded and continuous
$E[g(X_n)] \to E[g(X)]$」

① $\Rightarrow$

$E[g(X_n)]$ and $E[g(X)]$ only depend on the distribution of $g(X_n)$ and $g(X)$, hence without loss of generality, we may assume the probablity space $(\Omega, \mathcal{F}, P)$ is given as $\boxed{10}$.

Thus $X_n \xrightarrow{as} X$.

Therefore $g(X_n) \xrightarrow{as} g(X)$ ($\because$ g: continuous)

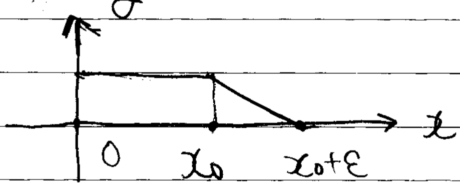
By bounded convergence theorem ($|g| \leq M < \infty$)

$E[g(X_n)] \to E[g(X)]$

② $\Leftarrow$

We define $g_1^{(\varepsilon)}$ and $g_2^{(\varepsilon)}$ as below.
(i) (ii)
($\varepsilon > 0$)

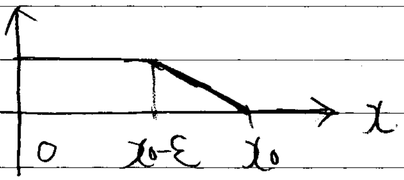
(i)



$$\begin{cases} 1 & (x \leqq x_0) \\ -\frac{1}{\varepsilon}(x-x_0) & (x_0 < x < x_0+\varepsilon) \\ 0 & (x \geqq x_0+\varepsilon) \end{cases}$$

$y = g_1^{(\varepsilon)}(x)$

(ii)



$$\begin{cases} 1 & (x \leqq x_0-\varepsilon) \\ -\frac{1}{\varepsilon}(x-x_0+\varepsilon) & (x_0-\varepsilon < x < x_0) \\ 0 & (x \geqq x_0) \end{cases}$$

$y = g_2^{(\varepsilon)}(x)$

Apply $E[g_1^{(\varepsilon)}(X_n)] \to E[g_1^{(\varepsilon)}(X)]$
$E[g_2^{(\varepsilon)}(X_n)] \to E[g_2^{(\varepsilon)}(X)]$

(i') $\mathbb{I}(x \leq x_0) \leqq g_1^{(\varepsilon)}(x)$

So $E[\mathbb{I}(X_n \leq x_0)] \leqq E[g_1^{(\varepsilon)}(X_n)]$

$\therefore$ $P(X_n \leq x_0) \leqq E[g_1^{(\varepsilon)}(X_n)]$

$$\limsup_{n\to\infty} P(X_n \leq x_0) \leqq \lim_{n\to\infty} E[g_1^{(\varepsilon)}(X_n)] = E[g_1^{(\varepsilon)}(X)]$$

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And then by taking $\varepsilon \downarrow 0$.

$$\lim_{\varepsilon \downarrow 0} E[g_1^{(\varepsilon)}(X)] = P(X \leq \lambda_0)$$

(g_1 is bounded. We may apply Bounded Convergence Theorem

$\varepsilon \downarrow 0 \quad g_1^{(\varepsilon)}(x) \to \mathbb{I}\{X \leq \lambda_0\}$)

So $\limsup_{n \to \infty} P(X_n \leq \lambda_0) \leq P(X \leq \lambda_0)$

(II) Similarly

$$g_2^{(\varepsilon)}(x) \leq \mathbb{I}\{X_n \leq \lambda_0\}$$

$$\therefore E[g_2^{(\varepsilon)}(X_n)] \leq E[\mathbb{I}\{X_n \leq \lambda_0\}]$$

$$\lim_{n \to \infty} E[g_2^{(\varepsilon)}(X_n)] \leq \liminf_{n \to \infty} P(X_n \leq \lambda_0)$$

$$\parallel$$

$$E[g_2^{(\varepsilon)}(X)]$$

By taking $\varepsilon \downarrow 0$, the left hand side =

$$\lim_{\varepsilon \downarrow 0} E[g_2^{(\varepsilon)}(X)] \leq E[\lim_{\varepsilon \downarrow 0} g_2^{(\varepsilon)}(X)] = E[\mathbb{I}\{X < \lambda_0\}] = P(X < \lambda_0)$$

(Bounded Convergence Theorem)

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So we have $P(X<x_0) \leq \liminf P(X_n \leq x_0)$

If $x_0 \in C(X)$ (ie $P(X=x_0)=0$)

Then $P(X<x_0) = P(X \leq x_0)$.

Hence $\lim_{n\to\infty} P(X_n \leq x_0) = P(X \leq x_0)$

for all $x_0 \in C(X)$.

Therefore $X_n \xrightarrow{d} X$

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$\boxed{12}$ Continuous Mapping Theorem

- $X_n \xrightarrow{d} X$

- f: Borel-Measurable function

$D \overset{\text{def}}{=} \{x \in \mathbb{R} \mid f(\cdot) \text{ is discontinuous at } x\}$

(Then D is also a borel measurable set)

$P(X \in D) = 0$.

Then $f(X_n) \xrightarrow{d} f(X)$.

(proof) We prove that for any bounded continuous function g, $E[g \circ f(X_n)] \to E[g \circ f(X)]$

Since $E[g \circ f(X_n)]$ and $E[g \circ f(X)]$ is determined by the distribution of $g \circ f(X_n)$ and $g \circ f(X)$, without loss of generality we may suppose that $(\Omega, \mathcal{F}, P)$ is given as $\boxed{10}$.

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So $X_n \to X$ (as)

$\therefore$ Hence $f(X_n) \to f(X)$ (as)

$(\because N_1 = \{ \omega \mid X_n(\omega) \nrightarrow X(\omega) \}$
$N_2 = \{ \omega \mid X(\omega) \in D \}$

$P(N_1 \cup N_2) = 0$. $\omega \notin N_1 \cup N_2 \Rightarrow f(X_n(\omega)) \to f(X(\omega)))$

Therefore by bounded convergence theorem

$E[g \circ f(X_n)] \to E[g \circ f(X)]$.

Now the proof is complete.

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$\boxed{13}$ ① $X_n \xrightarrow{d} X$

② $\forall G$: openset: $\liminf_{n\to\infty} P(X_n \in G) \geq P(X \in G)$

③ $\forall F$: closed set: $\limsup_{n\to\infty} P(X_n \in F) \leq P(X \in F)$

④ $\forall A \in \mathbb{B}(\mathbb{R})$: $P(X \in \partial A) = 0$ $\lim P(X_n \in A) = P(X \in A)$

$(\partial A = \bar{A} \setminus \mathring{A})$

① ~ ④ are all equivalent.

$\boxed{① \Rightarrow ②}$

$P(X_n \in G)$ and $P(X \in G)$ is determined by their distributions

Hence without loss of generality we may assume that $(\Omega, \mathcal{F}, P)$ is given as $\boxed{10}$.

Therefore $X_n \xrightarrow{as} X$.

When $X_n \xrightarrow{as} X$, then $\mathbb{I}\{X \in G\} \leq \mathbb{I}\{X_n \in G\}$ for sufficiently large $n \geq 1$ (a.s)

$\therefore \mathbb{I}\{X \in G\} \leq \liminf_{n\to\infty} \mathbb{I}\{X_n \in G\}$ (a.s)

$\Rightarrow E[\mathbb{I}\{X \in G\}] \leq E[\liminf_{n\to\infty} \mathbb{I}\{X_n \in G\}]$

$\quad \| $

CHA SHIN $P(X \in G)$

Moreover by Fatou's lemma

$$E[\liminf_{n\to\infty} \mathbb{I}\{X_n \in G\}] \leq \liminf_{n\to\infty} E[\mathbb{I}\{X_n \in G\}]$$

$$= \liminf_{n\to\infty} P(X_n \in G)$$

$\boxed{② \Leftrightarrow ③}$

This is quite easy.

($\Rightarrow$) F^c is an openset.

$$\liminf P(X_n \in F^c) \geq P(X \in F^c)$$

$$\liminf (1 - P(X_n \in F)) \geq 1 - P(X \in F)$$

$$\Leftrightarrow \limsup P(X_n \in F) \leq P(X \in F)$$

($\Leftarrow$) ⋯ similar

$\boxed{② \text{ and } ③ \Rightarrow ④}$

$\left(\begin{array}{l} A^{\circ} \cdots \text{ openset} \\ \bar{A} \cdots \text{ closed-set} \end{array} \right.$

$$P(X_n \in A^{\circ}) \leq P(X_n \in A) \leq P(X_n \in \bar{A})$$

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$\liminf P(X_n \in \mathring{A}) \geq P(X \in \mathring{A}) = P(X \in A)$

$(\because P(X \in \partial A) = 0)$

$\limsup (X_n \in \overline{A}) \leq P(X \in \overline{A}) = P(X \in A)$

$(\because P(X \in \partial A) = 0)$

So $\liminf P(X_n \in A) \geq \liminf P(X_n \in \mathring{A}) \geq P(X \in A)$

$\limsup P(X_n \in A) \leq \limsup P(X_n \in \overline{A}) \leq P(X \in A)$

$\therefore \lim_{n \to \infty} P(X_n \in A) = P(X \in A)$

$\boxed{④ \Rightarrow ①}$

$A = (-\infty, x]$

$\partial A = \{x\}$

$\therefore P(X = x) = 0 \Rightarrow \lim P(X_n \leq x) = P(X \leq x)$

So $X_n \xrightarrow{d} X$.

Now the proof is complete.

$\boxed{4}$ (Helly's selection system)

Let $\{F_n\}_{n\geq 1}$ be a sequence of distribution function.

(i.e $\forall n \in \mathbb{N}$ $F_n(x)$: right-continuous, monotone-increasing, $\lim_{x\to\infty} F_n(x) = 1$ and $\lim_{x\to -\infty} F_n(x) = 0$)

Then there exists a sub-sequence $\{F_{n_k}\}_{k\geq 1}$

and $F(\cdot)$: right-continuous, monotone-increasing function

such that $\lim_{k\to\infty} F_{n_k}(x) = F(x)$ for all $x \in C(F)$

(i.e $F_{n_k} \xrightarrow{w} F$)

(proof)

$\boxed{\text{Step 1}}$

Let $Q = \{q_n\}_{n\geq 1}$ be the set of rational numbers

Consider $\{F_n(q_1)\}_{n\geq 1}$.

Since $0 \leq F_n(q_1) \leq 1$ for all $n \geq 1$.
($F_n(q_1) \in [0,1]$ - bounded)

By Bolzano-Weierstrass's theorem. we may find a subsequence $\{n_1(m)\}_{m\geq 1} \subseteq \mathbb{N}$
s.t $F_{n_1(m)}(q_1)$ converges.

$$G(q_1) \overset{\text{def}}{=} \lim_{m\to\infty} F_{n_1(m)}(q_1)$$

Next we consider $\{F_{n_1(m)}(q_2)\}_{m\geq 1}$

In the same way we may consider the subsequence

$\{n_2(m)\}_{m\geq 1} \subseteq \{n_1(m)\}_{m\geq 1}$

such that $F_{n_2(m)}(q_2)$ converges.

We define $G(q_2) \overset{\text{def}}{=} \lim_{m\to\infty} F_{n_2(m)}(q_2)$

(Of course, $\lim_{m\to\infty} F_{n_2(m)}(q_1) = G(q_1)$)

By repeating this operation,

we obtain $\{n_1(m)\}, \{n_2(m)\}, \cdots$.

Now $\hat{n}(m) \overset{\text{def}}{=} n_m(m)$.

Then $\lim_{m\to\infty} F_{\hat{n}(m)}(q) = G(q)$

for all $q \in \mathbb{Q}$.

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$\boxed{\text{Step2}}$

We define $F(x) \overset{def}{=} \inf\{G(q) \mid q \in \mathbb{Q},\ q > x\}$

Then $F(\cdot)$ is a monotone increasing and right-continuous function.

- monotone-increasing $(x_1 < x_2)$

$\{G(q) \mid q \in \mathbb{Q},\ q > x_1\} \supseteq \{G(q) \mid q \in \mathbb{Q},\ q > x_2\}$

$\Rightarrow \inf\{\ \cdots\ \} \leq \inf\{\ \cdots\ \}$

$\therefore F(x_1) \leq F(x_2)$

- right-continuous

By definition of $F(\cdot)$, $\forall \varepsilon > 0$ $\forall x \in \mathbb{R}$ $\exists q^* > x$ $q^* \in \mathbb{Q}$ such that $F(x) \leq G(q^*) < F(x) + \varepsilon$

Now pick $\hat{x} \in (q^*, x)$

Then $F(x) \leq F(\hat{x})$ and $F(\hat{x}) \leq G(q^*)$

$\left(\because F(\hat{x}) \overset{def}{=} \inf\{G(q) \mid q > \hat{x},\ q \in \mathbb{Q}\} \right.$

$\left. \{G(q) \mid q > \hat{x},\ q \in \mathbb{Q}\} \ni G(q^*) \right)$

So $F(x) \leq F(\hat{x}) \leq G(q^*) < F(x) + \varepsilon$

$\therefore$ This implies $F(\cdot)$ is right continuous

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$\boxed{\text{Step3}}$

We show that $F_{\tilde{n}(m)} \xrightarrow{w} F$.

(i.e $\forall x \in C(F)$ $\quad F_{\tilde{n}(m)}(x) \to F(x)$ as $m \to \infty$)

We pick $x \in C(F)$. Since $F(\cdot)$ is continuous at x
$\forall \varepsilon > 0$. $\exists \delta > 0$ s.t $\forall y \in (x-\delta, x+\delta)$

$F(x) - \varepsilon < F(y) < F(x) + \varepsilon$.

We pick $r_1 < r_2 < x < s$ $\quad r_1, r_2, s \in (x-\delta, x+\delta) \cap \mathbb{Q}$

Then $F(x) - \varepsilon < F(r_1) \leq F(r_2) \leq F(x) \leq F(s) < F(x) + \varepsilon$
($\because F(\cdot)$ is monotone increasing)

① $F_{\tilde{n}(m)}(x) \leq F_{\tilde{n}(m)}(s)$

$\because$ (F_n: distribution function $\Rightarrow$ monotone-increasing)

$\therefore \limsup\limits_{m \to \infty} F_{\tilde{n}(m)}(x) \leq \lim\limits_{m \to \infty} F_{\tilde{n}(m)}(s) = G(s)$
($\because \boxed{\text{Step1}}$)

$F(s) \overset{\text{def}}{=} \inf\{G(q) \mid q > s \; q \in \mathbb{Q}\} \geq G(s)$

$\because$ $G(\cdot)$ is also monotone-increasing on $\mathbb{Q}$.
So $G(s) \le G(q)$ $\forall q \in \mathbb{Q}$ $q > s$
Hence $G(s) \le F(s)$.

$\left(\because G(q_1) = \lim_{m \to \infty} F_{\hat{n}(m)}(q_1) \le \lim_{m \to \infty} F_{\hat{n}(m)}(q_2) = G(q_2) \right.$ $(q_1 < q_2)$
$F_{\hat{n}(m)}$ is a distribution function $\Big)$

Therefore $\limsup_{m \to \infty} F_{\hat{n}(m)}(x) \le G(s) \le F(s) < F(x) + \varepsilon$

② $F_{\hat{n}(m)}(x) \ge F_{\hat{n}(m)}(r_2)$

$\therefore \liminf F_{\hat{n}(m)}(x) \ge \lim_{m \to \infty} F_{\hat{n}(m)}(r_2) = G(r_2)$.

$F(r_1) = \inf \{ G(q) \mid q > r_1,\ q \in \mathbb{Q} \} \le G(r_2)$

$\therefore \liminf F_{\hat{n}(m)}(x) \ge G(r_2) \ge F(r_1) > F(x) - \varepsilon$

①&② implies $\lim_{m \to \infty} F_{\hat{n}(m)}(x) = F(x)$ $\forall x \in C(F)$

So $F_{\hat{n}(m)} \xrightarrow{w} F$.

Now the proof is complete. ■

$\boxed{15}$

- $\{F_n\}_{n\geq 1}$ is tight means that

$\forall \varepsilon>0 \ \exists M(\varepsilon)>0$ s.t $\limsup_{n\to\infty} (1-F_n(M)+F_n(-M)) \leq \varepsilon$

($\{\mu_i\}_{i\in I}$ I ~ index-set is tight

means that $\forall \varepsilon>0 \ \exists M(\varepsilon)>0$ s.t $\sup_{i\in I} \mu_i([-M,M]^c) \leq \varepsilon$)

- Now we prove the following statement

Let $\{F_n\}_{n\geq 1}$ be a sequence of distribution functions

Then $\{F_n\}_{n\geq 1}$ is tight $\Leftrightarrow$ $\forall \{n_k\}_{k\geq 1} \subseteq \mathbb{N}$

if : $F_{n_k} \xrightarrow{w} F$ then F is also a distribution

function ⊥

(Of course, we may rewrite $\{F_n\}_{n\geq 1}$ to

probability measure $\{\mu_n\}_{n\geq 1}$)

(proof)

① ⇒

Suppose that $\{F_n\}_{n\geq 1}$ is tight.

Then $^{\forall} n_k$: sub-sequence, $\{F_{n_k}\}_{k\geq 1}$ is also tight.

because $\limsup_{k\to\infty} \left(1 - F_{n_k}(M) + F_{n_k}(-M)\right)$

$\leq \limsup_{n\to\infty} \left(1 - F_n(M) + F_n(-M)\right) \leq \varepsilon < \infty$

Let $F_{n_k} \xrightarrow{w} F$,

And we pick $x_1 < -M < M < x_2$ $\quad x_1, x_2 \in C(F)$

$\limsup_{k\to\infty} \left(1 - F_{n_k}(x_1) + F_{n_k}(x_2)\right)$

$\leq \limsup_{k\to\infty} \left(1 - F_{n_k}(M) + F_{n_k}(-M)\right) \leq \varepsilon$

$\therefore$ $1 - F(x_1) + F(x_2) \leq \varepsilon$ when x_2 is sufficiently large
x_1 is sufficiently small.

and $0 \leq F(x) \leq 1$ $\quad (x \in \mathbb{R})$

Hence $\lim_{x\to\infty} F(x) = 1$, $\lim_{x\to-\infty} F(x) = 0$ holds.

So $F(\cdot)$ is a distribution function

② $\Leftarrow$

We show that "$\{F_n\}_{n\ge1}$ is NOT tight

$\Rightarrow \exists\{n_k\}_{k\ge1}$ $F_{n_k} \xrightarrow{w} F$: F is not a distribution function"

$\{F_n\}_{n\ge1}$ is not tight means that

$\exists \varepsilon>0$ $\forall M>0$ $\limsup\limits_{n\to\infty}\left(1-F_n(M)+F_n(-M)\right)>\varepsilon$

· put $M=1$

$$\underbrace{\lim_{n\to\infty}\sup_{m\ge n}\left(1-F_m(1)+F_m(-1)\right)}_{=g_n^{(1)}}>\varepsilon$$

$g_n^{(1)}$ is decreasing so $g_1^{(1)}>\varepsilon$

$\therefore \sup\limits_{m\ge1}\left(1-F_m(1)+F_m(-1)\right)>\varepsilon$

So we may find $m_1\ge1$ st

$1-F_{m_1}(1)+F_{m_1}(-1)>\varepsilon$

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- put $M=2$

$$\lim_{n\to\infty} \underbrace{\sup_{m\geq n} \left(1-F_m(2)+F_m(-2)\right)}_{=g_n^{(2)}} > \varepsilon$$

$g_n^{(2)}$ is decreasing so $g_{m_1}^{(2)} > \varepsilon$

$\therefore \sup_{m\geq m_1} \left(1-F_m(2)+F_m(-2)\right) > \varepsilon$

We may find $m_2 > m_1$ such that

$1-F_{m_2}(2)+F_{m_2}(-2) > \varepsilon$

By repeating the similar argument we may find $\{m_k\}_{k\geq 1}$ such that $1-F_{m_k}(k)+F_{m_k}(-k) > \varepsilon$ ... (*)

for all $k\geq 1$.

$\{F_{m_k}\}_{k\geq 1}$ is also a sequence of distribution functions

And By Helly's Selection Theorem we may find

$\{m_{k_\ell}\}_{\ell\geq 1} \subset \{m_k\}_{k\geq 1}$ and F: monotone-increasing and right-continuous function. such that

$F_{m_{k_\ell}} \xrightarrow{w} F$ (i.e $F_{m_{k_\ell}}(x) \to F(x)$ as $\ell\to\infty$ $\forall x \in C(F)$)

However F is not a distribution function

It's because if we pick $x_1<0$, $x_2>0$

$(x_1, x_2 \in C(F))$

$$\lim_{\ell\to\infty} \left(1-F_{m_{k_\ell}}(x_2) + F_{m_{k_\ell}}(x_1)\right) \geq$$

$$\liminf_{\ell\to\infty} \left(1-F_{m_{k_\ell}}(k_\varepsilon) + F_{m_{k_\ell}}(-k_\varepsilon)\right) > \varepsilon \quad (\because \circledast)$$

$$\therefore 1-F(x_2)+F(x_1) > \varepsilon\ (>0)$$

So $\lim_{x_2\uparrow\infty} F(x) < 1$ or $\lim_{x_1\downarrow-\infty} F(x_1) > 0$

Hence $F(\cdot)$ is not a distribution function

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$\boxed{16}$

$$\int_{\mathbb{R}} \phi(x)\, dF_n \geq \int_{[-M,M]^c} \phi(x)\, dF_n \quad \left(\because \phi(\cdot) \geq 0,\ \mathbb{R} \supseteq [-M,M]^c \right)$$

$$\geq \int_{[-M,M]^c} \inf_{x \in [-M,M]^c} \phi(x)\, dF_n$$

$$= \inf_{|x|>M} \{\phi(x)\} \cdot \int_{[-M,M]^c} dF_n$$

$$= \inf_{|x|>M} \{\phi(x)\} \left(F_n([-M,M]^c) \right)$$

Hence $C \geq \inf_{|x|>M} \{\phi(x)\} \cdot F_n([-M,M]^c)$

$$\therefore \frac{C}{\inf_{|x|>M} \{\phi(x)\}} \geq F_n([-M,M]^c)$$

Take sufficiently large $M>0$ $\inf_{|x|>M} \{\phi(x)\} \uparrow \infty$

So $\forall \varepsilon>0$ $\exists M_\varepsilon$ s.t. $F_n([-M,M]^c) \leq \varepsilon$

$\therefore \limsup F_n([-M,M]^c) \leq \varepsilon$

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$\boxed{17}$

Consider $X_{m,k} \sim \text{Uniform}\left(\left(0, \frac{k}{m}\right) \cup \left(\frac{k+1}{m}, \frac{m+1}{m}\right) \right)$

$m = 1, 2, 3 \ldots \quad k = 0, 1 \ldots m-1$

And define $\{X_n\}_{n \geq 1} = \{X_{1,0}, X_{2,0}, X_{2,1}, X_{3,0}, X_{3,1}, X_{3,2} \cdots\}$

First, we show that $X_n \xrightarrow{d} \text{Uniform}(0,1)$.

Consider $P(X_{m,k} \leq x)$ $\left(\overset{\text{def}}{=} F_{m,k}(x)\right)$

$$F_{m,k}(x) = \begin{cases} x & x \in \left[0, \frac{k}{m}\right) \\ \frac{k}{m} & x \in \left[\frac{k}{m}, \frac{k+1}{m}\right] \\ x - \frac{1}{m} & x \in \left(\frac{k+1}{m}, \frac{m+1}{m}\right) \\ 1 & x \in \left[\frac{m+1}{m}, \infty\right) \end{cases}$$

And define

$$F(x) = \begin{cases} x & x \in [0, 1) \\ 1 & x \in [1, \infty) \end{cases}$$

$(F(x), F_{m,k}(x) = 0 \text{ if } x < 0)$

Then $|F_{m,k}(x) - F(x)| \leq \frac{1}{m}$

$\therefore \max_{k=0\sim m} |F_{m,k}(x) - F(x)| \leq \frac{1}{m}$

$\therefore \lim_{m\to\infty} \max_{k=0\sim m} |F_{m,k}(x) - F(x)| = 0$

So this implies that $F_n(x) \to F(x)$ $(x \in \mathbb{R})$

$\therefore F_n(x) \xrightarrow{w} F(x)$. $(\therefore X_n \xrightarrow{d} X)$

However we consider $f_{m,k}(x)$ (pdf)

$$f_{m,k}(x) = \begin{cases} 1 & x \in (0, \frac{k}{m}) \\ 0 & x \in [\frac{k}{m}, \frac{k+1}{m}] \\ 1 & x \in (\frac{k+1}{m}, \frac{m+1}{m}) \end{cases}$$

For any $^{\forall} x \in [0,1]$ and any $m \geq 1$

we may find $k = 0 \sim m-1$ such that $x \in [\frac{k}{m}, \frac{k+1}{m}]$.

$\Rightarrow f_{m,k}(x) = 0$.

Hence $^{\forall} x \in [0,1]$ there are infinitly many $n \geq 1$

s.t $f_n(x) = 0$. So $\liminf_{n\to\infty} f_n(x) = 0$

Therefore $f_n(x) \nrightarrow 1$ for any $x \in [0,1]$

$\boxed{18}$

(1) $y > 0$.

$P(M_n / n^{\frac{1}{\alpha}} \leq y) = P(M_n \leq y \cdot n^{\frac{1}{\alpha}})$

$= (1 - (y \cdot n^{\frac{1}{\alpha}})^{-\alpha})^n = (1 - y^{-\alpha} \cdot \frac{1}{n})^n$

$= (1 - \frac{1}{n y^{\alpha}})^n = \left\{ (1 - \frac{1}{n y^{\alpha}})^{(-n y^{\alpha})} \right\}^{\left(\frac{-1}{y^{\alpha}}\right)}$

$\rightarrow \exp(-y^{-\alpha})$ (as $n \to \infty$)

(2) $y < 0$

$P(n^{\frac{1}{\beta}} \cdot M_n \leq y) = P(M_n \leq y \cdot n^{-\frac{1}{\beta}})$ $(*_2)$

$= (1 - |y \cdot n^{-\frac{1}{\beta}}|^{\beta})^n = (1 - \frac{1}{n}|y|^{\beta})^n$

$= \left\{ (1 - \frac{1}{n}|y|^{\beta})^{\frac{-n}{|y|^{\beta}}} \right\}^{(-|y|^{\beta})}$

$\rightarrow \exp(-|y|^{\beta})$

$(*_2)$ When n is suffiently large, $y \cdot n^{-\frac{1}{\beta}} \in [-1, 0]$

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(3) $P(M_n - \log n \leq y)$

$= P(M_n \leq y + \log n) = \{1 - \exp(-(y+\log n))\}^n$ (※3)

(※3 when n is suffiently large $y + \log n \geq 0$)

$= \left(1 - \frac{1}{n}\exp(-y)\right)^n = \left\{\left(1 - \frac{1}{n}\exp(-y)\right)^{(-n\exp(y))}\right\}^{(-\exp(-y))}$

$\to \exp(-\exp(-y))$

So the proof is complete. ■

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$\boxed{19}$ Since $E[g(X_n)]$, $E[g(X)]$ only depend on the distributions of X_n and X, without loss of generality, we may assume that $X_n \xrightarrow{as} X$ by Skorokhod's representation theorem.

Therefore $E[\liminf g(X_n)] \leq \liminf E[g(X_n)]$

by Fatou's lemma and $\liminf g(X_n) = g(X)$ (a.s)

$\because$ $g(\cdot)$ is continuous and $X_n \to X$ (a.s)

So we have $E[g(X)] \leq \liminf E[g(X_n)]$

$\boxed{20}$ Suppose $X_n \sim F_n$ and $X \sim F$.

Since $\int g\,dF_n = E[g(X_n)]$, $\int h\,dF_n = E[h(X_n)]$

$\int h\,dF = E[h(X)]$ only depend on the distributions

of $\{X_n\}_{n \geq 1} \cup \{X\}$, without loss of generality

we may suppose that $X_n \to X$ (a.s.)

Now we may apply the result of $\boxed{28}$ in chapter 1

And we have the desired result.

$\boxed{21}$ $F_n \xrightarrow{w} F$ and $F(\cdot)$ is continuous

then we have $\sup_{x \in \mathbb{R}} |F_n(x) - F(x)| \to 0.$

(proof) Let $\varepsilon > 0$ be a positive number $(\varepsilon \ll 1)$

Now we take $m \in \mathbb{N}$ s.t. $\frac{1}{m} < \frac{\varepsilon}{2}$.

$x_k^{(m)} \in F^{-1}\left(\frac{k}{m}\right)$ $(k = 1, 2, 3 \cdots m-1)$

(Since $F(\cdot)$ is continuous, we may find such $x_k^{(m)}$)

$x_0^{(m)} \overset{\text{def}}{=} -\infty$ $x_m^{(m)} = +\infty.$

- If $x \in \{x_1^{(m)}, x_2^{(m)}, \cdots x_{m-1}^{(m)}\}$,

By taking large $n > N^{(\varepsilon)}$

$|F_n(x) - F(x)| < \frac{\varepsilon}{2}$

- If $x \in \mathbb{R} \setminus \{x_1^{(m)}, \cdots x_{m-1}^{(m)}\}$

there exists $k = 0 \sim m-1$ s.t.

$x \in (x_k^{(m)}, x_{k+1}^{(m)})$

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($F(\cdot)$ is monotone increasing)

$F_n(x) - F(x) \leq F_n(x_{k+1}^{(m)}) - F(x_k^{(m)})$

$= F_n(x_{k+1}^{(m)}) - \frac{k}{m}$ (by definition of $x_k^{(m)}$)

$= F_n(x_{k+1}^{(m)}) - \frac{k+1}{m} + \frac{1}{m}$

$= F_n(x_{k+1}^{(m)}) - F(x_k^{(m)}) + \frac{1}{m}$

$\leq |F_n(x_{k+1}^{(m)}) - F(x_k^{(m)})| + \frac{1}{m}$ ---- $(*)$

If we take $n > N^{(\varepsilon)}$

$|F_n(x_{k+1}^{(m)}) - F(x_k^{(m)})| < \frac{\varepsilon}{2}$

So $(*) < \frac{\varepsilon}{2} + \frac{1}{m} < \varepsilon$

Similarly $F(x) - F_n(x) \leq F(x_{k+1}^{(m)}) - F_n(x_k^{(m)})$

$= \left(\frac{1}{m} + \frac{k}{m}\right) - F_n(x_k^{(m)})$

$= \frac{1}{m} + F(x_k^{(m)}) - F_n(x_k^{(m)}) \leq \frac{1}{m} + |F(x_k^{(m)}) - F_n(x_k^{(m)})|$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

- Therefore $\sup_{x \in \mathbb{R}} |F_n(x) - F(x)| < \varepsilon$ $(\forall n > N^{(\varepsilon)})$

This implies $\lim_{n \to \infty} \sup_{x \in \mathbb{R}} |F_n(x) - F(x)| = 0$

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$\boxed{22}$. Glivenko–Cantelli's lemma

- $\hat{F}_n(x) \overset{\text{def}}{=} \frac{1}{n}\sum_{j=1}^{n} \mathbb{I}\{X_j \le x\}$

- $F(x) = P(X_n \le x)$ $(X_1, X_2 \cdots X_n \sim \text{iid})$

Then $\sup_{x\in\mathbb{R}} |\hat{F}_n(x) - F(x)| \to 0$ (as)

(proof)

$\tilde{\hat{F}}_n(x) \overset{\text{def}}{=} \frac{1}{n}\sum_{j=1}^{n} \mathbb{I}\{X_j < x\}$

$x_k^{(m)} \overset{\text{def}}{=} \inf\{x \mid F(x) \ge \frac{k}{m}\}$

$(m \in \mathbb{N} \quad k = 1, 2, \cdots m-1)$

Let $\varepsilon > 0$ be an arbitrary positive number.

We fix m s.t. $\frac{1}{m} < \frac{\varepsilon}{2}$. $(m \in \mathbb{N})$

For each $k = 1, 2, \cdots m-1$

$\hat{F}_n(x_k^{(m)}) \to F(x_k^{(m)})$ (as)

$\tilde{\hat{F}}_n(x_k^{(m)}) \to F(x_k^{(m)}-)$ (as)

(* $F(x-) = P(X < x)$)

($\because$ by Strong Law of Large Numbers)

Hence $\exists \Omega_k^{(m)} \in \mathcal{F} : P(\Omega_k^{(m)}) = 1$

$\forall \omega \in \Omega_k^{(m)}$ $\exists N_k(\omega)$ s.t. $|\hat{F}_n(x_k^{(m)})^{(\omega)} - F(x_k^{(m)})| < \frac{\varepsilon}{2}$

and $|\tilde{F}_n(x_k^{(m)})^{(\omega)} - F(x_k^{(m)}-)| < \frac{\varepsilon}{2}$

for all $n > N_k(\omega)$.

Therefore let $\Omega^{(m)} = \bigcap_{k=1}^{m-1} \Omega_k^{(m)} \in \mathcal{F}$; $P(\Omega^{(m)}) = 1$

$\forall \omega \in \Omega^{(m)}$ $\forall k = 1 \sim m-1$ $\exists N(\omega)$ $(= \max\{N_1(\omega), \dots, N_{m-1}(\omega)\})$

s.t. $|\hat{F}_n(x_k^{(m)})^{(\omega)} - F(x_k^{(m)})| < \frac{\varepsilon}{2}$ and

$|\tilde{F}_n(x_k^{(m)})^{(\omega)} - F(x_k^{(m)}-)| < \frac{\varepsilon}{2}$

for all $n > N(\omega)$.

For convenience we define $x_0^{(m)} = -\infty$
$x_m^{(m)} = \infty$.

So the statement above "$\forall k = 1 \sim m-1$"

can be written as "$\forall k = 0 \sim m$".

Now we pick $x \in \mathbb{R}$. (and suppose $\omega \in \Omega^{(m)}$)

Then we may find $k = 0 \sim m-1$ st

$x \in (x_k^{(m)}, x_{k+1}^{(m)})$ (if $x \notin \{x_1^{(m)}, \cdots x_{m-1}^{(m)}\}$)

① $\hat{F_n}(x) - F(x) \leq \hat{F_n}(x_{k+1}^{(m)}) - F(x_k^{(m)})$

$\quad \downarrow$ (※1)

$\leq \hat{F_n}(x_{k+1}^{(m)}) - \frac{k}{m}$

$= \hat{F_n}(x_{k+1}^{(m)}) - \frac{k+1}{m} + \frac{1}{m}$

$\quad \downarrow$ (※2)

$\leq \hat{F_n}(x_{k+1}^{(m)}) - F(x_{k+1}^{(m)}-) + \frac{1}{m}$

$\leq |\hat{F_n}(x_{k+1}^{(m)}) - F(x_{k+1}^{(m)}-)| + \frac{1}{m}$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ (if $n > N(\omega)$)

NOTE

(※1) ... See the definition of $x_k^{(m)}$ and be careful of the right-continuity of $F(\cdot)$.

$F(x_k^{(m)}) \stackrel{\text{def}}{=} \inf\{x \mid F(x) \geq \frac{k}{m}\}$

We may find $\{x_{k,l}^{(m)}\}_{l \geq 1}$ $x_{k,l}^{(m)} \downarrow x_k^{(m)}$

where $\{x_{k,l}^{(m)}\} \subseteq \{x \mid F(x) \geq \frac{k}{m}\}$

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$\lim_{\ell \to \infty} F(x_{k,\ell}^{(m)}) = F(x_k^{(m)}) \geq \frac{k}{m}$ $\quad (\because F(x_{k,\ell}^{(m)}) \geq \frac{k}{m})$

$F(\cdot)$ is right-continuous

② $F(x_{k+1}^{(m)}-) \leq \frac{k+1}{m}$

Suppose that $F(x_{k+1}^{(m)}-) > \frac{k+1}{m}$.

Let $\{x_{k+1,\ell}^{(m)}\}_{\ell \geq 1}$: $x_{k+1,\ell}^{(m)} \nearrow x_{k+1}^{(m)}$ $\quad (x_{k+1,\ell}^{(m)} < x_{k+1}^{(m)})$

$\lim_{\ell \to \infty} F(x_{k+1,\ell}^{(m)}) = F(x_{k+1}^{(m)}-) > \frac{k+1}{m}$

So there exists suffiently large $\ell > L$

s.t. $F(x_{k+1,\ell}^{(m)}) > \frac{k+1}{m}$ $\quad (x_{k+1,\ell}^{(m)} < x_{k+1}^{(m)})$

Hence $x_{k+1,\ell}^{(m)} \in \{x \mid F(x) \geq \frac{k+1}{m}\}$

but $x_{k+1,\ell}^{(m)} < x_{k+1}^{(m)} \overset{def}{=} \inf\{x \mid F(x) \geq \frac{k+1}{m}\}$

$\therefore$ Contradiction

Thus $F(x_{k+1}^{(m)}-) \leq \frac{k+1}{m}$

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Similarly,

② $F(x) - \hat{F}_n(x) \leq F(x_{k+1}^{(m)}-) - \hat{F}_n(x_k^{(m)})$

$\leq \frac{k+1}{m} - \hat{F}_n(x_k^{(m)})$ (similar to ①)

$= \frac{1}{m} + \frac{k}{m} - \hat{F}_n(x_k^{(m)})$

$= \frac{1}{m} + F(x_k^{(m)}) - \hat{F}_n(x_k^{(m)})$

$\leq \frac{1}{m} + |F(x_k^{(m)}) - \hat{F}_n(x_k^{(m)})|$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ (if $n > N(\omega)$)

Therefore, $\forall \varepsilon > 0$

$\exists\, \Omega^{(m(\varepsilon))} \in \mathcal{F} : P(\Omega^{(m)}) = 1$ such that

$\forall \omega \in \Omega^{(m)}$ $\exists N(\omega)$ s.t. $\forall n > N(\omega)$

$\sup_{x \in \mathbb{R}} |\hat{F}_n(x) - F(x)| < \varepsilon.$

This implies $\lim_{n \to \infty} \sup_{x \in \mathbb{R}} |\hat{F}_n(x) - F(x)| = 0.$ ■

$\boxed{23}$

① $X_n \xrightarrow{p} X \Rightarrow X_n \xrightarrow{d} X$

The easiest way to prove this is to use

Bounded Convergence Theorem and 「$X_n \xrightarrow{d} X$」 $\Leftrightarrow$

「$\forall g$: continuous and bounded $E[g(X_n)] \to E[g(X)]$」

So $X_n \xrightarrow{p} X \overset{\circledast}{\Rightarrow} g(X_n) \xrightarrow{p} g(X) \Rightarrow E[g(X_n)] \to E[g(X)]$

(Bounded Convergence Theorem)

$\Rightarrow X_n \xrightarrow{d} X$

⊛ $X_n \xrightarrow{p} X \Leftrightarrow$ 「$\forall \{n_k\}_{k\geq 1} \subseteq \mathbb{N}$

$\exists \{n_{k_\ell}\}_{\ell\geq 1}$: $X_{n_{k_\ell}} \to X$ (a.s)」

$\Rightarrow$ 「$\forall \{n_k\}_{k\geq 1} \subseteq \mathbb{N}$ $\exists \{n_{k_\ell}\}_{\ell\geq 1}$: $g(X_{n_{k_\ell}}) \to g(X)$ (a.s)」

(g : continuous)

$\Leftrightarrow g(X_n) \xrightarrow{p} g(X)$

② $X_n \xrightarrow{d} C \Rightarrow X_n \xrightarrow{P} C$

「$X_n \xrightarrow{d} C$」 $\Leftrightarrow$ 「$\liminf P(X_n \in G) \geq P(C \in G)$

for all G : open-set」

Let $G = (C-\varepsilon, C+\varepsilon)$.

Then $\liminf P(X_n \in (C-\varepsilon, C+\varepsilon))$

$= \liminf P(|X_n - C| < \varepsilon) \geq 1$

$\lim_{n\to\infty} P(|X_n - C| < \varepsilon) = 1$

$(\therefore \lim P(|X_n - C| \geq \varepsilon) = 0)$

$\therefore X_n \xrightarrow{P} C$

$\left(\begin{array}{l} \circledast \ X_n+Y_n \le x, \ C-Y_n \le \varepsilon \Rightarrow X_n + C \le x+\varepsilon \\ A \subset B \Rightarrow P(A) \le P(B) \end{array} \right.$

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$\boxed{\text{ex}}$ $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{d} C$

① $P(X_n+Y_n \le x) = P(X_n+Y_n \le x, \ |Y_n - C| > \varepsilon) + P(X_n+Y_n \le x, \ |Y_n - C| \le \varepsilon)$

$\le P(|Y_n - C| > \varepsilon) + P(X_n + C \le x+\varepsilon)$ $\downarrow \circledast$

$= P(|Y_n - C| > \varepsilon) + P(X_n \le x - C + \varepsilon)$

We may infinitely small $\varepsilon > 0$ so that

$x - C + \varepsilon \in C(X)$.

$\limsup P(X_n+Y_n \le x) \le \lim_{n\to\infty} \{P(|Y_n - C| > \varepsilon) + P(X_n \le x - C+\varepsilon)\}$

$= P(X \le x - C + \varepsilon)$ $\quad (X_n \xrightarrow{d} X)$

And $\varepsilon \downarrow 0$, we have $\limsup P(X_n+Y_n \le x) \le P(X \le x - C)$

$= P(X + C \le x)$

② $P(X_n + C \le x - \varepsilon) = P(X_n + C \le x - \varepsilon, \ |Y_n - C| > \varepsilon) + P(X_n + C \le x - \varepsilon, \ |Y_n - C| \le \varepsilon)$

$\le P(|Y_n - C| > \varepsilon) + P(X_n + Y_n \le x)$

Similarly we may find infinitely small $\varepsilon > 0$

st $x - C - \varepsilon \in C(X)$

$\therefore \lim P(X_n + c \le x - \varepsilon) \le \liminf \{ P(|Y_n - c| > \varepsilon) + P(X_n + Y_n \le x) \}$

$\therefore P(X \le x - c - \varepsilon) \le \liminf P(X_n + Y_n \le x)$

$\quad \parallel$

$P(X + c \le x - \varepsilon)$

if $x \in C(X+c)$ then $\varepsilon \downarrow 0$ we have

$P(X + c \le x) \le \liminf P(X_n + Y_n \le x)$

So $\lim\limits_{n \to \infty} P(X_n + Y_n \le x) = P(X + c \le x)$

for all $x \in C(X+c)$.

Hence $X_n + Y_n \xrightarrow{d} X + c$

$\boxed{25}$ In the previous question we have found that

「 $X_n - Z_n \xrightarrow{P} 0$, $X_n \xrightarrow{d} X$ then $Z_n \xrightarrow{d} X$. 」

So in this question, it's enough to show that

$X_n Y_n - CX_n \xrightarrow{P} 0$ because $CX_n \xrightarrow{d} CX$.

(proof)

$P(|X_n Y_n - CX_n| > \varepsilon) = P(|X_n||Y_n - C| > \varepsilon)$

$= P(|X_n||Y_n - C| > \varepsilon, \ |X_n| > M)$

$+ P(|X_n||Y_n - C| > \varepsilon, \ |X_n| \leqq M)$

$\leqq P(|X_n| > M) + P(|Y_n - C| > \frac{\varepsilon}{M})$

Suppose $M \in C(X)$.

And we may take infinitely large $M > 0$

($\because$ Points of discontinuity are only countable)

$n \nearrow \infty$ $\limsup_{n \to \infty} P(|X_n Y_n - CX_n| > \varepsilon) \leqq P(|X| > M)$.

And $M \nearrow \infty$, we have $\limsup_{n \to \infty} P(|X_n Y_n - CX_n| > \varepsilon) =$

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$\boxed{26}$ Let $g(x) = |x|^{\beta}$ $h(x) = |x|^{\alpha}$

Then $\lim_{|x| \to \infty} \frac{h(x)}{g(x)} = 0.$

$E[Y_n^{\beta}] = E[|Y_n|^{\beta}] = E[g(Y_n)]$

$(\because Y_n \geq 0)$

Since $\lim_{n \to \infty} E[g(Y_n)] \to 1$

hence for sufficiently large $n > N$

$E[g(Y_n)] \leq M < \infty$

By $\boxed{16}$ $\phi(x) \stackrel{def}{=} g(x)$

$\sup_{n > N} \int \phi(x) \, dF_n = \sup_{n > N} E[g(Y_n)] < \infty$

$(Y_n \sim F_n)$

And $\phi(x) \to \infty$ as $|x| \to \infty$

Hence $\{Y_n\}_{n > N}$ is tight.

$\Rightarrow \{Y_n\}_{n \geq 1}$ tight

(We may ignore finite items)

Let $\{n_k\}_{k\ge1} \subseteq \mathbb{N}$ an arbitrary subsequence

$\{Y_{n_k}\}_{k\ge1}$: is also tight

By $\boxed{I5}$, we may find a subsub-sequence $\{n_{k_\ell}\}_{\ell\ge1}$ and Y

$Y_{n_{k_\ell}} \xrightarrow{d} Y$. ($Y_{n_k} \sim F_{n_k}$ $Y \sim F$, F_{n_k}, F: df)

$\Leftrightarrow$ 「$\forall f$: an arbitrary bounded continuous function

$E[f(Y_{n_{k_\ell}})] \to E[f(Y)]$」

So 「$\forall \{n_k\}_{k\ge1} \subseteq \mathbb{N}$ $\exists \{n_{k_\ell}\}_{\ell\ge1}$ s.t.

$E[f(Y_{n_{k_\ell}})] \to E[f(Y)]$」

implies 「$E[f(Y_n)] \to E[f(Y)]$」 (see chapter 2)

Therefore $Y_n \xrightarrow{d} Y$. ($\because$ $\boxed{II}$)

Now $E[Y_n^{\beta}]$, $E[Y_n^{\alpha}]$ only depend

on the distribution of $\{Y_n\}_{n\ge1}$

Without loss of generality, we may suppose that $Y_n \xrightarrow{a.s} Y$. (By $\boxed{10}$)

We have
- $Y_n \to Y$ (a.s)
- $g(x), h(x)$. continuous
- $g(x) \to \infty$ as $|x| \to \infty$.
- $\frac{h(x)}{g(x)} \to 0$ as $|x| \to \infty$
- $\sup_{n \geq N} E[g(Y_n)] < \infty$

(without loss of generality we may assume $\sup_{n \geq 1} E[g(Y_n)] < \infty$)

By $\boxed{28}$ in Chapter 1

We have $E[h(Y_n)] \to E[h(Y)]$.
$= 1$ (by assumption)

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By Fatou's lemma

$$E[\liminf g(Y_n)] \leq \liminf E[g(Y_n)] = 1$$

$$\overset{\parallel}{E[g(Y)]} \qquad (\therefore E[g(Y)] \leq 1)$$

However, by Jensen's inequality, $(\varphi(x) = |x|^{\frac{\beta}{\alpha}})$

$$E[h(Y)^{\frac{\beta}{\alpha}}] \geq E[h(Y)]^{\frac{\beta}{\alpha}} = 1$$

$$\overset{\parallel}{E[g(Y)]} \; (\leq 1)$$

Hence we have $E[h(Y)^{\frac{\beta}{\alpha}}] = E[h(Y)]^{\frac{\beta}{\alpha}} = 1$

Equality holds when $h(Y) = 1$ (a.s)

(See Jensen's inequality)

$|Y|^{\alpha} = 1$ (a.s) $\Rightarrow$ $Y = 1$ (a.s) $\quad (\because Y \geq 0)$

$\therefore Y_n \xrightarrow{d} Y = 1$, ($\boxed{22}$)

$\therefore Y_n \xrightarrow{P} Y = 1$

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Inversion formula

$$2\pi\left\{\mu((a,b))+\frac{1}{2}\mu(\{a,b\})\right\}$$
$$=\lim_{T\to\infty}\int_{-T}^{T}\frac{1}{it}\left(e^{-iat}-e^{-ibt}\right)\cdot\phi(t)\,dt$$

where μ is a probability measure

and $\phi(t)=\int_{\mathbb{R}}e^{itx}\mu(dx)$

(proof)

$$\frac{1}{it}\left(e^{-iat}-e^{-itb}\right)=\int_a^b e^{-ity}\,dy$$

So $\left|\frac{1}{it}\left(e^{-iat}-e^{-itb}\right)\right|\le\int_a^b\left|e^{-ity}\right|dy=b-a$

$\left|e^{-ity}\right| = 1$

Hence $\int_{-T}^{T}\frac{1}{it}\left(e^{-ita}-e^{-itb}\right)\phi(t)\,dt$

$$= \int_{-T}^{T}\int_{\mathbb{R}} \frac{1}{it}\left(e^{-ita}-e^{-itb}\right) e^{itx}\,\mu(dx)\,dt$$

$$\left|\int_{-T}^{T}\int_{R}(\sim)\,\mu(dx)\,dt\right| \leq \int_{-T}^{T}\int_{R} \underbrace{|\cdot|}_{(b-a)}\,\mu(dx)\,dt$$

$$= \int_{-T}^{T}(b-a)\,dt \qquad (\because \mu(\mathbb{R})=1)$$

$$= (b-a)\cdot 2T < \infty$$

Hence the integral is finite.

So by Fubini's theorem we may swap $\int_{-T}^{T}$, $\int_{\mathbb{R}}$.

$$\therefore = \int_{\mathbb{R}} \int_{-T}^{T} \frac{1}{it}\left(e^{-ita}-e^{-itb}\right) e^{itx}\,dt\,\mu(dx)$$

$$= \int_{R}\int_{-T}^{T} \frac{1}{it}\left(e^{it(x-a)}-e^{it(x-b)}\right) dt\,\mu(dx)$$

$$\left(\begin{array}{l} e^{it(x-a)} = \cos(x-a)t + i\sin(x-a)t \\ \frac{\cos(x-a)t}{t} \text{ is an odd function} \\ \text{So } \int_{-T}^{T} \frac{\cos(x-a)t}{t}\,dt = 0 \end{array}\right)$$

$$= \int_{\mathbb{R}} 2\int_{0}^{T}\left(\frac{\sin(x-a)t}{t} - \frac{\sin(x-b)t}{t}\right) dt$$

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$$= 2\int_{R}\int_0^T \left(\frac{\sin(x-a)t}{t} - \frac{\sin(x-b)t}{t}\right) dt\, d\mu$$

Next we define $S(T) = \int_0^T \frac{\sin t}{t} dt$.

Then it's easy to verify that

$$\int_0^T \frac{\sin\theta t}{t} dt = \operatorname{sgn}(\theta) \cdot S(|\theta| T)$$

Thus $= 2\int_{\mathbb{R}} \{ \operatorname{sgn}(x-a) S(|x-a|T) - \operatorname{sgn}(x-b) S(|x-b|T) \} d\mu$

Consider $| \operatorname{sgn}(x-a) S(|x-a|T) - \operatorname{sgn}(x-b) S(|x-b|T) |$

$\leq 2 \sup_{y>0} S(y) < \infty$ (see $\boxed{28}$)

($\rightarrow$ not related to T and integrable)

It's μ-integrable.

Therefore by Bounded Convergence theorem (or L.D.C.T)

$$\lim_{T\to\infty} 2\int_{R} \{ \operatorname{sgn}(x-a) S(|x-a|T) - \operatorname{sgn}(x-b) S(|x-b|T) \} d\mu$$

$$= 2\int_{R} \lim_{T\to\infty} (\operatorname{sgn}(x-a) S(|x-a|T) - \operatorname{sgn}(x-b) S(|x-b|T)) d\mu$$

$= (*)$

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$$= \lim_{T\to\infty} \left(\operatorname{sgn}(x-a)\, S(|x-a|T) - \operatorname{sgn}(x-b)\, S(|x-b|T) \right)$$

$$\downarrow (*_2)$$

$$= \frac{\pi}{2} \left(\operatorname{sgn}(x-a) - \operatorname{sgn}(x-b) \right)$$

$$= \begin{cases} x<a \cdots & 0 \\ x=a \cdots & \frac{\pi}{2} \\ a<x<b \cdots & \pi \\ x=b \cdots & \frac{\pi}{2} \\ x>b \cdots & 0 \end{cases}$$

$(*_2)$ $\lim_{T\to\infty} \int_0^T \frac{\sin t}{t}\, dt = \lim_{T\to\infty} S(T) = \frac{\pi}{2}$
(see $\boxed{28}$)

$$= \frac{\pi}{2}\, \mathbb{I}_{\{a,b\}}(x) + \pi \cdot \mathbb{I}_{(a,b)}(x).$$

So

$$(*_1) = 2\int_{\mathbb{R}} \left(\frac{\pi}{2} \cdot \mathbb{I}_{\{a,b\}} + \pi\, \mathbb{I}_{(a,b)} \right)$$

$$= \pi\, \mu(\{a,b\}) + 2\pi\, \mu((a,b))$$

So we have the desired result.

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① $\left|\int_0^T \frac{\sin x}{x} - \frac{\pi}{2}\right| \leq \frac{T+1}{T^2}$

$\int_0^T \frac{\sin x}{x}dx = \int_0^T \sin x \int_0^\infty e^{-xy}dy\,dx$

$= \int_0^T\int_0^\infty \sin x\, e^{-xy}\,dy\,dx$

$|\sin x \cdot e^{-xy}| \leq x e^{-xy}$

$\int_0^T\int_0^\infty x e^{-xy}\,dy\,dx = \int_0^T dx = T < \infty$

Hence by Fubini's theorem.

$\int_0^T\int_0^\infty \sin x\, e^{-xy}\,dy\,dx$

$= \int_0^\infty\int_0^T \sin x\, e^{-xy}\,dx\,dy$ ... ⊛

(We may swap $\int_0^\infty$ and $\int_0^T$)

$$\int_0^T \sin x \cdot e^{-xy}\,dx = \left[\frac{-1}{1+y^2}(y\sin x+\cos x)e^{-xy}\right]_0^T$$

$$\left(\begin{aligned} \because\ &(\sin x\, e^{-xy})' = \cos x \cdot e^{-xy} - y\sin x\, e^{-xy} \\ &(\cos x\, e^{-xy})' = -\sin x\, e^{-xy} - y\cos x\, e^{-xy} \\ \Rightarrow\ &(y\sin x\, e^{-xy} + \cos x\, e^{-xy})' = -(1+y^2)\sin x\, e^{-xy} \end{aligned}\right.$$

$$\left((\sim)' = \frac{\partial}{\partial x}(\sim) \right)$$

$$= \frac{-1}{1+y^2}(y\sin T+\cos T)e^{-Ty} + \frac{1}{1+y^2}$$

Hence $\circledast = \int_0^\infty \left(\frac{1}{1+y^2} - \frac{1}{1+y^2}(y\sin T+\cos T)e^{-Ty}\right)dy$

$$= \frac{\pi}{2} - \sin T\int_0^\infty \frac{y}{1+y^2}e^{-Ty}\,dy - \cos T\int_0^\infty \frac{e^{-Ty}}{1+y^2}\,dy$$

$$\therefore \left|\int_0^T \frac{\sin x}{x} - \frac{\pi}{2}\right| \leq |\sin T|\int_0^\infty \frac{y}{1+y^2}e^{-Ty}\,dy + |\cos T|\int_0^\infty \frac{e^{-Ty}}{1+y^2}\,dy$$

$$\leq \int_0^\infty y e^{-Ty}\,dy + \int_0^\infty e^{-Ty}\,dy$$

$$= \frac{1}{T^2} + \frac{1}{T} = \frac{T+1}{T^2}$$

② $S(T) \stackrel{\text{def}}{=} \int_0^T \frac{\sin x}{x} dx$

We show that $\sup_{y>0} S(y) < \infty$

$\left| S(T) - \frac{\pi}{2} \right| \leq \frac{T+1}{T^2}$

So $|S(T)| \leq \left| S(T) - \frac{\pi}{2} \right| + \left| \frac{\pi}{2} \right|$

$\leq \frac{T+1}{T^2} + \frac{\pi}{2}$

$\sup_{T \geq 1} |S(T)| \leq 2 + \frac{\pi}{2}$

$\left(\because g(T) = \frac{T+1}{T^2} \right.$ is decreasing on $\left. (0, \infty) \right)$

consider $T \in (0,1)$

$\left| \int_0^T \frac{\sin x}{x} dx \right| \leq \int_0^T \left| \frac{\sin x}{x} \right| dx \leq \int_0^1 dx = 1$

$\therefore \sup_{T \in (0,1)} |S(T)| \leq 1$

$\therefore \sup_{T > 0} |S(T)| < \infty$

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$\boxed{29}$ $f(y) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-ity}\phi(t)\,dy$

if $\int_{\mathbb{R}} |\phi(t)|\,dt < \infty$

(proof)

We use inversion formula.

$\mu((a,b)) + \frac{1}{2}\mu(\{a,b\}) = \lim_{T\to\infty} \frac{1}{2\pi}\int_{-T}^{T} \frac{1}{it}(e^{-ita} - e^{-itb})\,\phi(t)\,dt$

Next, $\int_{-T}^{T} \frac{1}{it}(e^{-ita} - e^{-itb})\,\phi(t)\,dt$

$= \int_{-\infty}^{\infty} \frac{1}{it}(e^{-ita} - e^{-itb})\,\phi(t)\,\mathbb{I}_{(-T,T)}(t)\,dt$

Consider $\left|\frac{1}{it}(e^{-ita} - e^{-itb})\,\phi(t)\,\mathbb{I}_{(-T,T)}(t)\right| dt$

$\le (b-a)\,|\phi(t)|$ (by $\boxed{27}$ $\left|\frac{1}{it}(e^{-ita} - e^{-itb})\right| \le (b-a)$)

$\in L^1$ (integrable and not related to T)

By Lebesgue Dominated Convergence Theorem

$\lim_{T\to\infty} \frac{1}{2\pi}\int_{-T}^{T} \frac{1}{it}(e^{-ita} - e^{-itb})\,\phi(t)\,dt$

$= \lim_{T\to\infty} \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{it}(e^{-ita} - e^{-itb})\,\phi(t)\cdot\mathbb{I}_{(-T,T)}(t)\,dt$) L.D.C.T.

$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \lim_{T\to\infty}(\cdots)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{it}(e^{-ita} - e^{-itb})\,\phi(t)\,dt$

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$$\mu(\{a,b\})/2 + \mu((a,b))$$

Moreover $\quad \|$

$$\left| \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{it} \left(e^{-ita} - e^{-itb} \right) \phi(t)\, dt \right|$$

$\downarrow$ triangle inequality

$$\leq \frac{1}{2\pi} \int_{-\infty}^{\infty} (b-a) |\phi(t)|\, dt$$

$$= (b-a) \cdot \underbrace{\frac{1}{2\pi} \int_{-\infty}^{\infty} |\phi(t)|\, dt}_{<\infty}$$

By taking $b \downarrow a$,

$$\lim_{b \downarrow a} \left(\mu(\{a\})/2 + \mu((a,b)) \right) \left(= \mu(\{a\}) \right)$$

$$\leq \liminf_{b \downarrow a} \left\{ \mu(\{a,b\})/2 + \mu((a,b)) \right\}$$

$$\leq \lim_{b \downarrow a} (b-a) \frac{1}{2\pi} \int_{-\infty}^{\infty} |\phi(t)|\, dt = 0$$

Hence $\mu(\{a\}) = 0$

So $\forall a \in \mathbb{R} \quad \mu(\{a\}) = 0$

So we find out that there is not a point of discontinuity for μ

$$\therefore \mu((a,b)) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{it}\left(e^{-ita} - e^{-itb}\right)\phi(t)\,dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{a}^{b} e^{-ity}\,dy\ \phi(t)\,dt$$

Fubini's Theorem ⊛

$$= \frac{1}{2\pi}\int_{a}^{b}\int_{-\infty}^{\infty} e^{-ity}\phi(t)\,dt\,dy$$

(⊛ recall that $\left|e^{-ity}\phi(t)\right| \le |\phi(t)|$

$$\therefore \int_{-\infty}^{\infty}\int_{a}^{b} |\phi(t)|\,dy\,dt = (b-a)\int_{-\infty}^{\infty}|\phi(t)|\,dt < \infty$$

Hence we may apply Fubini's theorem)

$$\therefore \mu((a,b)) = \int_{a}^{b} \underbrace{\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ity}\phi(t)\,dt\right)}_{= f(y)} dy$$

$$\therefore f(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ity}\phi(t)\,dt$$

$\boxed{30}$ $X_1, X_2 \overset{iid}{\sim} \mu$

① $|\phi|^2 = \phi(t) \cdot \overline{\phi(t)}$

$= E[e^{itX_1}] \, \overline{E[e^{itX_2}]}$

$= E[e^{itX_1}] \, E[e^{-itX_2}]$ independent

$= E[e^{it(X_1-X_2)}]$

$\therefore |\phi|^2$ is chf of X_1-X_2.

② $\mathrm{Re}\,\phi(t)$

$\|$

$\frac{1}{2}\{\phi(t) + \overline{\phi(t)}\} = \frac{1}{2} E[e^{itX} + e^{-itX}]$

$= E[\frac{1}{2}e^{itX} + \frac{1}{2}e^{-itX}]$

$= E[E[e^{itXY} | X]] = E[e^{it(XY)}]$

where Y is a rv $P(Y=1) = P(Y=-1) = \frac{1}{2}$

which is independent with X,

So $\mathrm{Re}\,\phi(t)$ is chf of XY

$\boxed{31}$

① $\mu(\{a\}) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} e^{-ita}\phi(t)\,dt$

(proof)

$\frac{1}{2T}\int_{-T}^{T} e^{-ita}\phi(t)\,dt = \frac{1}{2T}\int_{-T}^{T} e^{-ita}\int_{\mathbb{R}} e^{itx}\,d\mu\,dt \quad (X\sim\mu)$

$= \frac{1}{2T}\int_{-T}^{T}\int_{\mathbb{R}} e^{it(x-a)}\,d\mu\,dt$

Since $|e^{it(x-a)}| = 1$ and $\int_{-T}^{T}\int_{\mathbb{R}} d\mu\,dt = 2T < \infty$,

hence by Fubini's theorem we may swap $\int_{-T}^{T}$ $\int_{\mathbb{R}}$

So $\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} e^{-ita}\phi(t)\,dt = \lim_{T\to\infty}\frac{1}{2T}\int_{\mathbb{R}}\int_{-T}^{T} e^{it(x-a)}\,dt\,d\mu$

$= \lim_{T\to\infty}\int_{\mathbb{R}} \underbrace{\frac{1}{2T}\int_{-T}^{T} e^{it(x-a)}\,dt}_{⊛}\,d\mu$

$|⊛| \le \frac{1}{2T}\int_{-T}^{T} dt = 1 < \infty$ $\Rightarrow$ μ-integrable

By Bounded Convergence Theorem (LDCT) we swap $\lim_{T\to\infty}$, $\int_{\mathbb{R}}$

$= \int_{\mathbb{R}}\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} e^{it(x-a)}\,dt\,d\mu$ ...... ⊛

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Now consider

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} e^{it(x-a)}\,dt$$

$$= \lim_{T\to\infty} \frac{1}{2T}\cdot\int_{-T}^{T} \left(\cos(x-a)t + i\sin(x-a)t\right) dt$$

$$= \lim_{T\to\infty} \frac{1}{T}\int_{0}^{T} \cos(x-a)t$$

$$\begin{cases} x \neq a & \lim_{T\to\infty} \frac{\sin(x-a)T}{T(x-a)} = 0 \\ x = a, & 1 \end{cases} \Rightarrow \mathbb{I}_{\{a\}}(x)$$

$$(*) = \int_{\mathbb{R}} \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} e^{it(x-a)}\,dt\,d\mu$$

$$= \int_{\mathbb{R}} \mathbb{I}_{\{a\}}(x)\,d\mu = \mu(\{a\})$$

Now the proof is complete.

(2) $P(X\in h\mathbb{Z})=1$ $(h>0)$ then $P(X=\lambda)=\frac{h}{2\pi}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{-it\lambda}\phi(t)\,dt$

(proof)

Let $\lambda=hk_0$ $(k_0\in\mathbb{Z})$

$$(\overline{RHS})=\frac{h}{2\pi}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{-ithk_0}\phi(t)\,dt$$

$$=\frac{h}{2\pi}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{-ithk_0}\int_{\mathbb{R}} e^{itx}\,d\mu\,dt$$

$$=\frac{h}{2\pi}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}}\int_{\mathbb{R}} e^{it(x-hk_0)}\,d\mu\,dt$$

↓ Fubini's Theorem

$$=\frac{h}{2\pi}\int_{\mathbb{R}}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{it(x-hk_0)}\,dt\,d\mu$$

↓ $e^{it(x-hk_0)} = \cos(\sim)+i\underbrace{\sin(\sim)}_{\text{odd function}}$

$$=\frac{h}{2\pi}\int_{\mathbb{R}} 2\int_{0}^{\frac{\pi}{h}} \cos(x-hk_0)t\,dt\,d\mu$$

$$=\int_{\mathbb{R}} \underbrace{\frac{h}{\pi}\int_{0}^{\frac{\pi}{h}} \cos(x-hk_0)t\,dt}_{(*)}\,d\mu$$

(*) $x=hk_0$: 1

$x\neq hk_0$: $\dfrac{\sin\frac{\pi}{h}(x-hk_0)}{\frac{\pi}{h}(x-hk_0)}$

(3) Similar to (2)

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Hence $(RHS) = \int_{\mathbb{R}} \left\{ \mathbb{I}_{\{hk_0\}}(x) + \frac{\sin\frac{\pi}{h}(x-hk_0)}{\frac{\pi}{h}(x-hk_0)} \cdot \mathbb{I}_{\mathbb{R}\setminus\{hk_0\}}(x) \right\} d\mu$

see $\boxed{\text{NOTE}}$

Since $x \in h\mathbb{Z}$ μ-a.e.

$= \int_{\mathbb{R}} \mathbb{I}_{\{hk_0\}}(x) + \underbrace{\sum_{k \neq k_0} \frac{\sin\pi(k-k_0)}{\pi(k-k_0)} \cdot \mathbb{I}_{\{hk\}}(x)}_{=0} \, d\mu$

$(\because \sin\pi(k-k_0) = 0)$

$= \int_{\mathbb{R}} \mathbb{I}_{\{hk_0\}}(x) \, d\mu = \mu(\{hk_0\}) = P(X = hk_0)$

$\boxed{\text{NOTE}}$

- $\int_{\mathbb{R}} \{\cdots\} d\mu = \int_{h\mathbb{Z}} \{\cdots\} d\mu \quad (\because \mu(h\mathbb{Z}) = 1)$

$= \int_{\mathbb{R}} \{\cdots\} \cdot \mathbb{I}_{h\mathbb{Z}}(x)$

- $\frac{\sin\frac{\pi}{h}(x-hk_0)}{\frac{\pi}{h}(x-hk_0)} \cdot \mathbb{I}_{\mathbb{R}\setminus\{hk_0\}}(x) \cdot \mathbb{I}_{h\mathbb{Z}}(x)$

$= \frac{\sin\frac{\pi}{h}(x-hk_0)}{\frac{\pi}{h}(x-hk_0)} \cdot \mathbb{I}_{h\mathbb{Z}\setminus\{hk_0\}}(x) = \sum_{k \neq k_0} \frac{\sin\frac{\pi}{h}(x-hk_0)}{\frac{\pi}{h}(x-hk_0)} \cdot \mathbb{I}_{\{hk\}}(x)$

$= \sum_{k \neq k_0} \frac{\sin\frac{\pi}{h}(hk-hk_0)}{\frac{\pi}{h}(hk-hk_0)} \cdot \mathbb{I}_{\{hk\}}(x)$

$\boxed{32}$

(1) $X \sim \text{Laplace}(\sigma=1, \mu=0)$

pdf of X : $f_X(x) = \frac{1}{2}\exp(-|x|)$

$$E[e^{itX}] = \int_{-\infty}^{\infty} \frac{1}{2}\exp(-|x|)\ \exp(itx)\, dx$$

$$= \int_0^{\infty} \frac{1}{2}\exp(-(1-it)x)\, dx$$

$$+ \int_{-\infty}^{0} \frac{1}{2}\exp(-(-1-it)x)\, dx$$

$$= \frac{1}{2}\cdot\frac{1}{1-it} + \int_0^{\infty} \frac{1}{2}\exp(-(1+it)x)\, dx$$

$$= \frac{1}{2}\left(\frac{1}{1-it} + \frac{1}{1+it}\right) = \frac{1}{2}\cdot\frac{2}{1+t^2} = \frac{1}{1+t^2}$$

$$\therefore \phi_X(t) = \frac{1}{1+t^2}$$

And $\int_{-\infty}^{\infty} |\phi_X(t)|\, dt = \pi < \infty$

Hence by $\boxed{29}$ $\frac{1}{2}\exp(-|y|) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{1+t^2}\exp(-ity)\, dt$

$t \to x$, $y \to -t$, then we have

$$\frac{1}{2}\exp(-|t|) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{1+x^2}\exp(itx)\, dx$$

Therefore we have

$$\exp(-|t|) = \int_{-\infty}^{\infty} \frac{1}{\pi}\frac{1}{1+x^2}\exp(itx)\,dx$$

So if $X^* \sim \text{Cauchy}(0,1)$

(pdf ... $f_{X^*}(x) = \frac{1}{1+x^2}\frac{1}{\pi}$)

Then $E[e^{itX^*}] = \exp(-|t|)$ ... chf of Cauchy distribution

(2) Let $X_1, X_2 \ldots \overset{iid}{\sim} \text{Cauchy}(0,1)$

$$E\left[e^{it\frac{1}{n}(X_1+\cdots+X_n)}\right] \underset{(iid)}{=} E\left[e^{i\left(\frac{t}{n}\right)X_1}\right]^n$$

$= \exp\left(-\left|\frac{t}{n}\right|\right)^n = \exp(-|t|)$ ... same as $E[e^{itX_1}]$

$\boxed{2\text{-}1}$, $\boxed{3}$ (1) implies chf determines its distribution

(uniqueness of chf)

So $\frac{X_1+\cdots X_n}{n} \sim \text{Cauchy}(0,1)$

$\boxed{33}$ Continuity theorem

(1) If $X_n \xrightarrow{d} X$ then $\phi_n(t) \to \phi(t)$

where $\phi_n(t)$: chf of X_n $(= E[e^{itX_n}])$

$\phi(t)$: chf of X $(= E[e^{itX}])$

(proof) Since 「$X_n \xrightarrow{d} X$」 $\Leftrightarrow$ 「$\forall g$: continuous and bounded . $E[g(X_n)] \to E[g(X)]$」.

let $g = e^{itX}$ then we have

「$X_n \xrightarrow{d} X$」 $\Rightarrow$ 「$E[e^{itX_n}] \to E[e^{itX}]$」 $\Leftrightarrow$

「$\therefore \phi_n(t) \to \phi(t)$」

(2) Let $\{X_n\}_{n \geq 1}$ be random variables,

and let $\phi_n(t) = E[e^{itX_n}]$ ($\because$ chf)

Suppose that $\phi_n(t) \to \phi(t)$

where $\phi(t)$ is continuous at $t = 0$.

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Then $\phi(t)$ is also a chf.

Let X be a random variable whose chf is $\phi(t)$.

Moreover $X_n \xrightarrow{d} X$.

(proof)

$\boxed{\text{Step 1}}$ $\{X_n\}_{n\geq 1}$ is tight

(i.e $F_n(x) \overset{\text{def}}{=} P(X_n \leq x)$ $\{F_n\}_{n\geq 1}$ is tight)

Let $\{\mu_n\}_{n\geq 1}$ be probability measures where

$\mu_n((-\infty, x]) = F_n(x)$.

- Since $\phi(t)$ is continuous at $t=0$.

$\forall \varepsilon > 0$ $\exists \delta > 0$ st $\sup_{t\in[-\delta,\delta]} |1-\phi(t)| < \frac{\varepsilon}{2}$ ... (※1)

- Next. Consider $\int_{[-\delta,\delta]} (1-\phi_n(t))\, dt$.

(Be careful of the fact that $|1-\phi_n(t)| \leq 2$

so it's integrable) $(\because |1-\phi_n(t)| = |E[1-e^{itX_n}]|$

$\leq E|1-e^{itX_n}| \leq 2)$

By Fubini's theorem

$$\int_{[-\delta,\delta]} (1-\phi_n(t)) = \int_{[-\delta,\delta]} \int_{\mathbb{R}} (1-e^{itx})\, \mu_n(dx)\, dt$$

$$= \int_{\mathbb{R}} \int_{[-\delta,\delta]} (1-e^{itx})\, dt\, \mu_n(dx)$$

(integral with respect to t)

Moreover, $= \int_{\mathbb{R}} \left(2\delta - \frac{2\sin\delta x}{x}\right) \mu_n(dx)$

So $\frac{1}{2\delta} \int_{[-\delta,\delta]} (1-\phi_n(t))\, dt = \int_{\mathbb{R}} \left(1-\frac{\sin\delta x}{\delta x}\right) d\mu_n$.

And $\int_{\mathbb{R}} \left(1-\frac{\sin\delta x}{\delta x}\right) \mu_n(dx) \geq \int_{\{|\delta x| \geq 2\}} \left(1-\frac{\sin\delta x}{\delta x}\right) \mu_n(dx)$

$$\geq \int_{\{|\delta x|\geq 2\}} \left(1-\frac{1}{|\delta x|}\right) d\mu_n \qquad \left(\because 1-\frac{\sin\delta x}{\delta x} \geq 0\right)$$

$$\geq \int_{\{|\delta x|\geq 2\}} \frac{1}{2}\, d\mu_n$$

$$= \frac{1}{2} \mu_n(\{|\delta x| \geq 2\})$$

$$\therefore \mu_n\left(\left\{|x| \geq \frac{2}{\delta}\right\}\right) \leq \frac{1}{\delta} \int_{[-\delta,\delta]} (1-\phi_n(t))\, dt$$

- By L.D.C.T $\lim_{n\to\infty} \frac{1}{\delta} \int_{[-\delta,\delta]} (1-\phi_n(t))\, dt$

$$= \frac{1}{\delta} \int_{[-\delta,\delta]} (1-\phi(t))\, dt$$

$$\leq \frac{1}{\delta} \int_{[-\delta,\delta]} \sup_{t\in[-\delta,\delta]} |1-\phi(t)|\, dt \leq \varepsilon$$

- Hence $\limsup_{n\to\infty} \mu_n\left(\left\{|x| \geq \underset{\overset{\shortparallel}{M}}{\frac{2}{\delta}}\right\}\right) \leq \varepsilon \quad \therefore \{\mu_n\}_{n\geq 1}$: tight

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$\boxed{\text{Step2}}$ Now we have already found that $\{X_n\}_{n\geq 1}$ ($\{\mu_n\}$) is tight.

Let $\{n_k\}_{k\geq 1} \subseteq \mathbb{N}$ be an arbitrary subsequence. Then $\{X_{n_k}\}_{k\geq 1}$ ($\{\mu_{n_k}\}_{k\geq 1}$) is also tight.

We may find a subsubsequence $\{n_{k_\ell}\}_{\ell\geq 1}$ and X s.t $X_{n_{k_\ell}} \xrightarrow{d} X$. (or $\mu_{n_{k_\ell}} \xrightarrow{d} \mu$, $X \sim \mu$) ($\boxed{15}$)

where X is a r.v. (μ is a probability measure)

$\therefore$ 「$X_{n_{k_\ell}} \xrightarrow{d} X$」 $\Leftrightarrow$ 「$\forall g$: bounded continuous $E[g(X_{n_{k_\ell}})] \to E[g(X)]$」

Hence "$\forall \{n_k\}$ $\exists \{n_{k_\ell}\}$ s.t $E[g(X_{n_{k_\ell}})] \to E[g(X)]$"

$\Leftrightarrow$ "$E[g(X_n)] \to E[g(X)]$" (ch2. $\boxed{40}$)

Thus $X_n \xrightarrow{d} X$.

CHA SHIN

By (1) $X_n \xrightarrow{d} X \Rightarrow E[e^{itX_n}] \to E[e^{itX}]$

We find that $\phi(t) = E[e^{itX}]$. ($\therefore \phi(t)$ is a chf)

Since chf is unique, we may conclude that

$X_n \xrightarrow{d} X$ where X is a r.v whose chf is $\phi(t)$. ∎

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$\boxed{34}$ $X_n \xrightarrow{d} X$ and $X_n \sim N(0, \sigma_n^2)$

We show that $\sigma_n^2 \to \sigma^2 \in [0, \infty)$

(proof)

$\phi_n(t) \overset{def}{=} E[e^{itX_n}] = \exp\left(-\frac{1}{2}\sigma_n^2 t^2\right)$

$\phi(t) \overset{def}{=} E[e^{itX}]$

Since $\lim_{n\to\infty} \phi_n(t) = \phi(t)$ ($\because X_n \xrightarrow{d} X$) and $\sigma_n^2 \to \sigma^2$

Hence $\phi(t) = \exp\left(-\frac{1}{2}\sigma^2 t^2\right)$

$\phi(t)$ is continuous at $t=0$ ($\because$ chf)

$\forall \varepsilon > 0$ $\exists M_\varepsilon > 0$ s.t. $\left|\exp\left(-\frac{1}{2}\sigma^2 t^2\right) - 1\right| \leqq \varepsilon$

for all $t \in [-M, M]$

$\therefore$ put $\varepsilon = 1/2$.

Then we have $\exists M > 0$ $\frac{1}{2} \leq \exp\left(-\frac{1}{2}\sigma^2 M^2\right) \leq \frac{3}{2}$

$\therefore -\ln 2 \leq -\frac{1}{2}\sigma^2 M^2$ $0 \leq \sigma^2 \leq \frac{2\ln 2}{M^2}$

And since $M \in (0, \infty)$, thus $\sigma^2 \in [0, \infty)$

CHA SHIN

$\boxed{35}$ Consider chf of X_n+Y_n and Use continuity theory

$$E\left[e^{it(X_n+Y_n)}\right] = E\left[e^{itX_n}\right]\cdot E\left[e^{itY_n}\right]$$

(independence)

$$\xrightarrow[(\text{as } n\to\infty)]{} E\left[e^{itX}\right]E\left[e^{itY}\right] = E\left[e^{it(X+Y)}\right] = \phi_{X+Y}(t)$$

By Continuity theorem $X_n+Y_n \xrightarrow{d} X+Y$.

※ In this question, we may suppose that X, Y are independent.

$\ulcorner X_n \xrightarrow{d} X,\ Y_n \xrightarrow{d} Y \Rightarrow X_n+Y_n \xrightarrow{d} X+Y \lrcorner$

though the question does not refer to that independence

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$\boxed{36}$

(1) $\boxed{\text{Step 1}}$

「 $e^{ix} - \sum_{m=0}^{n} \frac{(ix)^m}{m!} = \frac{i^{n+1}}{n!} \int_{s=0}^{s=x} e^{is} \cdot (x-s)^n ds$ 」

(proof)

• $n=0 \cdots$ $\boxed{LHS}$ $e^{ix} - 1$

$\boxed{RHS}$ $i \int_{s=0}^{s=x} e^{is} ds = i \cdot \left(\frac{e^{ix}-1}{i} \right) = e^{ix} - 1$

• $n=k$ $\boxed{LHS}$ $e^{ix} - \sum_{m=0}^{k} \frac{(ix)^m}{m!}$

$\boxed{RHS}$ $\frac{i^{k+1}}{k!} \int_{s=0}^{s=x} e^{is} (x-s)^k ds$

We suppose this statement is true

• $n=k+1$ $\boxed{LHS}$ $e^{ix} - \sum_{m=0}^{k+1} \frac{(ix)^m}{m!}$

$\boxed{RHS}$ $\frac{i^{k+2}}{(k+1)!} \int_{s=0}^{s=x} e^{is} (x-s)^{k+1} ds$

$= \frac{i^{k+2}}{(k+1)!} \left\{ \left[\frac{e^{is}}{i} (x-s)^{k+1} \right]_{s=0}^{s=x} + \int_{s=0}^{s=x} \frac{e^{is}}{i} (k+1)(x-s)^{k} ds \right.$

(Integration by parts)

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$$\frac{i^{k+2}}{(k+1)!}\left(\frac{-x^{k+1}}{i}+\int_{s=0}^{s=x}\frac{e^{is}}{i}(k+1)(x-s)^{k}ds\right)$$

$$=\frac{-(ix)^{k+1}}{(k+1)!}+\frac{i^{k+1}}{k!}\underbrace{\int_{s=0}^{s=x}e^{is}(x-s)^{k}ds}$$

$$\left(=e^{ix}-\sum_{m=0}^{k}\frac{(ix)^m}{m!}\right) \text{ (by assumption of } n=k)$$

$$=e^{ix}-\sum_{m=0}^{k+1}\frac{(ix)^m}{m!}=\boxed{LHS}$$

Therefore by induction the proof is complete ■

$\boxed{\text{Step 2}}$ By the equation in $\boxed{\text{Step 1}}$

$$\left|e^{ix}-\sum_{m=0}^{n}\frac{(ix)^m}{m!}\right|=\left|\frac{i^{n+1}}{n!}\int_{s=0}^{s=x}e^{is}(x-s)^{n}ds\right|$$

$$\leq\frac{1}{n!}\int_{s=0}^{s=x}\left|e^{is}(x-s)\right|^{n}ds$$

$$=\frac{1}{n!}\int_{s=0}^{s=|x|}s^{n}ds=\frac{|x|^{n+1}}{(n+1)!}$$

$$\therefore |LHS|\leq\frac{|x|^{n+1}}{(n+1)!} \quad ①$$

And also $\left| e^{ix} - \sum_{m=0}^{n} \frac{(ix)^m}{m!} \right|$

$= \left| \frac{i^{n+1}}{n!} \int_{s=0}^{s=x} e^{is}(x-s)^n ds \right| = \circledast$

By integral by parts

$\int_{s=0}^{s=x} e^{is}(x-s)^n ds = \left[\frac{e^{is}}{i}(x-s)^n \right]_{s=0}^{s=x}$

$+ \int_{s=0}^{s=x} \frac{e^{is}}{i} n(x-s)^{n-1} ds$

$= -\frac{e^{is}}{i} x^n + \frac{n}{i} \int_{s=0}^{s=x} e^{is}(x-s)^{n-1} ds$

$\therefore \circledast \leq \frac{1}{n!} \left\{ |x|^n + n \int_{s=0}^{s=|x|} s^{n-1} ds \right\} = \frac{2|x|^n}{n!}$

$\therefore$ We found that $\leq \min\left\{ \frac{2|x|^n}{n!}, \frac{|x|^{n+1}}{(n+1)!} \right\}$

(2) Next by putting $x = tX$

$\left| e^{itX} - \sum_{m=0}^{n} \frac{(itX)^m}{m!} \right| \leq \min\left\{ \frac{2|tX|^n}{n!}, \frac{|tX|^{n+1}}{(n+1)!} \right\}$

$$\left|E\left[e^{itX}-\sum_{m=0}^{n}\frac{(itX)^m}{m!}\right]\right| \leq E\left|e^{itX}-\sum_{m=0}^{n}\frac{(itX)^m}{m!}\right|$$

$$\leq E\min\left\{\frac{2|tX|^n}{n!}, \frac{|tX|^{n+1}}{(n+1)!}\right\}$$

$$\leq E\left[\frac{2|tX|^n}{n!}\right] \text{ or } E\left[\frac{|tX|^{n+1}}{(n+1)!}\right]$$

$$\Rightarrow \text{So} \leq \min\left\{E\left[\frac{2|tX|^n}{n!}\right], E\left[\frac{|tX|^{n+1}}{(n+1)!}\right]\right\}$$

Therefore $\left|\phi(t)-\sum_{m=0}^{n}\frac{E[(itX)^m]}{m!}\right| \leq E\min\left\{\frac{2|tX|^n}{n!}, \frac{|tX|^{n+1}}{(n+1)!}\right\}$

$$\leq \min\left\{E\left[\frac{2|tX|^n}{n!}\right], E\left[\frac{|tX|^{n+1}}{(n+1)!}\right]\right\}$$

$\boxed{37}$

① First we show the case of $n=1$.

$$\phi^{(1)}(t) = \lim_{h\to 0} \frac{\phi(t+h)-\phi(t)}{h}$$

$$= \lim_{h\to 0} \left(E[e^{i(t+h)X}] - E[e^{itX}]\right)/h$$

$$= \lim_{h\to 0} E\left[\frac{e^{i(t+h)X} - e^{itX}}{h}\right]$$

We prove that $\lim_{h\to 0}$ and $E[\cdot]$ can be swapped

$$\left|\frac{1}{h}(e^{i(t+h)X} - e^{itX})\right| \leq \frac{1}{|h|}|e^{itX}||e^{ihX}-1|$$

$$= \frac{1}{|h|}|e^{ihX}-1| \quad \circledast$$

By $\boxed{36}$ (1) $|e^{ihX}-1| \leq \min\{|hX|, 2\}$

$$\leq |hX|$$

$\therefore \circledast \leq \frac{1}{|h|}|hX| = |X|$

And $|X|$ is integrable $\therefore$ $\sup_{h\in\mathbb{R}} \left|\frac{1}{h}(e^{i(t+h)X} - e^{itX})\right|$

$$\leq |X| \in L^1$$

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By Lebesgue Dominated Convergence Theorem

$$\lim_{h\to 0} E[\cdot] = E\left[\lim_{h\to 0} \frac{e^{i(t+h)x} - e^{itx}}{h}\right]$$

$$= E\left[e^{itx} \cdot \lim_{h\to 0} \frac{e^{ihx}-1}{h}\right] = E[e^{itx} \cdot ix]$$

$$\therefore \phi^{(1)}(t) = E[e^{itx} ix] = \int_{\mathbb{R}} (ix) e^{itx} \mu(dx)$$

(Change Variable Formula in ch1.)

② $n \geq 2$

Similarly, suppose the statement holds when $n=k$.

(i.e. $\phi^{(k)}(t) = E[e^{itx} \cdot (ix)^k]$)

And $\phi^{(k+1)}(t) = \lim_{h\to 0} \frac{\phi^{(k)}(t+h) - \phi^{(k)}(t)}{h}$

$$= \lim_{h\to 0} E[\sim]$$

Again we use $|e^{ihx}-1| \leq |hx|$

and show that $\lim_{h\to 0}$ and $E[\,]$ can be swapped

Then we have the desired result for $n=k+1$.

The statement is proved by induction ∎

CHA SHIN

$\boxed{38}$ When $E[X^2] < \infty$, $\phi(t) = 1 + itE[X] - \frac{1}{2}t^2E[X^2] + o(t^2)$.

(proof) $o(t^2)$ means $\lim_{t\to 0} \frac{\phi(t^2)}{t^2} = 0$

So we show $\lim_{t\to 0} \frac{\phi(t) - 1 - itE[X] + \frac{t^2E[X^2]}{2}}{t^2} = 0$

• $\frac{\phi(t) - 1 - itE[X] + \frac{t^2}{2}E[X^2]}{t^2} = E\left[\frac{e^{itX} - 1 - itX - \frac{(itX)^2}{2}}{t^2}\right] \cdots \circledast$

• $\left|e^{itX} - 1 - itX - \frac{(itX)^2}{2}\right| = \left|e^{itX} - \sum_{m=0}^{2} \frac{(itX)^m}{m!}\right|$

$\le \min\left\{\frac{|tX|^3}{3!}, \frac{2|tX|^2}{2!}\right\} \le |tX|^2$ (by $\boxed{36}$)

So $\left|\frac{e^{itX} - 1 - itX - \frac{1}{2}(itX)^2}{t^2}\right| \le \frac{|tX|^2}{t^2} = |X^2| \ldots \in L^1$ (integrable)

$\therefore \sup_{t\in\mathbb{R}} \left|\frac{e^{itX} - 1 - itX - \frac{1}{2}(itX)^2}{t^2}\right| \le |X|^2$

By L.D.C.T $\circledast \ldots \lim_{t\to 0} E[\sim] = E[\lim_{t\to 0} \sim]$

(We may swap lim and $E[\cdot]$)

And $\lim_{t\to 0}(\sim) = 0$. Then we will have the desired result.

$\boxed{39}$ We show that $\varlimsup_{h \downarrow 0} \frac{\phi(h) - 2\phi(0) + \phi(-h)}{h^2} > -\infty$

implies that $E|X|^2 < \infty$

(proof) $\infty > -\varlimsup_{h\downarrow 0} (\sim) = \varliminf_{h \downarrow 0} \frac{2\phi(0) - \phi(h) - \phi(-h)}{h^2}$

$= \varliminf_{h\downarrow 0} E\left[\frac{2 - e^{ihX} - e^{-ihX}}{h^2}\right]$

$= \varliminf_{h\downarrow 0} E\left[\frac{2 - 2\cos hX}{h^2}\right]$ $\downarrow$ By Fatou's lemma $\left(\because \frac{2-2\cos hX}{h^2} \geq 0\right)$

$\geq E\left[\varliminf_{h \to 0} \frac{2 - 2\cos hX}{h^2}\right]$

$= E\left[X^2\right]$

So $E[X^2] < \infty$ ∎

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$\boxed{40}$ $E|X|^n < \infty$ for all $n \geq 1$

hence $\phi(t) = \sum_{n \geq 0}^{\infty} \frac{E[(itX)^n]}{n!} = \sum_{n \geq 0}^{\infty} \frac{(it)^n E[X^n]}{n!}$ ... ①

$\phi(t) = \exp\left(\frac{-t^2}{2}\right) = \sum_{m=0}^{\infty} \left(\frac{-t^2}{2}\right)^m \cdot \frac{1}{m!}$

$= \sum_{m=0}^{\infty} \frac{(-1)^m t^{2m}}{2^m m!}$ ... ②

By comparing ① and ②, (see coefficient of t^{2m})

put $n = 2m$ in ①.

$= \frac{(it)^{2m} E[X^{2m}]}{(2m)!} = \frac{(-1)^m t^{2m} E[X^{2m}]}{(2m)!} = \underbrace{\frac{(-1)^m t^{2m}}{2^m \cdot m!}}_{②}$

$i^2 = (-1)$

$\therefore E[X^{2m}] \cdot \frac{(-1)^m}{(2m)!} = \frac{(-1)^m}{2^m \cdot m!}$

$\therefore E[X^{2m}] = \frac{(2m)!}{m! \, 2^m}$

$\boxed{4}$

(1) Here the index set I is not necessary $\mathbb{N}$

Tightness is defined as " $\lim_{M\to\infty} \sup_{i\in I} \mu_i([-M,M]^c) = 0$ "

Now we show that $\phi_i(t) \left(= E_{X_i \sim \mu_i}[e^{itX_i}] \quad i \in I \right)$

are equicontinuous.

$$\cdot \; |\phi_i(t+h) - \phi_i(t)| = \left| E[e^{i(t+h)X_i}] - E[e^{itX_i}] \right|$$

$$\leq E\left| e^{i(t+h)X_i} - e^{itX_i} \right| = E\left| e^{ihX_i} - 1 \right| \leq E \min\{|hX_i|, 2\}$$

$$\therefore \sup_{t\in\mathbb{R}} |\phi_i(t+h) - \phi_i(t)| \leq E \cdot \min\{|h||X_i|, 2\}$$

$$\leq E \cdot \min\{\cdots\} \mathbb{I}\{|X_i| > M\} + E \min\{\cdots\} \mathbb{I}\{|X_i| \leq M\}$$

$$\leq 2\mu_i([-M,M]^c) + |h|M$$

$$\therefore \sup_{i\in I} \sup_{t\in\mathbb{R}} |\phi_i(t+h) - \phi_i(t)| \leq 2 \sup_{i\in I} \mu_i([-M,M]^c) + |h|M$$

Take large $M(\varepsilon)$ st $\sup_{i\in I} \mu_i([-M,M]^c) < \frac{\varepsilon}{4}$

And then take $|h| < \frac{\varepsilon}{2M_\varepsilon}$.

We have $\sup_{\substack{t\in\mathbb{R} \\ i\in I}} |\phi_i(t+h) - \phi_i(t)| < \varepsilon$

for all $h \in \left(\frac{-\varepsilon}{2M_\varepsilon}, \frac{\varepsilon}{2M_\varepsilon}\right)$.

∎

(2) Suppose $\mu_n \xrightarrow{d} \mu$

We show that $\phi_n(t) \to \phi(t)$ uniformly

on compact sets.

(proof) Without loss of generality

We may suppose that $F = [-K, K]$

($K > 0$. K is a natural number)

and show that $\lim_{n\to\infty} \sup_{t\in F} |\phi_n(t) - \phi(t)| = 0$.

$$|\phi_n(t) - \phi(t)| = \left|\phi_n(t) - \phi_n\left(\frac{k^{(t)}}{m}\right) + \phi_n\left(\frac{k^{(t)}}{m}\right) - \phi\left(\frac{k^{(t)}}{m}\right) + \phi\left(\frac{k^{(t)}}{m}\right) - \phi(t)\right|$$

$$\leq \underbrace{\left|\phi_n(t) - \phi_n\left(\frac{k^{(t)}}{m}\right)\right|}_{③} + \underbrace{\left|\phi_n\left(\frac{k^{(t)}}{m}\right) - \phi\left(\frac{k^{(t)}}{m}\right)\right|}_{②} + \underbrace{\left|\phi\left(\frac{k^{(t)}}{m}\right) - \phi(t)\right|}_{①}$$

(By triangle inequality)

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Here $m \in \mathbb{N}$ and $k^{(t)}$ is a function of t. and

$k^{(t)} \in \{-mK, -mK+1, \cdots, mK-1, mK\}$

and $\left|t - \frac{k^{(t)}}{m}\right| \leq \frac{1}{m}$. $(t \in F)$

(There always exists $k^{(t)}$ s.t $\left|t - \frac{k^{(t)}}{m}\right| \leq \frac{1}{m}$)

Now let $\varepsilon > 0$ be an arbitrary positive number

Suppose m is large enough $m < \min\{\delta_1, \delta_2\}$.

① … $\sup_{t \in F} \left|\phi\left(\frac{k^{(t)}}{m}\right) - \phi(t)\right| < \frac{\varepsilon}{3}$ because

$\phi(\cdot)$ is uniformly continuous and $\left|\frac{k^{(t)}}{m} - t\right| \leq \frac{1}{m} < \delta_2$

② … $\sup_{t \in F} \left|\phi_n\left(\frac{k^{(t)}}{m}\right) - \phi\left(\frac{k^{(t)}}{m}\right)\right| < \frac{\varepsilon}{3}$ for sufficiently large n

because $\left\{\frac{k^{(t)}}{m}\right\}_{t \in F} = \left\{-K, -K + \frac{1}{m}, -K + \frac{2}{m}, \cdots, K - \frac{1}{m}, K\right\}$

So there are only finite points. (K, m : already given)

hence $\phi_n(t) \to \phi(t)$ (pointwise)

$\Rightarrow$ $\sup_{t \in F} \left|\phi_n\left(\frac{k^{(t)}}{m}\right) - \phi\left(\frac{k^{(t)}}{m}\right)\right| < \frac{\varepsilon}{3}$ for large n.

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③ $\sup_{t\in F} \left| \phi_n(t) - \phi_n\left(\frac{k^{(t)}}{m}\right) \right| < \frac{\varepsilon}{3}$ for suffiently large n.

because in the previous question we have shown

that " $\sup_{\substack{i\in I \\ t\in\mathbb{R}}} | \phi_i(t+h) - \phi_i(t) | < \varepsilon$ " when $|h| < \delta_1$

$\therefore \limsup_{n\to\infty} \sup_{t\in F} \left| \phi_n(t) - \phi_n\left(\frac{k^{(t)}}{m}\right) \right|$

$\leq \sup_{n\geq 1} \sup_{t\in F} \left| \phi_n(t) - \phi_n\left(\frac{k^{(t)}}{m}\right) \right| < \frac{\varepsilon}{3}$

(because $\left| t - \frac{k^{(t)}}{m} \right| \leq \frac{1}{m} < \delta_1$)

Therefore when n is large enough

$\sup_{t\in F} | \phi_n(t) - \phi(t) | < \varepsilon$

$\therefore \lim_{n\to\infty} \sup_{t\in F} | \phi_n(t) - \phi(t) | = 0$

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$\boxed{42}$

(1) $E\left[e^{it\left(\frac{S_n}{n}\right)}\right] = E\left[e^{i\left(\frac{t}{n}\right)(S_n)}\right] = E\left[e^{i\left(\frac{t}{n}\right)X_1}\right]^n$

(i.i.d)

$= \phi\left(\frac{t}{n}\right)^n$

$= \left\{ \frac{\left(\phi\left(\frac{t}{n}\right) - \phi(0)\right) + \phi(0)}{\frac{t}{n}} \cdot \frac{t}{n} \right\}^n$

$= \left(\underbrace{\frac{\phi\left(\frac{t}{n}\right) - \phi(0)}{\left(\frac{t}{n}\right)}}_{\to ia} \cdot \frac{t}{n} + 1 \right)^n \to \exp(iat)$

(as $n \to \infty$)

So. $\frac{S_n}{n} \xrightarrow{d} a \Rightarrow \frac{S_n}{n} \xrightarrow{p} a$

(2) $\frac{S_n}{n} \xrightarrow{p} a \Rightarrow \frac{S_n}{n} \xrightarrow{d} a$

$\Rightarrow E\left[e^{it\left(\frac{S_n}{n}\right)}\right] \to \exp(iat)$

$\Leftrightarrow \phi\left(\frac{t}{n}\right)^n \to \exp(iat)$

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(3) $\lim_{h\to 0} \frac{\phi(h)-1}{h} = \lim_{n\to\infty} \frac{\phi(\frac{t}{n})-1}{(\frac{t}{n})}$ .... $\circledast$

(where $t\in\mathbb{R}$ and is fixed)

We show that $\frac{S_n}{n} \xrightarrow{P} a$ then $\circledast \to ia$.

$\frac{S_n}{n} \xrightarrow{P} a \Rightarrow \frac{S_n}{n} \xrightarrow{d} a \Rightarrow \phi(\frac{t}{n})^n \to e^{iat}$

$= \left(\phi(\frac{t}{n})-1+1\right)^n = \left(\frac{\phi(\frac{t}{n})-1}{(\frac{t}{n})}\cdot\frac{t}{n}+1\right)^n \to e^{iat}$

Hence $\frac{\phi(\frac{t}{n})-1}{(\frac{t}{n})}$ must $\to ia$ as $n\to\infty$. ∎

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$\boxed{43}$ $c \in \mathbb{R}$ so we may suppose that

$\mathrm{Re}(\phi(t)-1)/t^2 \to c$ (as $t \downarrow 0$)

$\Leftrightarrow$

$2 \cdot \mathrm{Re}(\phi(t)-1)/t^2 \to 2c$

$\parallel$

$$\lim_{t\downarrow 0} \frac{\phi(t)+\phi(-t)-2}{t^2} = 2c > -\infty$$

By $\boxed{39}$, we have $E[X^2] < \infty$ ($\Rightarrow E|X| < \infty$)

Hence $\boxed{37}$ implies that $\phi^{(1)}(0)$, $\phi^{(2)}(0)$ exist.

By L'hopital rule

$$= \lim_{t\downarrow 0} \frac{\phi'(t)-\phi'(-t)}{2t} = \lim_{t\downarrow 0} \frac{\phi''(t)+\phi''(-t)}{2} = \phi''(0) = 2c$$

$$\therefore \phi^{(2)}(0) = -E[X^2] = 2c \Rightarrow E[X^2] = -2c$$

(by $\boxed{37}$)

In particular $\phi(t) = 1 + o(t^2)$, this means that

$$\lim_{t \downarrow 0} \frac{\phi(t) - 1}{t^2} = 0.$$

$\therefore C = 0$. $\therefore E[X^2] = 0$ $(\because -2C = 0)$

$\Rightarrow X = 0$ (as) $(\because X^2 \geq 0)$

$\Rightarrow P(X=0) = 1$

$\Rightarrow E[e^{itX}] = 1$ $\therefore \phi(t) = 1$

$\boxed{44}$ $\{Y_n\}_{n\geq1}$: r.v , $\phi_n(t) = E[e^{itY_n}]$

① $Y_n \xrightarrow{d} 0 \Rightarrow \phi_n(t) \to \phi(t)$ where $\phi(t) = 1$ $|t| \leq \delta$

for some $\delta > 0$

This is obviously true since $Y_n \xrightarrow{d} 0$

$\Rightarrow \phi_n(t) \to 1$

② $\exists \delta > 0$ $\phi_n(t) \to \phi(t)$ where $\phi(t) = 1$ $|t| \leq \delta$.

· We use $\boxed{33}$ and $\boxed{43}$

First, $\phi(t)$ is continuous at $t=0$

So Y_n must converse in distribution. ($Y_n \xrightarrow{d} Y$.

Hence we may suppose $\phi(t)$ is a chf. $\phi(\cdot)$ is chf of Y)

Moreover by $\boxed{43}$: $\phi(t) - 1 = 0$ $(\forall t : |t| \leq \delta)$

$$\Rightarrow \lim_{t \to 0} \frac{\phi(t) - 1}{t^2} = 0$$

$$\Rightarrow E[Y] = 0$$

$$\Rightarrow Y = 0 \text{ (a.s)}$$

$$\Rightarrow Y_n \xrightarrow{d} Y$$

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45. Consider the chf of $S_n - S_m$ $(n>m)$

$$E[e^{itS_n}] = E[e^{it(S_n - S_m + S_m)}] = E[e^{it(S_n-S_m)}] \cdot E[e^{itS_m}]$$

$$\phi_{n,m}(t) = \phi_n(t)/\phi_m(t)$$

$$\left(\phi_{n,m}(t) \overset{\text{def}}{=} E[e^{it(S_n-S_m)}],\ \phi_n(t) = E[e^{itS_n}] \right)$$

Since S_n converges in distribution, $(S_n \xrightarrow{d} S)$

$\phi_n(t) \to \phi(t)$ (for all $t \in \mathbb{R}$)

And $\phi(t)$ is a chf. so it is continuous

at $t=0$ and $\phi(0)=1$.

We may find $\delta>0$ $|\phi(t)-1| \leq \frac{1}{2}$ $\forall t$: $|t| \leq \delta$.

$\Rightarrow$ $\phi(t) \neq 0$ $t \in [-\delta, \delta]$

(We want to guarantee that $\phi_m(t) \neq 0$

for $t \in [-\delta, \delta]$ in a closed interval)

$\therefore \lim_{\substack{n\to\infty \\ m\to\infty}} \phi_{n,m}(t) = \dfrac{\phi(t)}{\phi(t)} = 1 \quad (\phi(t) \neq 0) \quad t \in [-\delta, \delta]$.

By $\boxed{44}$ $\therefore$ $\phi_{n,m}(t) \to 1 \quad {}^{\forall} t \in [-\delta, \delta]$

$\Rightarrow S_n - S_m \xrightarrow{d} 0 \quad$ (as $n, m \to \infty$)

$\Rightarrow S_n - S_m \xrightarrow{P} 0 \quad$ (as $n, m \to \infty$)

Let us recall ch2. $\boxed{50}$ and $\boxed{51}$

$d(X, Y) \stackrel{def}{=} E\left[\dfrac{|X-Y|}{1+|X-Y|}\right]$

$d(S_n, S_m) = E\left[\dfrac{|S_n - S_m|}{1+|S_n - S_m|}\right]$

By Bounded convergence theorem $\left(\dfrac{|S_n - S_m|}{1+|S_n - S_m|} \leq 1\right)$

$S_n - S_m \xrightarrow{P} 0 \Rightarrow \lim_{\substack{n\to\infty \\ m\to\infty}} d(S_n, S_m) = 0$

From the conclusion of $\boxed{51}$,

$\{S_n\}_{n\geq 1}$: is a Cauchy sequence on $(\mathcal{X}, d)$

There exists S (r.v) s.t $S_n \xrightarrow{P} S$

$\boxed{46}$ Central Limit Theorem

Let $X_1, X_2, \ldots$ iid $E[X_i] = \mu$. $V(X_i) = \sigma^2$

We show that $\frac{S_n - n\mu}{\sigma n^{\frac{1}{2}}} \xrightarrow{d} N(0,1)$

(proof) $Z_i \stackrel{def}{=} \frac{X_i - \mu}{\sigma}$ $T_n = Z_1 + \cdots + Z_n$

We show $\frac{T_n}{n^{\frac{1}{2}}} \xrightarrow{d} N(0,1)$

$E\left[e^{it\left(\frac{T_n}{\sqrt{n}}\right)}\right] = E\left[e^{i\left(\frac{t}{\sqrt{n}}\right)Z_1}\right]^n$

$= \phi_Z\left(\frac{t}{\sqrt{n}}\right)^n$ $\left(\phi_Z(t) = E\left[e^{itZ}\right]\right)$

$E[Z_i] = 0$ $E[Z_i^2] = 1$ $(V[Z_i]) < \infty$

By $\boxed{38}$ $\phi_Z(t) = 1 - \frac{t^2}{2} + o(t^2)$

$\phi_Z\left(\frac{t}{\sqrt{n}}\right)^n = \left(1 - \frac{t^2}{2n} + \underset{\sim}{o\left(\frac{t^2}{n}\right)}\right)^n$

(∵ This goes to 0 faster than $\frac{t^2}{2n}$ so we may ignore this term)

$\lim_{n\to\infty} \phi_Z\left(\frac{t}{\sqrt{n}}\right)^n = \lim_{n\to\infty}\left(1 - \frac{t^2}{2n}\right)^n$

$= \exp\left(\frac{-t^2}{2}\right)$ (chf of $N(0,1)$)

Therefore, $\frac{S_n - n\mu}{\sigma n^{\frac{1}{2}}} = \frac{T_n}{n^{\frac{1}{2}}} \xrightarrow{d} N(0,1)$.

$\boxed{47}$

(1) $\{z_i\}_{i=1}^{n} \cup \{w_i\}_{i=1}^{n} \subseteq \mathbb{C}$

$|z_i| \leq \theta \quad |w_i| \leq \theta \quad (i=1\sim n)$

We prove that $\left|\prod_{m=1}^{n} z_m - \prod_{m=1}^{n} w_m\right| \leq \theta^{n-1} \sum_{m=1}^{n} |z_m - w_m|$

(proof)

$= \left|\prod_{m=1}^{n} z_m - w_1 \prod_{m=2}^{n} z_m + w_1 \prod_{m=2}^{n} z_m - \prod_{m=1}^{n} w_m\right|$) triangle inequality

$\leq |z_1 - w_1| \left|\prod_{m=2}^{n} z_m\right| + |w_1| \left|\prod_{m=2}^{n} z_m - \prod_{m=2}^{n} w_m\right|$

$(\leq \theta^{n-1}) \quad (\leq \theta)$

$\leq \theta^{n-1} |z_1 - w_1| + \theta \underbrace{\left|\prod_{m=2}^{n} z_m - \prod_{m=2}^{n} w_m\right|}_{\circledast}$

repeat the similar argument on $\circledast$

$\circledast \leq \theta^{n-2} |z_2 - w_2| + \theta \left|\prod_{m=3}^{n} z_m - \prod_{m=3}^{n} w_m\right| \cdots$

Finally we have $\leq \theta^{n-1} \sum_{m=1}^{n} |z_m - w_m|$

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(2) $b \in \mathbb{R}$ and $|b| \leq 1$ than $|e^b - (1+b)| \leq |b|^2$

(proof)

$e^b = \sum_{n\geq 0}^{\infty} \frac{b^n}{n!}$

$\therefore e^b - (1+b) = \sum_{n\geq 2}^{\infty} \frac{b^n}{n!}$

$|e^b - (1+b)| \leq \sum_{n\geq 2}^{\infty} \frac{|b|^n}{n!} \leq \sum_{n\geq 2} \frac{|b|^2}{n!} = |b|^2 \sum_{n\geq 2}^{\infty} \frac{1}{n!}$

$(\because |b| \leq 1) \quad \leq |b|^2$

(3) $\{C_n\}_{n\geq 1} \cup \{C\} \subseteq \mathbb{C}$, $C_n \to C$ (as $n \to \infty$) $\Rightarrow$

$(1+\frac{C_n}{n})^n \to e^c$ as $n \to \infty$

(proof)

Since $(1+\frac{C}{n})^n \to e^c$

hence $(1+\frac{C_n}{n})^n - e^{C_n} \to 0$

$z_m \stackrel{\text{def}}{=} (1+\frac{C_n}{n})$

$w_m \stackrel{\text{def}}{=} e^{\frac{C_n}{n}}$

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$$|z_m| \leq 1 + \frac{|c_n|}{n} \leq e^{\left|\frac{c_n}{n}\right|}$$

$$|w_m| \leq e^{\left|\frac{c_n}{n}\right|}$$

When n is large, $\left|\frac{c_n}{n}\right| \leq \left|\frac{c_n - c + c}{n}\right| \leq \left|\frac{c_n - c}{n}\right| + \left|\frac{c}{n}\right|$

$$\leq 2\left|\frac{c}{n}\right|$$

$(\because c_n \to c)$

So $|z_m|, |w_m| \leq e^{\frac{2|c|}{n}}$.

$$\theta \overset{\text{def}}{=} e^{\frac{2|c|}{n}}$$

$$\therefore \left|\prod_{m=1}^{n} z_m - \prod_{m=1}^{n} w_m\right| \leq \theta^{n-1} \sum_{m=1}^{n} |z_m - w_m| \qquad (\because (1))$$

$$\leq e^{2|c|(1-\frac{1}{n})} \sum_{m=1}^{n} \left|e^{\frac{c_n}{n}} - 1 - \frac{c_n}{n}\right|$$

$$= e^{2|c|(1-\frac{1}{n})} \cdot n \cdot \left|e^{\frac{c_n}{n}} - 1 - \frac{c_n}{n}\right|$$

$\downarrow (2)$

$$\leq e^{2|c||1-\frac{1}{n}|} \cdot n \cdot \left|\frac{c_n}{n}\right|^2$$

$$= e^{2|c|(1-\frac{1}{n})} \frac{|c_n|^2}{n} \to 0 \qquad (\text{as } n \to \infty)$$

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$\boxed{48}$ Let $\{\xi_i\}_{i\geq 1}$ be i.i.d random variables

$P(\xi_i = 1) = \frac{1}{2}$, $P(\xi_i = -1) = \frac{1}{2}$.

$S_{2^{n-1}} \overset{def}{=} \xi_1 (1+\xi_2)(1+\xi_3) \cdots (1+\xi_n)$

Let $X_1 = \xi_1$ $X_2 = \xi_1 \xi_2$ $X_3 = \xi_1 \xi_3$ $\cdots$

$X_4 = \xi_1 \xi_2 \xi_3$ $\cdots$ $\left(X_{2^{n-1}+j} \overset{def}{=} X_j \xi_{n+1} \quad n \geq 2 \right)$

And $S_{2^{n-1}} \overset{def}{=} X_1 + X_2 + \cdots + X_{2^{n-1}}$

$X_1 \sim X_{2^{n-1}}$ are pairwise independent and identically distributed

$\frac{S_{2^{n-1}}}{2^{n-1}} \to 0$ (as)

$(= E[X_i])$

$\left(\exists j \geq 2 \text{ st } \xi_j = -1 \right)$

However $P\left(\frac{S_{2^{n-1}}}{(2^{n-1})^{\frac{1}{2}}} = 0 \right)$

$= P(S_{2^{n-1}} = 0) = 1 - \left(\frac{1}{2}\right)^{n-1} \to 1$ (as $n \to \infty$).

Hence $\frac{S_{2^{n-1}}}{(2^{n-1})^{\frac{1}{2}}} \xrightarrow{d} 0$ (not converge in normal distribution)

So The Central Limit Theorem does not hold

$\boxed{49}$ $I_k = \begin{cases} 1 & \text{if the k-th is 6} \\ 0 & \text{else} \end{cases}$

$\therefore I_1, I_2 \cdots I_{180} \overset{iid}{\sim} \text{bernoulli}(\frac{1}{6})$ $\quad E[I_1] = \frac{1}{6} = \mu$

$V[I_1] = \frac{5}{36} = \sigma^2$

$S_n = \sum_{k=1}^{n} I_k$

$(n=180)$

By central limit theorem.

$Z_n = \frac{S_n - n\mu}{\sqrt{n}\,\sigma} \xrightarrow{d} N(0,1)$

fewer than 25 ⇒ at most 24 times

$P(\underset{\sim}{S_n < 24.5}) = P(S_n - n\mu \leq 24.5 - n\mu)$ (⇒ by rounding $24.4999 \doteq 24$)

(continuous modification)

$= P\left(\frac{S_n - n\mu}{\sqrt{n}\,\sigma} \leq \frac{24.5 - n\mu}{\sqrt{n} \cdot \sigma}\right) = P\left(Z_n \leq \frac{24.5 - 30}{\sqrt{180}\sqrt{\frac{5}{36}}}\right)$

$= P(Z_n \leq -1.1) \approx \Phi(-1.1)$

($\Phi(\cdot)$; cdf of $N(0,1)$)

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$\boxed{50}$ $2(\sqrt{S_n}-\sqrt{n}) = \dfrac{2(S_n-n)}{\sqrt{S_n}+\sqrt{n}} = \underbrace{\dfrac{(S_n-n)}{\sqrt{n}}}_{①} \cdot \underbrace{\dfrac{2\sqrt{n}}{\sqrt{S_n}+\sqrt{n}}}_{②}$

① By central limit theorem. $① \xrightarrow{d} N(0, \sigma^2)$

$② = \dfrac{2\sqrt{n}}{\sqrt{S_n}+\sqrt{n}} = \dfrac{2}{\sqrt{\frac{S_n}{n}}+1}$

By SLLN. $\dfrac{S_n}{n} \to 1$ (a.s)

$g(x) = \dfrac{2}{1+\sqrt{x}}$ is continuous at $x>0$

$\therefore g(\frac{S_n}{n}) \to 1$ (a.s) $\quad \therefore ② \to 1$ (a.s)

$\Rightarrow ② \xrightarrow{P} 1.$

We recall $\boxed{25}$ (in ch3) $\quad ① \xrightarrow{d} N(0, \sigma^2)$

$② \xrightarrow{P} 1$

Then $①② \xrightarrow{d} N(0, \sigma^2)\cdot 1 = \underset{\sim\sim\sim}{N(0, \sigma^2)}$

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$\boxed{5}$ $\sum_{m=1}^{n} X_m \Big/ \left(\sum_{m=1}^{n} X_m^2\right)^{\frac{1}{2}}$

$$= \underbrace{\frac{\sum_{m=1}^{n} X_m}{\sqrt{n}}}_{①} \cdot \underbrace{\frac{\sqrt{n}}{\left(\sum_{m=1}^{n} X_m^2\right)^{\frac{1}{2}}}}_{②}$$

$① \xrightarrow{d} N(0, \sigma^2)$

$② = \left(\frac{X_1^2 + \cdots X_n^2}{n}\right)^{-\frac{1}{2}}$

By SLLN, $\frac{X_1^2 + \cdots + X_n^2}{n} \xrightarrow{a.s} \sigma^2 > 0$

Let $g(x) = x^{-\frac{1}{2}}$ $(x>0)$

Then $g\left(\frac{X_1^2 + \cdots + X_n^2}{n}\right) \to \frac{1}{\sigma}$ (a.s) ($\Rightarrow$ in Probability)

$\Rightarrow ② \xrightarrow{p} \frac{1}{\sigma}$

$\therefore$ By $\boxed{25}$, $①\cdot② \xrightarrow{d} N(0,1)$

$\boxed{52}$ We use $\boxed{24}$.

- $S_{a_n}/\sigma\sqrt{a_n} \xrightarrow{d} N(0,1)$ by central limit theorem
- If $S_{N_n}/\sigma\sqrt{a_n} - S_{a_n}/\sigma\sqrt{a_n} \xrightarrow{P} 0$

Then $S_{N_n}/\sigma\sqrt{a_n} \xrightarrow{d} N(0,1)$ (by $\boxed{24}$)

- It's enough to show that $S_{N_n}/\sigma\sqrt{a_n} - S_{a_n}/\sigma\sqrt{a_n} \xrightarrow{P} 0$.

$\Leftrightarrow$ $\dfrac{S_{N_n}-S_{a_n}}{\sqrt{a_n}} \xrightarrow{P} 0$.

Let $\varepsilon>0$. Consider $P\left(\left|\dfrac{S_{N_n}-S_{a_n}}{\sqrt{a_n}}\right|>\varepsilon\right)$
$\delta>0$

$= P\left(\cdot\,,\ \dfrac{N_n}{a_n}\in[1,1+\delta]\right) + P\left(\cdot\,,\ \dfrac{N_n}{a_n}\notin[1,1+\delta]\right)$

$\le P\left(\left|\dfrac{S_{N_n}-S_{a_n}}{\sqrt{a_n}}\right|>\varepsilon,\ \dfrac{N_n}{a_n}\in[1,1+\delta]\right)$ ... ①

$+ P\left(\dfrac{N_n}{a_n}\notin[1,1+\delta]\right)$ ... ②

② .. Since $\dfrac{N_n}{a_n}\xrightarrow{P}1$. so ② $\to 0$ (as $n\to\infty$)

Next consider ①.

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$$① = P\left(\left| \frac{S_{N_n} - S_{a_n}}{\sqrt{a_n}} \right| > \varepsilon,\ N_n \in [a_n, (1+\delta)a_n] \right)$$

$$\leq P\left(\max_{k \in [a_n, (1+\delta)a_n] \cap \mathbb{N}} |S_k - S_{a_n}| > \varepsilon \sqrt{a_n} \right) \quad ①^*$$

By Kolmogorov's Maximal Inequality (ch2. $\boxed{62}$)

$$\leq \frac{1}{(\varepsilon \cdot \sqrt{a_n})^2} \cdot V\left[S_{[(1+\delta)a_n]} - S_{a_n} \right]$$

$$\leq \frac{1}{\varepsilon^2 a_n} \cdot \sigma^2 \cdot \delta \cdot a_n = \frac{\sigma^2 \delta}{\varepsilon^2}.$$

Therefore $\limsup P\left(\left| \frac{S_{N_n} - S_{a_n}}{\sqrt{a_n}} \right| > \varepsilon \right) \leq \frac{\sigma^2 \delta}{\varepsilon^2}$

Since $\delta > 0$ is arbitrary, hence by taking $\delta \downarrow 0$

We have the desired result. ∎

$\boxed{53}$ $\boxed{\text{Step 1}}$

· $S_{N_t} \leq t < S_{N_t+1}$ (by definition)

$$\frac{S_{N_t}}{N_t} \leq \frac{t}{N_t} < \frac{S_{N_t+1}}{N_t} = \frac{S_{N_t+1}}{N_t+1} \cdot \frac{N_t+1}{N_t}$$

By taking $t \uparrow \infty$. $N_t \uparrow \infty$. ($\forall \omega \in \Omega$)

($\because 0 < Y_i(\omega) < \infty \quad \therefore S_m < \infty$)

Hence by SLLN, $t \to \infty$ $\frac{S_{N_t}}{N_t} \to \mu$ (a.s)

$\frac{S_{N_t+1}}{N_t} \to \mu$ (a.s) $\left(\because \frac{N_t+1}{N_t} \to 1\right)$

So $\frac{t}{N_t} \to \mu$ (a.s)

$\therefore \frac{t}{\mu N_t} \to 1$ (a.s) (hence in Probability)

$\boxed{\text{Step 2}}$

By Central Limit Theorem, and $t \uparrow \infty \Rightarrow N_t \uparrow \infty$

$$\frac{S_{N_t} - \mu N_t}{\sigma (N_t)^{\frac{1}{2}}} \xrightarrow{d} N(0,1)$$

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By $\boxed{52}$ and $\boxed{\text{Step 1}}$, $\left(\dfrac{\left(\frac{t}{\mu}\right)}{N_t} \to 1 \right.$ (a.s) (hence in probability)$\left.\right)$

We have $\dfrac{S_{N_t} - \mu N_t}{\sigma \cdot \left(\frac{t}{\mu}\right)^{\frac{1}{2}}} \xrightarrow{d} N(0,1)$

$$\Rightarrow \quad \frac{\mu N_t - S_{N_t}}{\sigma\left(\frac{t}{\mu}\right)^{\frac{1}{2}}} \xrightarrow{d} N(0,1) \quad \cdots$$

$\boxed{\text{Step 3}}$

In order to obtain the desired result,

It's enough to show $\dfrac{\mu N_t - t}{\sigma\left(\frac{t}{\mu}\right)^{\frac{1}{2}}} - \dfrac{\mu N_t - S_{N_t}}{\sigma\left(\frac{t}{\mu}\right)^{\frac{1}{2}}} \xrightarrow{P} 0$

$\Leftrightarrow \quad \dfrac{S_{N_t} - t}{\sigma\left(\frac{t}{\mu}\right)^{\frac{1}{2}}} \xrightarrow{P} 0. \quad \Leftrightarrow \quad \underbrace{\dfrac{t - S_{N_t}}{\sqrt{t}}}_{} \xrightarrow{P} 0$

We show that $\dfrac{t - S_{N_t}}{\sqrt{t}} \xrightarrow{P} 0$, (as $t \to \infty$)

$$P\left(\frac{|t - S_{N_t}|}{\sqrt{t}} > \varepsilon \right) \leq P\left(\frac{Y_{N_t+1}}{\sqrt{t}} > \varepsilon \right) \quad \cdots \circledast$$

because $S_{N_t} \leq t < S_{N_t+1}$ and $Y_1, Y_2, \cdots$ are positive.

$$⊛ \cdots P\left(\frac{Y_{N_t+1}}{\sqrt{t}} > \varepsilon\right) = P\left(\ \cdot\ ,\ \frac{\mu(N_t+1)}{t} \in [1, 1+\delta]\right)$$

$$+ P\left(\ \cdot\ ,\ \frac{\mu(N_t+1)}{t} \notin [1, 1+\delta]\right) \quad (\delta > 0)$$

$$\leq P\left(\frac{Y_{N_t+1}}{\sqrt{t}} > \varepsilon,\ N_t+1 \in \left[\frac{t}{\mu}, \frac{t(1+\delta)}{\mu}\right]\right) \cdots ①$$

$$+ P\left(\frac{\mu(N_t+1)}{t} \notin [1, 1+\delta]\right) \cdots ②$$

② ∵ Since $\frac{t}{\mu N_t} \to 1$ (as) $\Rightarrow \frac{t}{\mu N_t} \cdot \frac{N_t}{N_t+1} \to 1$ (as)

$\Rightarrow \frac{\mu(N_t+1)}{t} \to 1$ (as) $\quad \therefore$ ② $\to 0$ (as $t \to \infty$)

$$① \cdots \leq P\left(\max_{1 \leq k \leq \frac{t(1+\delta)}{\mu}} \frac{Y_k}{\sqrt{t}} > \varepsilon\right) = P\left(\bigcup_{k=1}^{\frac{t}{\mu}(1+\delta)} \left\{\frac{Y_k}{\sqrt{t}} > \varepsilon\right\}\right)$$

(sub-additivity
i.i.d

$$\leq \frac{t(1+\delta)}{\mu} \cdot P\left(Y_1 > \varepsilon\sqrt{t}\right) = \frac{t(1+\delta)}{\mu} \cdot E\left[\mathbb{I}\{Y_1 > \varepsilon\sqrt{t}\}\right]$$

$$= \frac{t(1+\delta)}{\mu} \cdot E\left[\left(\frac{Y_1}{\varepsilon\sqrt{t}}\right)^2 \cdot \mathbb{I}\{Y_1 > \varepsilon\sqrt{t}\}\right] \qquad \left(\frac{Y_1}{\varepsilon\sqrt{t}} > 1\right)$$

$$= \frac{1+\delta}{\mu \cdot \varepsilon^2} \cdot E\left[Y_1^2 \cdot \mathbb{I}\{Y_1 > \varepsilon\sqrt{t}\}\right] \to 0 \quad (\text{as } t \to \infty)$$

$\boxed{*}$ Because Y_1^2 is integrable, so by L.D.C.T

$$E\left[\lim_{t \to \infty} (\sim)\right] = E[0]$$

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$\boxed{5-1}$ $\varphi_{n,m}(t) \overset{\text{def}}{=} E[e^{itX_{n,m}}]$

$\varphi_n(t) \overset{\text{def}}{=} E[e^{it(X_{n,1}+\dots+X_{n,n})}] = \prod_{m=1}^{n} \varphi_{n,m}(t)$

We show that $\varphi_n(t) \to \varphi(t) = \exp\left(\frac{-\sigma^2 t^2}{2}\right)$ (as $n\to\infty$)

$\boxed{\text{step 1}}$ $\sigma_{n,m}^2 \overset{\text{def}}{=} E[X_{n,m}^2]$ (See $\boxed{9}$ in ch3)

- $C_{n,m} \overset{\text{def}}{=} \sigma_{n,m}^2 \cdot \left(\frac{-t^2}{2}\right)$

Then $\sum_{m=1}^{n} C_{n,m} = \frac{-t^2}{2} \sum_{m=1}^{n} \sigma_{n,m}^2 \to \frac{-\sigma^2 t^2}{2}$ (as $n\to\infty$) ... ①

(by assumption)

- $\sum_{m=1}^{n} |C_{n,m}| = \frac{t^2}{2} \cdot \sum_{m=1}^{n} \sigma_{n,m}^2$

For sufficiently large n $\sum_{m=1}^{n} \sigma_{n,m}^2 < \infty$

($\because \sum_{m=1}^{n} \sigma_{n,m}^2 \to \sigma^2 < \infty$ as $n\to\infty$)

Without loss of generality, we may suppose

$\sup_{n\geq 1} \sum_{m=1}^{n} |C_{n,m}| < \infty$. ... ②

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- $\max_{1\le m\le n} E[X_{n,m}^2]$

$= \max_{1\le m\le n} \left\{ E[X_{n,m}^2 \cdot \mathbb{I}\{|X_{n,m}|>\varepsilon\}] + E[X_{n,m}^2 \cdot \mathbb{I}\{|X_{n,m}|\le\varepsilon\}] \right\}$

$\le \varepsilon^2 + \max_{1\le m\le n} E[X_{n,m}^2 \cdot \mathbb{I}\{|X_{n,m}|>\varepsilon\}]$

$\le \varepsilon^2 + \sum_{m=1}^{n} E[X_{n,m}^2 \cdot \mathbb{I}\{|X_{n,m}|>\varepsilon\}]$

$\limsup_{n\to\infty} E[X_{n,m}^2] \le \varepsilon^2$ and by taking $\varepsilon \downarrow 0$

we have $\max_{1\le m\le n} E[X_{n,m}^2] \to 0$ $\left(\therefore \max_{1\le m\le n} \sigma_{n,m}^2 \to 0\right)$

Hence $\max_{1\le m\le n} \left|\frac{(-t^2)}{2} \cdot \sigma_{n,m}^2\right| = \max_{1\le m\le n} |C_{n,m}| \to 0$ (as $n\to\infty$)

... ③

By ①~③

$$\begin{cases} \cdot \max_{1\le m\le n} |C_{n,m}| \to 0 \\ \cdot \sum_{m=1}^{n} C_{n,m} \to \lambda = \frac{-\sigma^2 t^2}{2} \\ \cdot \sup_{n\ge 1} \sum_{m=1}^{n} |C_{n,m}| < \infty \end{cases}$$

So $\prod_{m=1}^{n}(1+C_{n,m}) \to \exp\left(\frac{-\sigma^2 t^2}{2}\right)$ (by $\boxed{9}$)

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$$\underset{\wedge}{\varphi(t)}$$

$\boxed{\text{Step 2}}$ Consider $\left|\prod_{m=1}^{n}\varphi_{m,n}(t)-\prod_{m=1}^{n}(1+C_{n,m})\right|$ (for each fixed $t\in\mathbb{R}$)

$\cdot\ |\varphi_{m,n}(t)|\leq 1 \quad (\because \text{chf})$

$\cdot\ |1+C_{n,m}|=\left|1-\frac{1}{2}\sigma_{n,m}^2t^2\right|\leq 1$

By $\boxed{41}$ (2), $\left|\prod_{m=1}^{n}\varphi_{m,n}(t)-\prod_{m=1}^{n}(1+C_{n,m})\right|\leq\sum_{m=1}^{n}\left|\varphi_{n,m}(t)-1-C_{n,m}\right|$

$$=\sum_{m=1}^{n}\left|E\left[e^{itX_{n,m}}\right]-E\left[1+itX_{n,m}+\frac{(it)^2}{2}X_{n,m}^2\right]\right|$$

$$\leq\sum_{m=1}^{n}E\left|e^{itX_{n,m}}-1-itX_{n,m}-\frac{(it)^2}{2}X_{n,m}^2\right|$$

$$=\sum_{m=1}^{n}E\left|e^{itX_{n,m}}-\sum_{k=0}^{2}\frac{(itX_{n,m})^k}{k!}\right| \quad \downarrow \boxed{36}\ (1)$$

$$\leq\sum_{m=1}^{n}E\left[\min\{|t|^3|X_{n,m}|^3,\ 2t^2|X_{n,m}|^2\}\right] \quad \downarrow \text{separate into two terms}$$

$$=\sum_{m=1}^{n}E\left[(\sim)\cdot\mathbb{I}\{|X_{n,m}|>\varepsilon\}+(\sim)\cdot\mathbb{I}\{|X_{n,m}|\leq\varepsilon\}\right]$$

$$\leq\sum_{m=1}^{n}2t^2\cdot E\left[|X_{n,m}|^2\cdot\mathbb{I}\{|X_{n,m}|>\varepsilon\}\right]$$

$$+\sum_{m=1}^{n}|t|^3\cdot E\left[|X_{n,m}|^3\cdot\mathbb{I}\{|X_{n,m}|\leq\varepsilon\}\right] \qquad (|X_{n,m}|^3=|X_{n,m}|^2\cdot|X_{n,m}|)$$

$$\leq 2t^2\sum_{m=1}^{n}E\left[|X_{n,m}|^2\cdot\mathbb{I}\{|X_{n,m}|>\varepsilon\}\right]$$

$$\varepsilon\cdot|t|^3\sum_{m=1}^{n}E\left[|X_{n,m}|^2\cdot\mathbb{I}\{|X_{n,m}|\leq\varepsilon\}\right]$$

$$\leq 2t^2\sum_{m=1}^{n}E\left[|X_{n,m}|^2\cdot\mathbb{I}\{|X_{n,m}|>\varepsilon\}\right]$$

$$+\ \varepsilon|t|^3\sum_{m=1}^{n}E\left[|X_{n,m}|^2\right]$$

So $\limsup_{n\to\infty} \left| \prod_{m=1}^{n} \varphi_{n,m}(t) - \prod_{m=1}^{n} (1+C_{n,m}) \right|$

$\leq \varepsilon |t|^3 \cdot \sigma^2$.

Hence by taking $\varepsilon \downarrow 0$. we have $\limsup_{n\to\infty} \left| \prod_{m=1}^{n} \varphi_{m,n}(t) - \prod_{m=1}^{n} (1+C_{n,m}) \right|$

$= 0$.

So by $\boxed{\text{Step 1}}$, $\prod_{m=1}^{n} \varphi_{m,n}(t) \to \exp\left(-\frac{1}{2}\sigma^2 t^2\right)$

$\boxed{55}$ We use Lindeberg-Feller Theorem

$$X_{n,m} \overset{def}{=} \frac{Y_m - \frac{1}{m}}{\log(n)^{\frac{1}{2}}}$$

$$E[X_{n,m}] = 0$$

$$E[X_{n,m}^2] = \frac{1}{\log(n)} V[Y_m] = \frac{1}{m} - \frac{1}{m^2}$$

$$\therefore \sum_{m=1}^{n} E[X_{n,m}^2] = \frac{\sum_{m=1}^{n}\left(\frac{1}{m} - \frac{1}{m^2}\right)}{\log(n)}$$

Since $\log(n) \leq \sum_{m=1}^{n} \frac{1}{m} \leq \log(n)+1$ and $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

$$\therefore \frac{\log n - \frac{\pi^2}{6}}{\log n} \leq \sum_{m=1}^{n} E[X_{n,m}^2] \leq \frac{\log n + 1}{\log n}$$

$$\therefore n \to \infty, \quad \sum_{m=1}^{n} E[X_{n,m}^2] \to 1 \qquad \therefore \sigma^2 = 1$$

Next let $\varepsilon > 0$. (an arbitrary positive number)

Consider $\sum_{m=1}^{n} E[X_{n,m}^2 \cdot \mathbb{I}\{|X_{n,m}| > \varepsilon\}]$

$$= \sum_{m=1}^{n} E[X_{n,m}^2 \, \mathbb{I}\{|Y_m - \tfrac{1}{m}| > \varepsilon \cdot (\log n)^{\frac{1}{2}}\}]$$

Since $|Y_m - \frac{1}{m}| \leq 2$ when n is large $\varepsilon(\log n)^{\frac{1}{2}} \geq 2$

Then $X_{n,m}^2 \cdot \mathbb{I}\{|X_{n,m}| > \varepsilon\} = 0$. So $\lim_{n\to\infty} \sum_{m=1}^{n} E[X_{n,m}^2 \cdot \mathbb{I}\{|X_{n,m}| > \varepsilon\}] = 0$

By Lindeberg-Feller's theorem $\sum_{m=1}^{n} X_{n,m} \xrightarrow{d} N(0,1)$

So $\dfrac{S_n - \sum_{m=1}^{n} \frac{1}{m}}{(\log n)^{\frac{1}{2}}} \xrightarrow{d} N(0,1)$.

And $\left| \dfrac{S_n - \log n}{(\log n)^{\frac{1}{2}}} - \dfrac{S_n - \sum_{m=1}^{n} \frac{1}{m}}{(\log n)^{\frac{1}{2}}} \right| \le \dfrac{1}{\{\log(n)\}^{\frac{1}{2}}} \to 0$

(as $n \to \infty$)

$\Rightarrow \dfrac{S_n - \log(n)}{(\log n)^{\frac{1}{2}}} - \dfrac{S_n - \sum_{m=1}^{n} \frac{1}{m}}{(\log n)^{\frac{1}{2}}} \xrightarrow{P} 0$

By $\boxed{24}$, $\dfrac{S_n - \log(n)}{\{\log(n)\}^{\frac{1}{2}}} \xrightarrow{d} N(0,1)$

$\boxed{56}$ Here we give an alternative proof of Kolmogorov's three series theorem

(theorem)

- $\sum_{n=1}^{\infty} X_n(w)$: converge (as)

$\Downarrow \ ^{\forall}A>0 \qquad \Uparrow \ ^{\exists}A>0$

$(Y_n = X_n \cdot \mathbb{I}\{|X_n| \le A\})$

$$\begin{cases} \bullet \ \sum_{n=1}^{\infty} P(|X_n|>A) < \infty & \cdots ① \\ \bullet \ \sum_{n=1}^{\infty} E[Y_n] : \text{converges} & \cdots ② \\ \bullet \ \sum_{n=1}^{\infty} V[Y_n] < \infty & \cdots ③ \end{cases}$$

Proof of ($\Uparrow$) is same as $\boxed{70}$ (ch2)

Now we give alternative proof of ($\Downarrow$).

$\boxed{\text{Step 1}}$ Suppose $\sum_{n=1}^{\infty} X_n(w)$: converges (a.s)

If ① does not hold $\Leftrightarrow$ $^{\exists}A>0$ $\sum_{n=1}^{\infty} P(|X_n|>A) = +\infty$

Then by Borel-Cantelli's lemma (II), $P(|X_n|>A \ \text{i.o}) = 1$

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$\#\{n \mid |X_n| > A\} = \infty$ (a.s)

$\limsup_{n\to\infty} |X_n| \geq A > 0$ (a.s)

$\Rightarrow \sum_{n=1}^{\infty} X_n(\omega)$ does not converge (a.s)

$\Rightarrow$ Contradict the assumption.

Hence ① must hold.

Now that ① holds, we have $\sum_{n=1}^{\infty} Y_n(\omega)$ converge (a.s)

because $\sum_{n=1}^{\infty} P(|X_n| > A) = \sum_{n=1}^{\infty} P(X_n \neq Y_n) < \infty$

$\therefore P(\{X_n \neq Y_n\} \text{ i.o}) = 0$

$\Rightarrow \#\{n \mid X_n \neq Y_n\} < \infty$ (a.s)

$\Rightarrow \sum_{n=1}^{\infty} |X_n - Y_n| < \infty$ (a.s)

(Only finite number of $|X_n - Y_n| > 0$.)

$\Rightarrow \sum_{n=1}^{\infty} Y_n = \sum_{n=1}^{\infty} (Y_n - X_n + X_n) = \sum_{n=1}^{\infty} (Y_n - X_n) + \sum_{n=1}^{\infty} X_n$: converges (a.s)

$(\because \sum_{n=1}^{\infty} |X_n - Y_n| < \infty)$

So we have $\sum_{n=1}^{\infty} Y_n(w)$ converges (a.s)

Next we prove ③.

Suppose $\sum_{n=1}^{\infty} V[Y_n] = \infty$ and derive contradiction.

Let $a_n = \sum_{m=1}^{n} V[Y_m]$. Then $a_n \uparrow \infty$ (as $n \to \infty$)

Let $Z_{n,m} = \frac{Y_m - E[Y_m]}{a_n^{\frac{1}{2}}}$ (Y_m is bounded so $E[Y_m]$ exists)

$E[Z_{n,m}] = 0$

$E[Z_{n,m}^2] = \frac{V[Y_m]}{a_n}$ $\quad \sum_{m=1}^{n} E[Z_{n,m}^2] = 1 \quad \therefore \sigma^2 = 1$

Let $\varepsilon > 0$. (An arbitrary positive number)

Consider $\sum_{m=1}^{n} E[Z_{n,m}^2 \, \mathbf{I}\{|Z_{n,m}| > \varepsilon\}]$

$= \sum_{m=1}^{n} E[Z_{n,m}^2 \cdot \mathbf{I}\{|Y_m - E[Y_m]| > \varepsilon a_n^{\frac{1}{2}}\}]$

$\leq \sum_{m=1}^{n} E[Z_{n,m}^2 \cdot \mathbf{I}\{2A > \varepsilon a_n^{\frac{1}{2}}\}]$ ($|Y_m| \leq A$) ($\Rightarrow |E[Y_m]| \leq A$)

So when n is large $\varepsilon \cdot a_n^{\frac{1}{2}} \geq 2A$.

Hence $\lim_{n \to \infty} \sum_{m=1}^{n} E[Z_{n,m}^2 \cdot \mathbf{I}\{|Z_{n,m}| > \varepsilon\}] = 0$

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By Lindenberg-Feller's theorem

$$\sum_{m=1}^{n} \frac{Y_m - E[Y_m]}{a_n^{\frac{1}{2}}} \xrightarrow{d} N(0,1) \quad \cdots \boxed{*_1}$$

And $\sum_{n=1}^{\infty} Y_n$: converge (a.s)

So $\sum_{m=1}^{n} \frac{Y_m}{a_n^{\frac{1}{2}}} \to 0$ (a.s) $\cdots \boxed{*_2}$

($\therefore$ in probability)

$\boxed{*_1} - \boxed{*_2} \xrightarrow{d} N(0,1)$

$\therefore \sum_{m=1}^{n} \frac{-E[Y_m]}{a_n^{\frac{1}{2}}} \xrightarrow{d} N(0,1)$ $\cdots$ contradiction

($\because a_n, E[Y_m]$ are constant)

So $\sum_{n=1}^{\infty} V[Y_n] < \infty$ $\cdots$ ③ holds

Finally $\sum_{n=1}^{\infty} (Y_n - E[Y_n])$: converses (a.s)

by Kolmogorov's two series theorem.

And $\exists w \in \Omega$ s.t $\sum_{n=1}^{\infty} (Y_n(w) - E[Y_n])$ converses

$\sum_{n=1}^{\infty} Y_n(w)$ converses

So we have $\sum_{n=1}^{\infty} E[Y_n]$ converses $\cdots$ ②

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$\boxed{57}$ We cannot apply Linderberg-Feller's Theorem

(We use $\boxed{24}$.)

Consider $\{Y_m\}_{m\geq 2}$:

$$Y_m \overset{\text{def}}{=} \begin{cases} 1 & (\text{if } X_m > 0) \\ -1 & (\text{if } X_m < 0) \end{cases}$$

Then $\{X_m \neq Y_m\} = \{X_m = \pm m\}$

$\therefore P(X_m \neq Y_m) = \frac{1}{m^2}$

$\therefore \sum_{m\geq 2} P(X_m \neq Y_m) < \infty$

By Borel-Cantelli's lemma $P(\{X_m \neq Y_m \text{ i.o}\}) = 0$

$\#\{m \geq 2 \mid X_m \neq Y_m\} < \infty$ (as)

So $\sum_{m\geq 2}^{\infty} |X_m - Y_m| < \infty$ (as)

$\therefore \frac{1}{\sqrt{n}}|S_n - T_n| \leq \frac{1}{\sqrt{n}} \underbrace{\sum_{k=1}^{\infty} |X_k - Y_k|}_{<\infty} \to 0$ (as) $\left(T_n = \sum_{m=1}^{n} Y_m\right)$

$\therefore \left|\frac{S_n}{\sqrt{n}} - \frac{T_n}{\sqrt{n}}\right| \xrightarrow{P} 0$ $\quad \frac{T_n}{\sqrt{n}} \xrightarrow{d} N(0,1)$ $\quad \therefore \frac{S_n}{\sqrt{n}} \xrightarrow{d} N(0,1)$

(by $\boxed{24}$)

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$\boxed{58}$ We apply Lindeberg-Feller's Theorem

$X_{n,m} \overset{def}{=} (X_m - E[X_m]) / \sqrt{V[S_n]}$

$E[X_{n,m}] = 0 \qquad \sum_{m=1}^{n} E[X_{n,m}^2] = \sum_{m=1}^{n} \frac{V[X_m]}{V[S_n]} = 1 \quad (\sigma^2 = 1)$

Let $\varepsilon > 0$.

$\sum_{m=1}^{n} E[X_{n,m}^2 \cdot \mathbb{I}\{|X_{n,m}| > \varepsilon\}]$

$= \sum_{m=1}^{n} E[X_{n,m}^2 \cdot \mathbb{I}\{|X_m - E[X_m]| > \varepsilon \cdot \sqrt{V[S_n]}\}]$

$\leq \sum_{m=1}^{n} E[X_{n,m}^2 \cdot \mathbb{I}\{2M > \varepsilon \cdot \sqrt{V[S_n]}\}] \;\; \cdots \circledast$

Since $V[S_n] \uparrow \infty \; (= \sum_{m=1}^{n} V[X_m])$ and $M < \infty$

when n is large, $\varepsilon \sqrt{V[S_n]} \geq 2M$.

So $\circledast = 0$. (each $E[\sim] = 0$)

By Lindeberg Feller's theorem $\sum_{m=1}^{n} \frac{X_m - E[X_m]}{V[S_n]} \xrightarrow{d} N(0,1)$

$\therefore \frac{S_n - E[S_n]}{V[S_n]} \xrightarrow{d} N(0,1)$

$\boxed{59}$ We apply Lindeberg-Feller's Theorem.

Let $X_{n,m} = \frac{X_m}{\sqrt{n}}$. $E[X_{n,m}] = 0$.

$\sum_{m=1}^{n} E[X_{n,m}^2] = \sum_{m=1}^{n} \frac{E[X_m^2]}{n} = 1 \quad \therefore \sigma^2 = 1$

(for all $n \geq 1$)

Now let $\varepsilon > 0$.

$\sum_{m=1}^{n} E[X_{n,m}^2 \cdot \mathbb{I}\{|X_{n,m}| > \varepsilon\}]$

$= \sum_{m=1}^{n} E[\frac{|X_m|^2}{n} \cdot \mathbb{I}\{|X_m| > \varepsilon\sqrt{n}\}]$

$= \sum_{m=1}^{n} E[\frac{1}{n} |X_m|^{2+\delta} \cdot |X_m|^{-\delta} \cdot \mathbb{I}\{|X_m| > \varepsilon\sqrt{n}\}]$

$\leq \sum_{m=1}^{n} E[\frac{1}{n} |X_m|^{2+\delta} \cdot \frac{1}{(\varepsilon\sqrt{n})^\delta}]$

$= \sum_{m=1}^{n} \frac{1}{\varepsilon^\delta \cdot n^{1+\frac{\delta}{2}}} \cdot E[|X_m|^{2+\delta}]$

$= C \cdot \frac{1}{\varepsilon^\delta} \frac{1}{n^{\frac{\delta}{2}}} \to 0$ (as $n \to \infty$)

So $\sum_{m=1}^{n} \frac{X_m}{\sqrt{n}} \xrightarrow{d} N(0,1)$

$\boxed{60}$ (Lyapunov's theorem)

Let $X_{n,m} = \dfrac{X_m - E[X_m]}{\alpha_n}$,

$E[X_{n,m}] = 0$, $\displaystyle\sum_{m=1}^{n} E[X_{n,m}^2] = \sum_{m=1}^{n} \frac{V[X_m]}{\alpha_n^2} = \frac{V[X_1] + \cdots V[X_n]}{V[S_n]} = 1$

So $\sigma^2 = 1$.

Let $\varepsilon > 0$. Consider $\displaystyle\sum_{m=1}^{n} E[X_{n,m}^2 \cdot \mathbb{I}\{|X_{n,m}| > \varepsilon\}]$

$$= \sum_{m=1}^{n} E\left[\frac{|X_m - E[X_m]|^2}{\alpha_n^2} \cdot \mathbb{I}\{|X_m - E[X_m]| > \varepsilon \cdot \alpha_n\}\right]$$

$$= \sum_{m=1}^{n} E\left[\frac{|X_m - E[X_m]|^{2+\delta}}{\alpha_n^{2+\delta}} \cdot \underbrace{\frac{\alpha_n^{\delta}}{|X_m - E[X_m]|^{\delta}}}_{\leq \frac{1}{\varepsilon^{\delta}}} \cdot \mathbb{I}\{|X_m - E[X_m]| > \varepsilon \cdot \alpha_n\}\right]$$

$\left(\because \dfrac{1}{\varepsilon^{\delta} \alpha_n^{\delta}} \geq \dfrac{1}{|X_m - E[X_m]|^{\delta}}\right)$

$$\leq \frac{1}{\varepsilon^{\delta}} \cdot \sum_{m=1}^{n} E\left[\frac{|X_m - E[X_m]|^{2+\delta}}{\alpha_n^{2+\delta}}\right] \to 0 \quad (\text{as } n \to \infty)$$

So $\displaystyle\sum_{m=1}^{n} \frac{X_m - E[X_m]}{\alpha_n} = \frac{S_n - E[S_n]}{\alpha_n} \xrightarrow{d} N(0,1)$