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chapter 2. Solution by Toshinari Morimoto

$\boxed{1}$

(1) $P(A \cap B) = P(A)P(B)$

(2) ${}^\forall B_1, B_2 \in \mathcal{B}(\mathbb{R})$ $P(X \in B_1, Y \in B_2)$
$= P(X \in B_1) P(Y \in B_2)$

(3) ${}^\forall A \in \mathcal{F}, {}^\forall B \in \mathcal{G}$ $P(A \cap B) = P(A)P(B)$

(4) ${}^\forall A_1 \in \mathcal{F}_1 \ldots A_n \in \mathcal{F}_n$ $P(\bigcap_{i=1}^{n} A_i) = \prod_{i=1}^{n} P(A_i)$

(5) ${}^\forall B_1 \in \mathcal{B}(\mathbb{R}) \ldots B_n \in \mathcal{B}(\mathbb{R})$

$$P\left(\bigcap_{i=1}^{n} X_i \in B_i\right) = \prod_{i=1}^{n} P(X_i \in B_i)$$

(6) ${}^\forall I \subseteq \{1, 2 \ldots n\}$ $P(\bigcap_{i \in I} A_i) = \prod_{i=1}^{n} P(A_i)$

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$\boxed{2}$.

(1) $\sigma[X] \overset{def}{=} \{ X^{-1}(B) \mid B \in \mathcal{B}(\mathbb{R}) \}$
$\sigma[Y] = \{ Y^{-1}(B) \mid B \in \mathcal{B}(\mathbb{R}) \}$

pick $A_1 \in \sigma[X]$ $A_2 \in \sigma[Y]$

There exists $B_1 \in \mathcal{B}(\mathbb{R})$ and $B_2 \in \mathcal{B}(\mathbb{R})$

s.t $A_1 = X^{-1}(B_1)$ $A_2 = Y^{-1}(B_2)$

$P(A_1 \cap A_2) = P(X^{-1}(B_1) \cap Y^{-1}(B_2))$

$= P(X \in B_1 \cap Y \in B_2)$

$= P(X \in B_1) P(Y \in B_2)$

$= P(A_1) P(A_2)$

(2) Let $B_1, B_2 \in \mathcal{B}(\mathbb{R})$

$P(X \in B_1, Y \in B_2) = P(X^{-1}(B_1) \cap Y^{-1}(B_2))$

$X^{-1}(B_1) \in \mathcal{F}$ $Y^{-1}(B_2) \in \mathcal{G}$

$= P(X^{-1}(B_1)) P(Y^{-1}(B_2)) = P(X \in B_1) P(Y \in B_2)$

$\boxed{3}$

(1) $P(A^c \cap B) = P(B) - P(A\cap B) = P(B) - P(A)P(B)$
$= P(B)(1-P(A)) = P(B)P(A^c)$

$P(A\cap B^c)$ ... same as above

$P(A^c\cap B^c) = 1 - P(A\cup B) = 1 - P(A) - P(B) + P(A\cap B)$
$= 1 - P(A) - P(B) + P(A)P(B)$
$= (1-P(A))(1-P(B))$
$= P(A^c)P(B^c)$

(2) $P(I_A \in G_1, I_B \in G_2)$

$G_1, G_2 \in \mathbb{B}(\mathbb{R})$

$\{I_A \in G_1\} = \phi, A, A^c, \Omega$
$\{I_B \in G_2\} = \phi, B, B^c, \Omega$

Since ϕ. and Ω are independent of any (measurable) sets, we just have to consider A, A^c, B, B^c

$\therefore P(I_A\in G_1, I_B \in G_2) = P(A\cap B)$ or $P(A^c\cap B)$ or $P(A\cap B^c)$ or $P(A^c\cap B^c)$

By (1) they are all independent of each other,
So the proof is complete.

$\boxed{4}$ Consider $P(I_{A_{i_1}} \in G_1, \cdots I_{A_{i_k}} \in G_k)$

$1 \leq i_1 < i_2 < \cdots < i_k \leq n$

Since $(I_{A_{i_j}} \in G_j) = \phi, \Omega, A_{i_j}, A_{i_j}^c$

and ϕ, Ω are independent of any measurable

sets, we just consider independence of

$\{A_{i_1}, A_{i_1}^c\} \times \cdots \times \{A_{i_k}, A_{i_k}^c\}$.

$\{A_1 \cdots A_n\}$: independent $\Rightarrow$ $\{A_{i_1} \cdots A_{i_k}\}$: independent

Without loss of generality, we just have to

prove that if $\{A_1 \cdots A_n\}$ are independent

then $\{A_1, A_1^c\} \times \cdots \times \{A_n, A_n^c\}$: are independent.

(step 1) $A_1^c, A_2 \cdots A_n$ are independent

$P(A_1^c \cap A_2 \cdots A_n) = P(A_2 \cap \cdots \cap A_n) - P(A_1 \cap \cdots A_n)$

$= (1 - P(A_1)) P(A_2) \cdots P(A_n)$

$= P(A_1^c) P(A_2) \cdots P(A_n)$

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(Step 2) By repeating the similar discussion (at most 2^n-times)

we have $\forall A_1^* \in \{A_1, A_1^c\} \cdots A_n^* \in \{A_n, A_n^c\}$

$A_1^*, \cdots, A_n^*$ are independent

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$\boxed{5}$ Pairwise independence does not imply independence.

Suppose $P(X_i=0)=\frac{1}{2}$ $P(X_i=1)=\frac{1}{2}$ and

$X_1 \sim X_3$ are iid.

Consider events $A_1=\{X_1=X_2\}$
$A_2=\{X_2=X_3\}$
$A_3=\{X_3=X_1\}$

Then $\{A_1, A_2, A_3\}$ are pairwise independent

but not independent

$P(A_1=A_2)=P(X_1=X_2=X_3)=\left(\frac{1}{2}\right)^3 \times 2=\frac{1}{4}$

$P(A_1)\,P(A_2)=\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}$

So A_1 A_2 are independent (So are $\{A_2, A_3\}$, $\{A_3, A_1\}$)

However, $P(A_1 \cap A_2 \cap A_3)=P(X_1=X_2=X_3)=\frac{1}{4}$

$P(A_1)\,P(A_2)\,P(A_3)=\frac{1}{8}$

So $\{A_1, A_2, A_3\}$ are not independent

$\boxed{6}$ $\mathcal{A}_1, \cdots, \mathcal{A}_n$ independent and π-system

$\boxed{\text{step 1}}$ $\sigma[\mathcal{A}_1], \mathcal{A}_2 \cdots \mathcal{A}_n$ are independent

(proof) pick arbitrary sets from $\mathcal{A}_2 \sim \mathcal{A}_n$ and fix them

(Let $A_2 \in \mathcal{A}_2, \cdots, A_n \in \mathcal{A}_n$)

Consider $\mathcal{D}_1 = \{ A_1 \in \sigma[\mathcal{A}_1] \mid P(\bigcap_{i=1}^{n} A_i) = \prod_{i=1}^{n} P(A_i) \}$

$\mathcal{D}_1$ contains $\mathcal{A}_1$ because $\forall A_1 \in \mathcal{A}_1$, $A_1 \sim A_n$ are independent.

$\therefore \mathcal{A}_1 \subseteq \mathcal{D}_1$

And $\mathcal{D}_1$ is a Dynkin-system (λ-system)

- $\Omega \in \mathcal{D}_1 \cdots$ trivial
- $A_1, \hat{A}_1 \in \mathcal{D}_1$ $\quad P(\hat{A}_1 \cap A_2 \cdots \cap A_n) = P(\hat{A}_1) P(A_2) \cdots P(A_n)$
  $P(A_1 \cap \cdots \cap A_n) = P(A_1) P(A_2) \cdots P(A_n)$
  We have $P((\hat{A}_1 \setminus A_1) \cap A_2 \cdots A_n) = P(\hat{A}_1 \setminus A_1) \cdots P(A_n)$
- $\{A_{1,k}\}_{k \geq 1} \subseteq \mathcal{D}_1$ $\quad A_{1,k} \nearrow A_1$ $\quad P(A_{1,k} \cap \cdots \cap A_n)$
  $= P(A_{1,k}) \cdots P(A_n)$
  $k \to \infty$ we have $P(A_1 \cap \cdots \cap A_n) = P(A_1) \cdots P(A_n)$

So $\mathcal{D}_1$ is a Dynkin-system

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Since $\mathcal{A}_1 \subseteq \mathcal{D}_1$ and $\mathcal{D}_1$ is a Dynkin-system

$\lambda[\mathcal{A}_1] \subseteq \mathcal{D}_1$. And by π-λ theorem, we have

$\lambda[\mathcal{A}_1] = \sigma[\mathcal{A}_1] \subseteq \mathcal{D}_1$ ($\because \mathcal{A}_1$: π-system)

So we find out that $\sigma[\mathcal{A}_1], \mathcal{A}_2 \cdots \mathcal{A}_n$ are independent.

$\boxed{\text{step2}}$ In the same way, chose arbitrary sets from $\sigma[\mathcal{A}_1], \mathcal{A}_3, \mathcal{A}_4 \cdots \mathcal{A}_n$, and fix them

Let $A_1 \in \sigma[\mathcal{A}_1]$, $A_3 \in \mathcal{A}_3 \cdots A_n \in \mathcal{A}_n$

Consider $\mathcal{D}_2 = \{ A_2 \in \sigma(\mathcal{A}_2) \mid P(\bigcap_{i=1}^{n} A_i) = \prod_{i=1}^{n} P(A_i) \}$

Similary $\mathcal{A}_2 \subseteq \mathcal{D}_2$. $\mathcal{D}_2$: Dynkin-system

So $\lambda[\mathcal{A}_2] = \sigma(\mathcal{A}_2) \subseteq \mathcal{D}_2$
(π-λ system)

And we have $\sigma[\mathcal{A}_1]\ \sigma[\mathcal{A}_2] \cdots \mathcal{A}_3 \cdots \mathcal{A}_n$

are independent.

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$\boxed{\text{Step3}}$ Repeating the similar arguments again and finally we will prove that $\sigma(\alpha_1) \cdots \sigma(\alpha_n)$ are independent.

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$\boxed{7}$ We apply theorem 21.7

$\mathcal{G}_i = \{ X_i^{-1}((-\infty, a]) \mid a \in \mathbb{R} \}$ $(i=1 \sim n)$

$\mathcal{G}_1 \sim \mathcal{G}_n$ are independent and π-systems.

$\left(X_i^{-1}((-\infty, a]) \cap X_i^{-1}((-\infty, b]) = X_i^{-1}((-\infty, a \wedge b]) \right.$

$\left. a \wedge b = \min\{a, b\} \right)$

So by theorem 21.7, we have $\sigma[\mathcal{G}_1] \cdots \sigma[\mathcal{G}_n]$

are independent.

We still have to show that

$\sigma[\mathcal{G}_i] = \sigma[X_i] \overset{\text{def}}{=} \{ X_i^{-1}(B) \mid B \in \mathcal{B}(\mathbb{R}) \}$

We prove the follow fact. Let f: measurable function,

$\mathcal{E}$: family of measurable sets.

$f^{-1}(\mathcal{E}) \overset{\text{def}}{=} \{ f^{-1}(E) \mid E \in \mathcal{E} \}$
$f^{-1}(\sigma[\mathcal{E}]) \overset{\text{def}}{=} \{ f^{-1}(B) \mid B \in \sigma[\mathcal{E}] \}$

Then $\sigma[f^{-1}(\mathcal{E})] = f^{-1}(\sigma[\mathcal{E}])$, (proof) Since $\sigma[\mathcal{E}]$ is a σ-algebra, thus $f^{-1}(\sigma[\mathcal{E}])$ is also a σ-algebra. (easy to verify)

And $f^{-1}(\mathcal{E}) \subseteq f^{-1}(\sigma[\mathcal{E}])$ therefore we have

$\sigma[f^{-1}(\mathcal{E})] \subseteq f^{-1}(\sigma[\mathcal{E}])$.

Next we prove $\sigma[f^{-1}(\mathcal{E})] \supseteq f^{-1}(\sigma[\mathcal{E}])$

We consider $\{B \mid f^{-1}(B) \in \sigma[f^{-1}(\mathcal{E})]\}$

This is also a σ-algebra. ($\because$ $\sigma[f^{-1}(\mathcal{E})]$ is a σ-algebra)

This contains $\mathcal{E}$, thus it also contains $\sigma[\mathcal{E}]$.

So $\forall B \in \sigma[\mathcal{E}]$ $f^{-1}(B) \in \sigma[f^{-1}(\mathcal{E})]$

This implies $f^{-1}(\sigma[\mathcal{E}]) \subseteq \sigma[f^{-1}(\mathcal{E})]$. ∎

Using this fact $\sigma[\mathcal{G}_i] \overset{\text{equal}}{=} \{X_i^{-1}(B) \mid B \in \sigma[\mathcal{E}]$ $\quad \mathcal{E} = \{(-\infty, a] \mid a \in \mathbb{R}\}$

$= \{X_i^{-1}(B) \mid B \in \mathcal{B}(\mathbb{R})\}$

$= \sigma(X_i)$

Now the proof is complete.

$\boxed{8}$ Consider $\mathcal{A}_i = \{ \bigcap_{j=1}^{m_i} A_{ij} \mid A_{ij} \in \mathcal{F}_{ij} \}$
$(i = 1 \sim n)$

Then $\{\mathcal{A}_i\}_{i=1}^{n}$ are π-systems and independent

(I) $\mathcal{A}_i$ is a π-system. :

$$\underbrace{\left(\bigcap_{j=1}^{m_i} A_{ij} \right)}_{\in \mathcal{A}_i} \cap \underbrace{\left(\bigcap_{j=1}^{m_i} B_{ij} \right)}_{\in \mathcal{A}_i}$$

$$= \bigcap_{j=1}^{m_i} (A_{ij} \cap B_{ij})$$

Since $\mathcal{F}_{ij}$ is a σ algebra for each (i,j)

So $A_{ij} \cap B_{ij} \in \mathcal{F}_{ij}$. So $\bigcap_{j=1}^{m_i} A_{ij} \cap B_{ij} \in \mathcal{A}_i$

(II) $\mathcal{A}_1 \sim \mathcal{A}_n$ are independent

Let $I \subset \{1 \cdots n\}$ and $A_{ij} \in \mathcal{F}_{ij}$.

$$P\left(\bigcap_{i \in I} \bigcap_{j=1}^{m_i} A_{ij} \right) = \prod_{i \in I} \prod_{j=1}^{m_i} P(A_{ij}) \quad (\because \{\mathcal{F}_{ij}\} \text{ are indep})$$

$$= \prod_{i \in I} P\left(\bigcap_{j=1}^{m_i} A_{ij} \right)$$

So $\mathcal{A}_1 \sim \mathcal{A}_n$ are independent

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By theorem 2.1.7 $\sigma[\mathcal{A}_1) \cdots \sigma(\mathcal{A}_n)$ are independent.

$\boxed{\text{claim}}$ $\mathcal{G}_i = \sigma[\mathcal{A}_i]$ $(i=1 \sim n)$

① $\mathcal{G}_i \subseteq \sigma[\mathcal{A}_i]$

We prove $\bigcup_{j=1}^{m_i} \mathcal{F}_{ij} \subseteq \sigma[\mathcal{A}_i]$

If $A \in \bigcup_{j=1}^{m_i} \mathcal{F}_{ij}$, there exists j_0 s.t

$A \in \mathcal{F}_{ij_0}$. We rewrite A as A_{ij_0}.

Since $A_{ij_0} = \Omega \cap \Omega \cap \cdots \underbrace{A_{ij_0}}_{\in \mathcal{F}_{ij_0}} \cap \cdots \Omega \in \mathcal{A}_i$

$\Rightarrow A_{ij_0} \in \sigma[\mathcal{A}_i]$

Thus $\bigcup_{j=1}^{m_i} \mathcal{F}_{ij} \subseteq \sigma(\mathcal{A}_i)$ $\left(\Rightarrow \underbrace{\sigma[\bigcup_{j=1}^{m_i} \mathcal{F}_{ij})}_{= \mathcal{G}_i} \subseteq \sigma(\mathcal{A}_i)\right)$

② $\mathcal{G}_i \supseteq \sigma[\mathcal{A}_i]$

It's enough to show that $\mathcal{A}_i \subseteq \mathcal{G}_i$

$A_i = A_{i1} \cap A_{i2} \cdots \cap A_{im_i}$

$\{A_{i1} \cdots A_{im_i}\} \subseteq \mathcal{G}_i$ and $\mathcal{G}_i$ is a σ-algebra

Thus $\bigcap_{j=1}^{m_i} A_{ij} \in \mathcal{G}_i$. $\therefore \mathcal{A}_i \subseteq \mathcal{G}_i$ $(\Rightarrow \sigma(\mathcal{A}_i) \subseteq \mathcal{G}_i)$

Now the proof is complete ■

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$\boxed{9}$ $\mathcal{F}_{ij} \overset{\text{def}}{=} \sigma[X_{ij}] = \{X_{ij}^{-1}(B) \mid B \in \mathcal{B}(\mathbb{R})\}$

$\mathcal{G}_i \overset{\text{def}}{=} \sigma\left[\bigcup_{j=1}^{m_i} \mathcal{F}_{ij}\right]$

By theorem 2.1.9. $\mathcal{G}_1 \sim \mathcal{G}_n$ are independent

We prove that $f_i(X_{i1}, \dots X_{i,m_i})$

is a $\mathcal{G}_i$-measurable random variable

claim) $f_i(X_{i1} \cdots X_{i,m_i})$ is $\mathcal{G}_i$-measurable

Let $B \in \mathcal{B}(\mathbb{R})$. $(f_i(X_{i1} \dots X_{i,m_i}))^{-1}(B)$

$= (X_{i1}, \dots, X_{i,m_i})^{-1} \circ f_i^{-1}(B)$

Since f_i is $\mathbb{R}^{m_i}/\mathbb{R}$ measurable
thus $f_i^{-1}(B) \in \mathcal{B}(\mathbb{R}^{m_i})$.

So we prove that $\forall B \in \mathcal{B}(\mathbb{R}^{m_i})$,

$(X_{i1} \cdots X_{i,m_i})^{-1}(B) \in \mathcal{G}_i$.

However. $\left\{\prod_{i=1}^{m_i} (-\infty, a_i] \mid a_i \in \mathbb{R}\right\}$ generates $\mathcal{B}(\mathbb{R}^{m_i})$,

it's enough to show that $\forall \prod_{i=1}^{m_i} (-\infty, a_i]$.

$(X_{i,1} \cdots X_{i,m_i})^{-1}\left(\prod_{j=1}^{m_i}(-\infty, a_{ij}]\right) \in \mathcal{G}_i$

(see , $\boxed{\text{NOTE}}$)

And $(X_{i,1} \cdots X_{i,m_i})^{-1}\left(\prod_{j=1}^{m_i}(-\infty, a_{ij}]\right)$

$= \bigcap_{j=1}^{m_i} \underbrace{X_{i,j}^{-1}((-\infty, a_{ij}])}_{\in \mathcal{F}_{ij}} \in \mathcal{A}_i = \left\{ \bigcap_{j=1}^{m_i} A_{ij} \mid A_{ij} \in \mathcal{F}_{ij} \right\}$

$\subseteq \sigma[\mathcal{A}_i] = \mathcal{G}_i$

(see $\boxed{8}$)

So the proof is complete.

$\boxed{\text{NOTE}}$ $(S, \mathcal{A}), (T, \mathcal{B})$ -- measurable spaces.

$f: S \to T$ is a $\mathcal{A}/\mathcal{B}$-measurable function

$\mathcal{B} = \sigma[\mathcal{G}]$.

Then $\forall G \in \mathcal{G}\ f^{-1}(G) \in \mathcal{A} \Rightarrow \forall B \in \mathcal{B}\ \ f^{-1}(B) \in \mathcal{A}$.

(Proof) $\{B \mid f^{-1}(B) \in \mathcal{A}\}$ is a σ-algebra containing
($\because \mathcal{A}$: σ-algebra)
$\mathcal{G}$. So $\sigma[\mathcal{G}]$ is also contained by it.
(by assumption).
Hence $\forall B \in \sigma[\mathcal{G}]\ \ f^{-1}(B) \in \mathcal{A}$.

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$\boxed{10}$ Consider a measure $\hat{\mu}$ on $(\mathbb{R}, \mathbb{B}(\mathbb{R}^n))$

which satisfies $\hat{\mu}\left(\prod_{i=1}^{n}(-\infty, x_i]\right) = P\left(\bigcap_{i=1}^{n}\{X_i \le x_i\}\right)$

Let $g = \{(-\infty, x_1] \times \dots \times (-\infty, x_n] \mid x_1 \sim x_n \in \mathbb{R}\}$

g is a π-system and $\sigma[g] = \mathbb{B}(\mathbb{R}^n)$

By assumption, $\hat{\mu}\left(\prod_{i=1}^{n}(-\infty, x_i]\right)$

$= P\left(\bigcap_{i=1}^{n}\{X_i \le x_i\}\right)$ (by independence)

$= \prod_{i=1}^{n} P(X_i \le x_i) = \prod_{i=1}^{n} \mu_i((-\infty, x_i])$

Thus on g, $\hat{\mu} = \mu_1 \times \mu_2 \times \dots \mu_n$.

And $(g, \hat{\mu})$ is σ-finite (finite) and g is a π-system.

By uniqueness of measure, $\hat{\mu} = \mu_1 \times \dots \times \mu_n$

$\boxed{\text{NOTE}}$

(The definition of $\mu_1 \times \dots \times \mu_n$ is a measure on $\mathbb{B}(\mathbb{R}^n)$

which satisfies $(\mu_1 \times \dots \mu_n)\left(\prod_{i=1}^{n}(-\infty, x_i]\right) = \prod_{i=1}^{n} \mu_i((-\infty, x_i])$

Here $\hat{\mu}$ also satisfies this condition and

We have proved such a measure is unique

So $\hat{\mu} = \mu_1 \times \dots \times \mu_n$

[1] Suppose $h \geq 0$.

$$E[h(X,Y)] = \int_\Omega h(X,Y)\,dP$$

By change Variable Formula. (We will show it again)

$$= \int_{\mathbb{R}^2} h(x,y)\,(\mu\times\nu)(dx\,dy)$$

By Tonelli's theorem.

$$= \int_{\mathbb{R}}\int_{\mathbb{R}} h(x,y)\,\mu(dx)\,\nu(dy) \quad \blacksquare$$

$\boxed{\text{NOTE}}$ We show the change variable formula for two random variables.

Let $h : \mathbb{R}^2 \to [0,\infty)$ $\quad \exists \{h_n\}_{n\geq 1}$: nonnegative simple measurable s.t. $h_n \uparrow h$

$$E[h(X,Y)] = \lim_{n\to\infty} E[h_n(X,Y)] \quad (\text{By MCT})$$

$$E[h_n(X,Y)] = \int_\Omega h_n(X,Y)\,dP$$

$$= \int_\Omega \sum_{j=1}^{J_n} a_j^{(n)} I_{A_j^{(n)}}(X(\omega), Y(\omega))\,dP \qquad \{a_j\} \subseteq [0,\infty) \quad \{A_j\} \subseteq \mathcal{B}(\mathbb{R}^2)$$

$$= \sum_{j=1}^{J_n} a_j^{(n)} \cdot P((X,Y) \in A_j^{(n)})$$

Let $\mu \sim X$, $\nu \sim Y$. By the previous exercise,

$$P \circ (X,Y)^{-1}(\cdot) = (\mu\times\nu)(\cdot)$$

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We can rewrite it as

$$= \sum_{j=1}^{J_n} a_j^{(n)} (\mu \times \nu)(A_j^{(n)})$$

$$= \int_{\mathbb{R}^2} h_n(x,y)\, (\mu\times\nu)(dx\,dy)$$

So $\lim_{n\to\infty} (\cdots) = \int_{\mathbb{R}^2} h(x,y)\, (\mu\times\nu)(dx\,dy)$

Finally by Tonelli's theorem

$$\int_{\mathbb{R}} \int_{\mathbb{R}} h(x,y)\, \mu(dx)\, \nu(dy)$$

Now the proof is complete. ∎

[12] Consider (X, Y) below.

$X \backslash Y$	-1	0	1
-1	0	b	0
0	a	c	a
1	0	b	0

$2a + 2b + c = 1$ (probability) $(a>0,\ b>0,\ c>0)$

$E[X]E[Y] = 0 \cdot 0 = 0$

$\left(E[X] = \sum_{x=-1,0,1} x\, P(X=x) = (-1) \cdot b + 0(2a+c) + 1 \cdot (b) = 0 \right.$

$E[XY] = 0$

$\left(E[XY] = \sum_{(x,y) \in \{(-1,1),(1,-1),(-1,-1),(1,1)\}} xy\, P(X=x, Y=y) = 0 \right)$

However $P(X=1, Y=1) = 0$

$P(X=1)P(Y=1) = ab > 0$

So they are not independent

[3] $X \sim \mu,\ Y \sim \nu$ (i.e. $P(X \le \lambda) = \mu((-\infty, \lambda]) = F(x)$)

$$P(X+Y \le Z) = \int_\Omega \mathbb{I}\{X+Y \le Z\}\, dP$$

By Change Variable Formula

$$= \int_{\mathbb{R}^2} \mathbb{I}\{x+y \le Z\}\, \mu \times \nu(dx\, dy)$$

By Tonelli's Theorem ($\because \mathbb{I}\{\sim\} \ge 0$: non-negative)

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{I}\{x \le Z-y\}\, \mu(dx)\, \nu(dy)$$

$$= \int_{\mathbb{R}} \mu((-\infty, Z-y])\, \nu(dy)$$

$$= \int_{\mathbb{R}} F(Z-y)\, \nu(dy)$$

(The same meaning)

$$= \int_{\mathbb{R}} F(Z-y)\, dG$$

$\boxed{14}$ By $\boxed{13}$. $P(X+Y \leq Z) = \int F(Z-y)\, dG(y)$

Since X has a probability density function f,

$$= \int_{\mathbb{R}} \int_{-\infty}^{Z-y} f(x)\, dx\, dG(y) \quad ; \; x \to x-y$$

$$= \int_{\mathbb{R}} \int_{-\infty}^{Z} f(x-y)\, dx\, dG(y) \quad ; \; \text{Tonelli's theorem}$$

$$= \int_{-\infty}^{Z} \int_{\mathbb{R}} f(x-y)\, dG(y)\, dx$$

So. $g(x) \overset{\text{def}}{=} \int_{\mathbb{R}} f(x-y)\, dG(y)$

then $P(X+Y \leq Z) = \int_{-\infty}^{Z} g(x)\, dx$

Hence $g(\cdot)$ is a pdf of Z

$\boxed{15}$ $\mathcal{G} \overset{def}{=} \{ \prod_{i=1}^{n} (a_i, b_i] \times \mathbb{R} \times \mathbb{R} \cdots \mid n \in \mathbb{N}\ \ a_i, b_i \in \mathbb{R} \}$
$\mathcal{G}$ is a semi-algebra

$G_0[\cdot]$ is a smallest algebra containing $\cdot$.

We define a set function $P_0 : \mathcal{G} \to [0, \infty]$.
as $P_0(G) = \mu_n(\prod_{i=1}^{n} (a_i, b_i])$
where $G \in \mathcal{G}$, $G = \prod_{i=1}^{n} (a_i, b_i] \times \mathbb{R} \times \mathbb{R} \times \cdots$

$\boxed{\text{Step 1}}$ First we extend $(\mathcal{G}, P_0)$ on an algebra $G_0[\mathcal{G}]$. : $(G_0[\mathcal{G}], P_0)$ which has finite additivity.

As we have proved in chapter 1, when $\mathcal{G}$ is a semi-algebra $G_0[\mathcal{G}] = \mathcal{A} \overset{def}{=} \{A \mid A$ is a finite disjoint union of sets from $\mathcal{G}\}$

And we define $P_0(A)$ $(A \in G_0[\mathcal{G}] \overset{\text{eg}}{=} \mathcal{A})$

as $P_0(A) \overset{def}{=} \sum_{j=1}^{J} P_0(G_j)$ $(A = \sum_{j=1}^{J} G_j)$

Then $(G_0[\mathcal{G}]\ P_0)$ has finite additivity and $P_0(\cdot)$ is well-defined.

(proof:) $A_1, A_2 \in \mathcal{G}$ and $A_1 \cap A_2 = \phi$
$A_1 = \sum_{j=1}^{J_1} G_{1j}$ $A_2 = \sum_{j=1}^{J_2} G_{2j}$

Σ .. disjoint union

$$P_0(A_1 \cup A_2) = P_0\left(\sum_{i=1}^{2}\sum_{j=1}^{J_i} G_{(ij)}\right) = \sum_{i=1}^{2}\sum_{j=1}^{J_i} P_0(G_{(ij)})$$
$$= \sum_{i=1}^{2} P_0(A_i)$$

So it has finite additivity.

If $A = \sum_{j=1}^{J} G_j = \sum_{l=1}^{L} H_l$ $\quad \left(\{G_j \cap H_l\}_{j=1 \sim J,\ l=1 \sim L} \subseteq \mathcal{G}\right.$
$$= \sum_{j=1}^{J}\sum_{l=1}^{L} G_j \cap H_l \quad \left.(\because \mathcal{G} : \pi\text{-system})\right)$$

So $P_0(A) = \sum_{j=1}^{J}\sum_{l=1}^{L} P_0(G_j \cap H_l) = \sum_{l=1}^{L}\sum_{j=1}^{J} P_0(G_j \cap H_l)$
$$\sum_{j=1}^{J} P_0(G_j) \qquad\qquad \sum_{l=1}^{L} P_0(H_l)$$

$$\left(\because G_j = \sum_{l=1}^{L} G_j \cap H_l,\ H_l = \sum_{j=1}^{J} G_j \cap H_l\right.$$

So it's well-defined

$\boxed{\text{Step2}}$ Next we want to show that $(G_0[\mathcal{G}], P_0)$ can be extended $(G[\mathcal{G}], P)$

We show that 「$(G_0[\mathcal{G}], P_0)$ has countable additivity」

$\Leftrightarrow$ 「$\forall \{B_n\}_{n\geq 1} \subseteq G_0[\mathcal{G}]$ $\ B_n \downarrow \phi \rightarrow P_0(B_n) \downarrow 0$.」

(We will show this in the next exercise)

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Let $\mathcal{F}_n$ be a sub-σ-algebra of $\mathcal{B}(\mathbb{R}^{\mathbb{N}})$ containing $\{B \mid B = B^* \times \mathbb{R} \times \mathbb{R} \cdots,\ B^* \in \mathcal{B}(\mathbb{R}^n),\ n = 1, 2, 3, \cdots\}$

And if $B \in \mathcal{B}(\mathbb{R}^n)$, then $B^* \in \mathcal{B}(\mathbb{R}^n)$ is a set such that $B = B^* \times \mathbb{R} \times \mathbb{R} \cdots$

in $\boxed{\text{Step 2}}$ We first prove that without loss of generality we may suppose $B_n \in \mathcal{F}_n$ for each $n = 1, 2, \ldots$

(proof) Since $B_n \in G_0[\mathcal{g}]$, B_n can be written as $= \sum_{j=1}^{J_n} G_j^{(n)}$ $\{G_j^{(n)}\}_{j=1}^{J_n} \subseteq \mathcal{g}$

And each $\{G_j^{(n)}\}$ can be written as $G_j^{(n)} = \left(\prod_{i=1}^{k_n} (a_{j,i}^{(n)}, b_{j,i}^{(n)}]\right) \times \mathbb{R} \times \mathbb{R} \cdots$

k_n-dim

Thus $\{G_j^{(n)}\}_{j=1}^{J_n} \subseteq \mathcal{F}_{k_n}$

Be careful of the fact that $\mathcal{F}_1 \subset \mathcal{F}_2 \subseteq \cdots$

Hence without loss of generality we may assume

$k_1 < k_2 < \cdots$. (increasing)

Now we consider $\{\tilde{B}_n\}_{n\geq 1}$

$\tilde{B}_1 = \mathbb{R}^N$ $\tilde{B}_2 = \mathbb{R}^N \cdots$ $\tilde{B}_{k_1-1} = \mathbb{R}^N$ and
$\tilde{B}_{k_1} = B_1$ $\tilde{B}_{k_1+1} = B_1 \cdots$ $\tilde{B}_{k_2-1} = B_1$ and
$\tilde{B}_{k_2} = B_2$, $\tilde{B}_{k_2+1} = B_2 \cdots$ $\tilde{B}_{k_3-1} = B_2 \cdots$

Then $\tilde{B}_n \in \mathcal{F}_n$ for all $n \geq 1$ $\tilde{B}_n \downarrow \phi$ as $n \to \infty$.

So $\boxed{\text{Step2}}$ is complete ■

$\boxed{\text{Step3}}$ We try to derive a contradiction by assuming that $B_n \downarrow \phi$ but $\exists \delta > 0$ s.t. $P_0(B_n) \geq \delta > 0$ for all $n \geq 1$.

(As $\boxed{\text{Step2}}$ shows, we assume $B_n \in \mathcal{F}_n$)

We may find $E_n \subseteq B_n$: $E_n = \bigcup_{j=1}^{J_n} [\tilde{a}_{j1}^{(n)}, \tilde{b}_{j1}^{(n)}] \times \cdots \times [\tilde{a}_{jn}^{(n)}, \tilde{b}_{jn}^{(n)}] \times \mathbb{R} \times \mathbb{R} \cdots$

($\because B_n \in \mathcal{F}_n$)

s.t. $\mu_n(B_n^* \setminus E_n^*) \leq \frac{\delta}{2^{n+1}}$.

$F_n \overset{\text{def}}{=} \bigcap_{m=1}^{n} E_m$.

Then $B_n \setminus F_n = B_n \setminus \left(\bigcup_{m=1}^{n} E_m^c\right) = \bigcup_{m=1}^{n} (B_n \setminus E_m) \subset \bigcup_{m=1}^{n} (B_m \setminus E_m)$
($\because B_{n+1} \subseteq B_n \subseteq \cdots$)

So $\mu_n(B_n^* \setminus F_n^*) \leq \sum_{m=1}^{n} \mu_m(B_m^* \setminus F_m^*) \leq \sum_{n=1}^{\infty} \frac{\delta}{2^{n+1}} = \frac{\delta}{2}$

$\because \mu_n(B_n^*) = P_0(B_n) \geq \delta$

$\therefore \mu_n(F_n^*) \geq \frac{\delta}{2}$. F_n^* is not empty for all $n \geq 1$.

Moreover $F_n^* = F_n^* \cap (F_{n-1}^* \times \mathbb{R}) \cap \cdots (F_1^* \times \mathbb{R} \cdots \times \mathbb{R})$

is a compact set.

$\boxed{\text{Step 4}}$ Now we take a sequence of points from F_m.

(i.e $\omega^m \in F_m$ and $\omega^m = (\omega_1^m, \omega_2^m, \dots)$)

Since $\omega^m \in F_m \subseteq F_1$ thus $\omega_1^m \in F_1^*$ and

F_1^* is a compact set (by $\boxed{\text{Step 3}}$), by Bolzano-Weierstrass

Theorem. we may find a subsequence $\{m_1(j)\}_{j \geq 1} \subseteq \mathbb{N}$

s.t $\omega_1^{m_1(j)} \to \theta_1 \in F_1^*$.

In the same way, when $m \geq 2$.

$(\omega_1^m, \omega_2^m) \in F_2^*$ ($F_2 \subseteq F_1$) and F_2^* is a compact set.

Thus we may take a subsequence of $\{m_1(j)\}_{j\geq 2}$ (say $m_2(j)$) s.t. $(\omega_1^{m}, \omega_2^{m})$ converges.

So $(\omega_1^{m_2(j)}, \omega_2^{m_2(j)}) \to (\theta_1, \theta_2) \in F_2^*$.

By repeating the similar argument, $\theta = (\theta_1, \theta_2, \dots) \in F_n$ for all $n \geq 1$. So $\theta \in \bigcap_{n=1}^{\infty} F_n$.

Thus F_n is not empty. $\bigcap_{n=1}^{\infty} F_n \subseteq \bigcap_{n=1}^{\infty} E_n \subseteq \bigcap_{n=1}^{\infty} B_n$

So $B_n \not\downarrow \phi$. It contradicts our assumption. ■

CHA SHIN

16 Let $\mathcal{R}$ be a ring. (i.e. $\forall A, B \in \mathcal{R}$: $A \cup B,\ A \setminus B \in \mathcal{R}$)
Let $\mu : \mathcal{R} \to [0, \infty]$ and have finite additivity.

First, we show that the following are equivalent.

① $(\mathcal{R}, \mu)$ has countable additivity.
② $(\mathcal{R}, \mu)$ has sub-countable additivity
③ $(\mathcal{R}, \mu)$ is continuous from below.

① ⇒ ② trivial

① ⇒ ③ Suppose $A_n \uparrow A$ $(\{A_n\} \cup \{A\} \subseteq \mathcal{R})$

Now let $B_1 = A_1$, $B_n = A_n \setminus A_{n-1}$ $(\in \mathcal{R})$ Then $A = \sum_{n=1}^{\infty} B_n$

$$\mu(A) = \sum_{n=1}^{\infty} \mu(B_n) = \sum_{n=1}^{\infty} \left(\mu(A_n) - \mu(A_{n-1})\right) \quad (A_0 = \emptyset)$$
$$= \lim_{n\to\infty} \mu(A_n)$$

② ⇒ ① $\{A_n\}_{n \geq 1} \cup \{A\} \subseteq \mathcal{R}$ and $A = \sum_{n=1}^{\infty} A_n$

$\mu(A) \leq \sum_{n=1}^{\infty} \mu(A_n)$ (by sub-additivity)

$\forall N \geq 1$ $\bigcup_{n=1}^{N} A_n \subseteq A$. $\left(\bigcup_{n=1}^{N} A_n \in \mathcal{R}\right)$

$$\therefore \mu\left(\bigcup_{n=1}^{N} A_n\right) = \sum_{n=1}^{N} \mu(A_n) \leq \mu(A) \qquad N \uparrow \infty.$$

(finite additive)

(finite additivity implies monotonicity) □

③ ⇒ ① $\{A_n\} \cup \{A\} \subseteq \mathcal{R}$: $A = \sum_{n=1}^{\infty} A_n$

$B_n \stackrel{\text{def}}{=} \sum_{m=1}^{n} A_m \quad (\in \mathcal{R}) \qquad B_n \nearrow A$

$\mu(A) = \lim_{n\to\infty} \mu(B_n) = \lim_{n\to\infty} \sum_{m=1}^{n} \mu(A_m) = \sum_{n=1}^{\infty} \mu(A_n)$

(by ③'s assumption) (finite additivity of μ) (So ① ~ ③ are equivalent)

Now we prove ① (⇔ ②, ③) ⇒ ④ ⇒ ⑤

④ μ is continuous from above.
⑤ μ is continuous from above at ϕ.

(④ : $\{A_n\} \cup \{A\} \subseteq \mathcal{R}$ $A_n \downarrow A$, $\mu(A_1) < \infty$
$\Rightarrow \lim_{n\to\infty} \mu(A_n) = \mu(A_1)$)

(⑤ $\{A_n\} \subseteq \mathcal{R}$ $A_n \downarrow \phi$ $\mu(A_1) < \infty$
$\Rightarrow \lim_{n\to\infty} \mu(A_n) = 0$)

We show ③ ⇒ ④

Suppose $A_n \downarrow A$ $\mu(A_1) < \infty$ $(\{A_n\} \cup \{A\} \subseteq \mathcal{R})$

Let $B_n = A_1 \setminus A_n$ then $B_n \nearrow A_1 \setminus A$

So $\lim_{n\to\infty} \mu(B_n) = \mu(A_1 \setminus A) = \underset{<\infty}{\mu(A_1)} - \mu(A)$

$\parallel$

$\lim_{n\to\infty} (\underset{<\infty}{\mu(A_1)} - \mu(A_n))$ Hence we have $\lim_{n\to\infty} \mu(A_n) = \mu(A)$

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④ ⇒ ⑤ trivial $(A=\phi)$

Finally we prove that if $(\mathcal{R}, \mu)$ is finite

(i.e $^{\forall}A \in \mathcal{R}.\ \mu(A) < \infty$) then ⑤ ⇒ ①.

Suppose $\{A_n\}_{n\ge1} \cup \{A\} \subset \mathcal{R}$. $A = \sum_{n=1}^{\infty} A_n$

$B_n = A \setminus \sum_{m=1}^{n} A_m$ $B_n \downarrow \phi$ $\mu(B_1) < \infty$

$\therefore \lim_{n\to\infty} \mu(B_n) = 0$ ($\because$ ⑤)

$= \lim_{n\to\infty} \left(\mu(A) - \mu\left(\sum_{m=1}^{n} A_m\right)\right) = 0$

$= \lim_{n\to\infty} \left(\mu(A) - \sum_{m=1}^{n} \mu(A_m)\right) = 0$

$\therefore \sum_{n=1}^{\infty} \mu(A_n) = \mu(A)$ ■

CHA SHIN

$\boxed{17}$ (1) "$(S, \mathcal{S})$ is nice" means that

there exists function $\varphi : S \mapsto \mathbb{R}$

s.t φ and φ^{-1} are both measurable

(i.e $\forall B \in \mathcal{B}(\mathbb{R})$ $\varphi^{-1}(B) \in \mathcal{S}$,
$\forall A \in \mathcal{S}$ $\varphi(A) \in \mathcal{B}(\mathbb{R})$)

(2) $\boxed{\text{step 1}}$ Let $S = [0,1)^{\mathbb{N}}$ $\mathcal{S} = \mathcal{B}([0,1)^{\mathbb{N}})$

We show that there exists a desired function φ_0

$x \in S$. $x = (x^1, x^2, x^3, \ldots)$ $(x^j \in [0,1)$ $j \geq 1)$

We consider binary extension of $x^1 = 0.x_1^1 x_2^1 \cdots$.

We also define a metric ρ on S as

$\rho(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} |x^n - y^n|$

Then we define $\varphi_0(x) = 0.x_1^1 \, x_2^1 x_1^2 \, x_3^1 x_2^2 x_1^3 \cdots$

$\varphi_0(x)$ and $\varphi_0^{-1}(x)$ is continuous (hence measurable)

$\left(\because \rho(x,y) \to 0 \Rightarrow |\varphi_0(x) - \varphi_0(y)| \to 0 \text{ and } |\varphi_0(x) - \varphi(y)| \to 0 \right.$
(I)
$\left. \Rightarrow \rho(x,y) \to 0 \right)$
(II)

(I) 「 $\rho(x,y) \to 0 \Rightarrow |\varphi_0(x) - \varphi_0(y)| \to 0$ 」

We show if x is fixed and $^{\exists}\delta > 0$ $|\varphi_0(x) - \varphi_0(y)| \geq \delta > 0$

then $^{\exists}\varepsilon > 0$ st $\rho(x,y) \geq \varepsilon > 0$

Since $|\varphi_0(x) - \varphi_0(y)| > \delta > 0$ we may find $n, k \in \mathbb{N}$

st $|x_k^n - y_k^n| = 1$. thus we may find $\varepsilon > 0$ st

$\rho(x,y) \geq \varepsilon$

(II) 「 $|\varphi_0(x) - \varphi_0(y)| \to 0$ then $\rho(x,y) \to 0$ 」

The argument is similar to (I).

$\boxed{\text{Step 2}}$ Next suppose that a metric space

$(S, \tilde{d})$ is given.

When $\tilde{d}$ is a metric, $d = \frac{\tilde{d}}{1+\tilde{d}}$ is also
a metric.

So we define $d(x,y) = \frac{\tilde{d}(x,y)}{1+\tilde{d}(x,y)}$

$0 \leq d(x,y) < 1$.

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By assumption, we can take a countable dense set
$Q = \{q_1, q_2 \ldots q_n \ldots\} \subseteq S$.

We define $\psi(x) : S \to [0,1]^N$ as

$\psi(x) = (d(x, q_1), d(x, q_2), d(x, q_3) \cdots)$

Now we show that ψ, ψ^{-1} are both continuous
(thus measurable)

ρ is a metric on $[0,1]^N$ defined at $\boxed{\text{step 1}}$..

(I) ψ is continuous

$$\rho(\psi(x), \psi(y)) = \sum_{n=1}^{\infty} \frac{1}{2^n} |d(x, q_n) - d(y, q_n)|$$ triangle inequality

$$\leq \sum_{n=1}^{\infty} \frac{1}{2^n} d(x, y) = d(x, y)$$

So $y \to x$ (in d) then $\psi(y) \to \psi(x)$ (in ρ)

(II) ψ^{-1} is continuous

If $\psi(x_n) \to \psi(x)$ (in ρ) (as $n \to \infty$)

Then for each $m \geq 1$, $|d(x_n, q_m) - d(x, q_m)| \to 0$
(as $n \to \infty$)