Chapter 1 Solutions by Toshinari Morimoto

$\boxed{1}$.

(1) monotonicity: If $A, B \in \mathcal{F}$ and $A \subseteq B$

then $P(A) \leq P(B)$

(proof) $B = (B \setminus A) \cup (A)$

Since μ is measure and $(B \setminus A)$, A are disjoint

thus $\mu(B) = \mu(B \setminus A) + \mu(A)$ and $P(B \setminus A) \geq 0$

So $\mu(B) \geq \mu(A)$

(2) sub-additivity: $\{A_n\}_{n \geq 1} \subseteq \mathcal{F}$

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} P(A_n)$$

(proof) $B_1 \overset{\text{def}}{=} A_1$ and $B_n \overset{\text{def}}{=} A_n \setminus \bigcup_{j=1}^{n-1} A_j$ $(n \geq 2)$

Then $\bigcup_{n=1}^{\infty} A_n = \sum_{n=1}^{\infty} B_n$ (= disjoint countable union)

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu\left(\sum_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \mu(B_n) \leq \sum_{n=1}^{\infty} \mu(A_n)$$

($\because$ measure) ($\because$ $B_n \subseteq A_n$)

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(3) (A)

- If $A_n \nearrow A$ $(\{A_n\} \subseteq \mathcal{F}.)$
$A_n \subseteq A_{n+1}$.

$\lim_{n\to\infty} \mu(A_n) = \mu(A)$

(proof) $B_n = A_n \setminus A_{n-1}$ $(B_1 = A_1)$

Then $A = \bigcup_{n=1}^{\infty} A_n = \sum_{n=1}^{\infty} B_n$

$\mu(A) = \mu(\sum_{n=1}^{\infty} B_n) = \sum_{n=1}^{\infty} \mu(B_n) = \lim_{n\to\infty} \sum_{m=1}^{n} \mu(B_m)$

$= \lim_{n\to\infty} \sum_{m=1}^{n} \{\mu(A_n \setminus A_{n-1})\} = \lim_{n\to\infty} \sum_{m=1}^{n} (\mu(A_n) - \mu(A_{n-1}))$

$= \lim_{n\to\infty} \mu(A_n)$

- If $A_n \searrow A$ and $\mu(A_1) < \infty$ $\lim_{n\to\infty} \mu(A_n) = \mu(A)$

(proof)
$B_n = A_1 \setminus A_n$ then $B_n \nearrow B$.
$B = A_1 \setminus A$

By the previous result, $\lim_{n\to\infty} \mu(B_n) = \mu(B)$

(left) $\lim_{n\to\infty} \mu(A_1 \setminus A_n) = \lim_{n\to\infty} (\mu(A_1) - \mu(A_n))$
(right) $= \mu(A_1) - \mu(A)$

Since $\mu(A_1) < \infty$, we have $\lim_{n} \mu(A_n) = \mu(A)$

$\boxed{2}$

(1) $F(x) \overset{\text{def}}{=} P(\{w \mid X(w) \leq x\})$

If $x_1 < x_2$ then $\{w \mid X(w) \leq x_1\} \subseteq \{w \mid X(w) \leq x_2\}$

By monotonicity of measure (probability)

$P(\{w \mid X(w) \leq x_1\}) \leq P(\{w \mid X(w) \leq x_2\})$

thus $F(x_1) \leq F(x_2)$

(2) $x \nearrow \infty \cdots \{w \mid X(w) \leq x\} \nearrow \{w \mid X(w) < \infty\} = \Omega$

$\therefore \lim_{x \to \infty} P(\{w \mid X(w) \leq x\}) = P(\{w \mid X(w) < \infty\}) =$

$\lim_{x \to \infty} F(x) = P(\Omega) = 1$

($\because$ continuity of measure (I))

$x \downarrow -\infty \quad \{w \mid X(w) \leq x\} \downarrow \phi$

And $P(\Omega) = 1 < \infty$ thus $\lim_{x \to -\infty} P(\{w \mid X(w) \leq x\}) = P(\phi) = 0$

($\because$ continuity of measure) (II)

(3) Suppose $x_n \downarrow x_0$ (as $n \to \infty$)

$$\lim_{n\to\infty} P(\{\omega \mid X(\omega) \le x_n\}) = P\left(\bigcap_{n=1}^{\infty} \{\omega \mid X(\omega) \le x_n\}\right)$$

$$= P(\{\omega \mid X(\omega) \le x_0\}) = F(x_0)$$

(left) $\lim_{n\to\infty} F(x_n) = F(x_0)$ (right)

(4) $x_n \uparrow x$

$$\lim_{n\to\infty} F(x_n) = \lim_{n\to\infty} P(\{\omega \mid X(\omega) \le x_n\}) = P\left(\bigcup_{n=1}^{\infty} \{\omega \mid X(\omega) \le x_n\}\right)$$

$$\lim_{n\to\infty} F(x_n) = F(x-)$$

$$= P(\{\omega \mid X(\omega) < x\})$$

$$= P(X < x)$$

(5) $P(X=x) = P(\{\omega \mid X(\omega) \in (-\infty, x]\} \setminus \{\omega \mid X(\omega) \in (-\infty, x)\})$

$$= P(\{\omega \mid X(\omega) \in (-\infty, x]\}) - P(\{\omega \mid X(\omega) \in (-\infty, x)\})$$

$$P(X \le x) - P(X < x)$$

$$= F(x) - F(x-)$$

Borel measurable set.

$\boxed{3}$ $\Omega = (0,1)$ $\mathcal{F} = \mathcal{B}((0,1))$ P: Lebesgue Measure.

$$X(w) \overset{\text{def}}{=} \inf\{x \in \mathbb{R} \mid F(x) \geq w\}$$

Then $P(X(w) \leq x) = F(x)$ $\left(= P(\{w \mid w \leq F(x)\})\right)$

(proof) We show that $\{w \mid X(w) \leq x\} = \{w \mid w \leq F(x)\}$

① $\{w \mid X(w) \leq x\} \subseteq \{w \mid w \leq F(x)\}$

By definition of $X(w)$, $\exists \{x_n\}_{n \geq 1}$ $x_n \downarrow X(w)$: $F(x_n) \geq w$.

Right continuity : "$w \leq \lim_n F(x_n) = F(X(w))$" ... (a)

And $X(w) \leq x \Rightarrow$ "$F(X(w)) \leq F(x)$" ... (b)

By (a), (b), we have $w \leq F(x)$.

② $\{w \mid X(w) \leq x\} \supseteq \{w \mid w \leq F(x)\}$

$w \leq F(x) \Rightarrow x \in \{y \in \mathbb{R} \mid F(y) \geq w\}$

$\Rightarrow x \geq \inf\{y \in \mathbb{R} \mid F(y) \geq w\} = X(w)$.

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$\boxed{4}$

① $\int_x^\infty \exp\left(\frac{-y^2}{2}\right) dy$

$z = y - x \qquad \frac{dz}{dy} = 1$

$\int_0^\infty \exp\left(\frac{-1}{2}(z+x)^2\right) dz$

$= \int_0^\infty \underbrace{\exp\left(\frac{-z^2}{2}\right)}_{\leq 1} \cdot \exp(-zx) \exp\left(\frac{-x^2}{2}\right) dz$

$\int_0^\infty \exp(-zx)\, dz \cdot \exp\left(\frac{-x^2}{2}\right) = x^{-1} \exp\left(\frac{-x^2}{2}\right)$

② $\frac{d}{dx}\left((x^{-1} - x^{-3}) \exp\left(\frac{-x^2}{2}\right)\right)$

$= (-x^{-2} + 3x^{-4}) \exp\left(\frac{-x^2}{2}\right) - (1 - x^{-2}) \exp\left(\frac{-x^2}{2}\right)$

$= (-1 + 3x^{-4}) \exp\left(\frac{-x^2}{2}\right)$

Thus $\int_x^\infty (1 - 3y^{-4}) \exp\left(\frac{-y^2}{2}\right) dy = (x^{-1} - x^{-3}) \exp\left(\frac{-x^2}{2}\right)$

$\leq \int_x^\infty 1 \cdot \exp\left(\frac{-y^2}{2}\right) dy$

Now the proof is completed

$\boxed{5}$

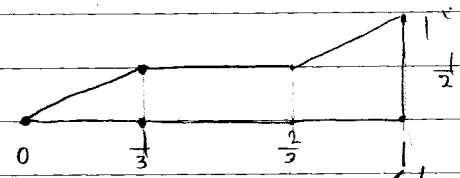
(1) cdf : $F(x) = P(\{w \mid X(w) \leq x\})$

If there exists a Borel measurable function such that $F(x) = \int_{-\infty}^{x} f(y)\,dy$.

Then $f(\cdot)$ is called a probability density function of X.

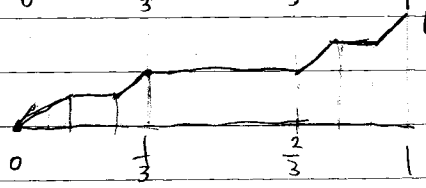
(2) No. It does not always exists

$F_1(x) =$



$(F_n \to F)$

$F_2(x) =$



(F can be a cdf)

⋮

We consider the Cantor - Lebesgue function : $F(x)$

As we see $F(x)$ is flat almost everywhere.

Thus $F'(x) = 0$ (ae). Suppose $P(X \leq x) = F(x)$

So $\int_{-\infty}^{x} F'(y)\,dy = 0$ (for all x)

Thus its density function does not exist

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[6] $\mathcal{G} \overset{def}{=} \{ (a,b] \mid a,b \in \mathbb{R} \}$

$\mu((a,b]) \overset{def}{=} F(b) - F(a)$.

where $F(\cdot)$ is a right-continuous real-valued function. Then $(\mathcal{G}, \mu)$ can be extended to $(\sigma[\mathcal{G}], \mu)$.

(Proof) If $\mathcal{G}$: semi-algebra, $(\mathcal{G}, \mu)$: finite additive and $\forall \{G_n\}_{n\geq 1} \cup \{G\} \subseteq \mathcal{G}$: $G = \sum_{n=1}^{\infty} G_n$ (disjoint union)

$\mu(G) \leq \sum_{n=1}^{\infty} \mu(G_n)$. then $(\mathcal{G}, \mu)$ can be extended to $(\sigma[\mathcal{G}], \mu)$.

Here we prove that $G = \sum_{n=1}^{\infty} G_n$ ($\{G\} \cup \{G_n\}_{n\geq 1} \subset \mathcal{G}$)

then $\mu(G) = \sum_{n=1}^{\infty} \mu(G_n)$. (thus $\mu(G) \leq \sum_{n=1}^{\infty} \mu(G_n)$)

(You may refer to the textbook 测度与概率论基础 程士宏 (北京大学) chapter 2, Prop 2.12

$\cup$ ... union. $\sum$ ... disjoint union

We separate the proof into $\boxed{\text{Step 1}}$ ~ $\boxed{\text{Step 4}}$

$\boxed{\text{Step 1}}$ $(a,b] = \sum_{j=1}^{n} (a_j, b_j]$ $(n<\infty)$

Then $\mu((a,b]) = \sum_{j=1}^{n} \mu((a_j, b_j])$ holds

(proof) We rearrange the order : $\{(a_j, b_j]\}_{j=1}^{n}$

$\rightarrow \{(a_{(j)}, b_{(j)}]\}_{j=1}^{n}$ so that

$$a = a_{(1)} \leq b_{(1)} = a_{(2)} \leq b_{(2)} \cdots \leq b_{(n)} = b.$$

$$\sum_{j=1}^{n} \mu((a_j, b_j]) = \sum_{j=1}^{n} \mu((a_{(j)}, b_{(j)}])$$

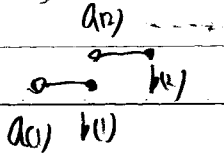
$$= \sum_{j=1}^{n} F(b_{(j)}) - F(a_{(j)})$$

$$= F(b_{(n)}) - F(a_{(n)}) + F(b_{(n-1)}) - F(a_{(n-1)}) + \cdots$$

$$\cdots F(b_{(1)}) - F(a_{(1)})$$

$$= F(b_{(n)}) - F(a_{(1)}) = F(b) - F(a) = \mu((a,b])$$

$$\left(\begin{array}{ll} * \; F(a_{(n)}) = F(b_{(n-1)}) & (\because a_{(n)} = b_{(n-1)}) \\ F(a_{(n-1)}) = F(b_{(n-2)}) & \vdots \\ \vdots & \\ F(a_{(2)}) = F(b_{(1)}) & \end{array} \right)$$



disjoint union

$\boxed{\text{Step2}}$ $(a,b] \supseteq \sum_{n=1}^{\infty} (a_n, b_n]$ Then $\mu((a,b]) \geq \sum_{n=1}^{\infty} \mu((a_n, b_n])$

In the same way, we rearrange the order.

But there are infinite intervals. thus.

$b(1) = a(2)$, $b(2) = a(3)$ ... does not alway hold.

We can only say that "$a(1) \leq b(1) \leq a(2) \leq b(2)$ ..."

(proof) pick $1 \leq N < \infty$

$$\mu((a,b]) = F(b) - F(a) = \underbrace{F(b) - F(b(N))}_{\geq 0} + \sum_{n=1}^{N} F(b(n)) - F(a(n)) + \underbrace{F(a(1)) - F(a)}_{\geq 0}$$

$$\geq \sum_{n=1}^{N} F(b(n)) - F(a(n)) \quad (\text{for all } N)$$

$N \uparrow \infty$ we have $\mu((a,b]) \geq \sum_{n=1}^{N} \mu((a_n, b_n])$

It is not necessarily disjoint union

$\boxed{\text{step3}}$ If $(a,b] \subseteq \bigcup_{n=1}^{N} (a_n, b_n]$ then

$$\mu((a,b]) \leq \sum_{n=1}^{N} \mu((a_n, b_n]) \qquad (N<\infty)$$

(proof) We prove by induction

$N=1$ ... It's trivial.

$N=k$ We assume the statement above holds.

$N=k+1$ ... $(a,b] \subseteq \bigcup_{n=1}^{k+1} \mu((a_n, b_n])$

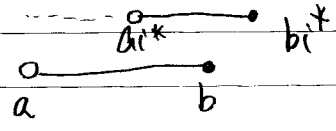
We pick $b_{i^*} \overset{\text{def}}{=} \max_{1 \leq n \leq k+1} \{b_n\}$.

(case I) $a_{i^*} \leq a$. then $(a,b] \subseteq (a_{i^*}, b_{i^*}]$

$$\sum_{n=1}^{k+1} \mu((a_n, b_n]) \geq \mu((a_{i^*}, b_{i^*}]) \geq \mu((a,b])$$

(case II) $a_{i^*} > a$ then $(a, a_{i^*}] \subseteq \bigcup_{n \neq i^*} (a_n, b_n]$.

(*)



$\rightarrow$ $(a, a_{i^*}]$ must be covered by intervals $\{(a_n, b_n]\}_{n \neq i^*}$.

By assumption $(N=k)$, we have.

$$\mu((a, a_i^*]) \leq \sum_{n \neq i^*} \mu((a_n, b_n]) .$$ (∵ there are k-intervals)

$$\Rightarrow \mu((a, a_i^*]) + \underbrace{\mu((a_i^*, b_i^*])}$$

$$\leq \sum_{n \neq i^*} \mu((a_n, b_n]) + \underbrace{\mu((a_i^*, b_i^*])}$$

$$= \sum_{n=1}^{k+1} \mu((a_n, b_n])$$

The left hand side $= F(b_i^*) - F(a)$

$$\geq F(b) - F(a) = \mu((a, b])$$

So we have $\mu((a, b]) \leq \sum_{n=1}^{k+1} \mu((a_n, b_n])$

Thus the statement is true for $N=k+1$.

disjoint union.

$\boxed{\text{step 4}}$ $(a,b] = \sum_{n=1}^{\infty} (a_n, b_n] \Rightarrow$

$$\mu((a,b]) = \sum_{n=1}^{\infty} \mu((a_n, b_n])$$

(proof) $(a,b] \supseteq \sum_{n=1}^{\infty} (a_n, b_n]$. So by $\boxed{\text{step 2}}$

we have $\mu((a,b]) \geq \sum_{n=1}^{\infty} \mu((a_n, b_n])$.

Next we prove $\mu((a,b]) \leq \sum_{n=1}^{\infty} \mu((a_n, b_n])$.

Since $F(\cdot)$ is right continuous. for all $\varepsilon > 0$ and

for each $n \geq 1$ there exists $\delta_n > 0$. such that

$F(b_n + \delta_n) - F(b_n) \leq \frac{\varepsilon_n}{2^n}$.

Let $\eta > 0$ be an arbitrary positive number.

It's obviously true that $(a+\eta, b] \subseteq [a+\eta, b]$

$\subseteq (a,b] = \sum_{n=1}^{+\infty} (a_n, b_n] \subseteq \bigcup_{n=1}^{\infty} (a_n, b_n + \delta_n)$

So $[a+\eta, b]$: bounded closed set is covered by

$\{(a_n, b_n + \delta_n)\}_{n \geq 1}$: open-intervals.

By Heine-Borel Cover theorem.

We may cover $[a+\eta, b]$ by finite number of

$\{(a_n, b_n+\delta_n)\}_{n=1}^{N}$.

So $(a+\eta, b]$ is covered by $\{(a_n, b_n+\delta_n]\}_{n=1}^{N}$

By $\boxed{\text{Step 3}}$ $(a+\eta, b] \subseteq \bigcup_{n=1}^{N} (a_n, b_n+\delta_n] \Rightarrow$

$$\mu((a+\eta, b]) \leq \sum_{n=1}^{N} \mu((a_n, b_n+\delta_n])$$

$$\begin{aligned}
\therefore F(b) - F(a+\eta) &\leq \sum_{n=1}^{N} F(b_n+\delta_n) - F(a_n) \\
&\leq \sum_{n=1}^{N} \{F(b_n+\delta_n) - F(b_n) + F(b_n) - F(a_n)\} \\
&\leq \sum_{n=1}^{N} \left(\frac{\varepsilon}{2^n} + F(b_n) - F(a_n)\right) \\
&\leq \varepsilon + \sum_{n=1}^{\infty} (F(b_n) - F(a_n)) \\
&= \sum_{n=1}^{\infty} \mu((a_n, b_n]) + \varepsilon
\end{aligned}$$

$\eta \downarrow 0$ (Left) $\Rightarrow F(b) - F(a)$ ($\because$ right continuous)

$\varepsilon \downarrow 0$ (Right) $\rightarrow \sum_{n=1}^{\infty} \mu((a_n, b_n])$ $\blacksquare$

[7] Dynkin's π-λ theorem.

- $\mathcal{G}$ ... π-system (ie $\forall A, B$ if $A, B \in \mathcal{G} \Rightarrow A \cap B \in \mathcal{G}$)
- $\lambda[\cdot]$ - the smallest λ-system (Dynkin system) containing $\cdot$.
- $\sigma[\cdot]$ the smallest σ-algebra containing $\cdot$.

Then $\lambda[\mathcal{G}] = \sigma[\mathcal{G}]$.

① σ-algebra is also a λ-system (easy to verify)

thus $\lambda[\mathcal{G}] \subseteq \sigma[\mathcal{G}]$.

② If a λ-system is also a π-system

then it's a σ-algebra. (easy to verify)

When $\mathcal{G}$ is a π-system, then $\lambda[\mathcal{G}]$ is also a

π-system (We prove it in ③) ... ⊛

So $\lambda[\mathcal{G}]$ is a λ-system and a π-system

$\Rightarrow$ $\lambda[\mathcal{G}]$ is a σ-algebra. $\therefore \lambda[\mathcal{G}] \supseteq \sigma[\mathcal{G}]$ ∎

The proof is done but we should show ⊛.

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③ Finally we prove $\mathcal{G}$ = π-system $\Rightarrow$ $\lambda[\mathcal{G}]$ = π-system

$\mathcal{D}_1 \overset{def}{=} \{A \mid \forall B \in \mathcal{G} \quad A \cap B \in \lambda[\mathcal{G}]\}$
$\mathcal{D}_2 \overset{def}{=} \{B \mid \forall A \in \lambda[\mathcal{G}] \quad A \cap B \in \lambda[\mathcal{G}]\}$

Since $\mathcal{G}$ is a π-system, thus $\mathcal{G} \subseteq \mathcal{D}_1$. $(*_1)$

And $\mathcal{D}_1$ is a λ-system. $(*_2)$ So $\lambda[\mathcal{G}] \subseteq \mathcal{D}_1$

Since $\lambda[\mathcal{G}] \subseteq \mathcal{D}_1$, we find that $\mathcal{G} \subseteq \mathcal{D}_2$. $(*_3)$

$\mathcal{D}_2$ is also a λ-system. $(*_4)$ thus $\lambda[\mathcal{G}] \subseteq \mathcal{D}_2$

this implies $\lambda[\mathcal{G}]$ is a π-system. $(*_5)$ ■

$(*_1)$ $\mathcal{G}$ is a π-system $\Rightarrow$ Let $A \in \mathcal{G}$ and $\forall B \in \mathcal{G}$

$A \cap B \in \mathcal{G} \subseteq \lambda[\mathcal{G}] \Rightarrow A \in \mathcal{D}_1$. Thus $\mathcal{G} \subseteq \mathcal{D}_1$.

$(*_2)$ $\mathcal{D}_1$ is a λ-system because

- $\Omega \in \mathcal{D}_1$ ... $\forall B \in \mathcal{G} \quad \Omega \cap B = B \in \mathcal{G} \subseteq \lambda[\mathcal{G}]$
- $A, \hat{A} \in \mathcal{D}_1 \quad (A \subseteq \hat{A}) \quad A \cap B \in \lambda[\mathcal{G}]$
  $\hat{A} \cap B \in \lambda[\mathcal{G}]$
  $\Rightarrow (\hat{A} \cap B) \setminus (A \cap B) = (\hat{A} \setminus A) \cap B \in \lambda[\mathcal{G}]$

If $\{A_n\}_{n\geq 1} \subseteq \mathcal{D}_1$ $A_n \nearrow A$. Then,

Since $A_n \cap B \in \lambda(\mathcal{G})$ and $A_n \cap B \nearrow A \cap B$

So $A \cap B \in \lambda(\mathcal{G})$ (for all $B \in \mathcal{G}$)

$\Rightarrow A \in \mathcal{D}_1$

(*3) $\lambda(\mathcal{G}) \subseteq \mathcal{D}_1$ implies $\mathcal{G} \subseteq \mathcal{D}_2$

because $\lambda(\mathcal{G}) \subseteq \mathcal{D}_1 \Rightarrow {}^\forall A \in \lambda(\mathcal{G})$ ${}^\forall B \in \mathcal{G}$ $A \cap B \in \lambda(\mathcal{G})$.

This means ${}^\forall B \in \mathcal{G}$ $B \in \mathcal{D}_2$. $\Rightarrow \mathcal{G} \subseteq \mathcal{D}_2$.

(*4) $\mathcal{D}_2$ is a λ-system.. The proof is similar to (*2)

(*5) $\lambda(\mathcal{G}) \subseteq \mathcal{D}_2$... implies ${}^\forall A, B \in \lambda(\mathcal{G})$ $A \cap B \in \lambda(\mathcal{G})$

So $\lambda(\mathcal{G})$ is a λ-system

$\boxed{8}$ Since $(\mathcal{P}, \mu_1)$ is σ-finite, there exists

$\{\Omega_n\}_{n\geq 1} \subseteq \mathcal{P}$ $\Omega_n \uparrow \Omega$: $\mu_1(\Omega_n) < \infty$

Define $\mathcal{D}^{(n)} = \{A \in \mathcal{F} \mid \mu_1(A \cap \Omega_n) = \mu_2(A \cap \Omega_n)\}$

- $\mathcal{P} \subset \mathcal{D}^{(n)}$; $\forall A \in \mathcal{P}$. $A \cap \Omega_n \in \mathcal{P}$

  ($\because$ $\Omega_n, A \in \mathcal{P}$ and $\mathcal{P}$ is a π-system)

  So $\mu_1(A \cap \Omega_n) = \mu_2(A \cap \Omega_n)$

  hence $A \in \mathcal{D}^{(n)}$ for all $A \in \mathcal{P}$.

- $\mathcal{D}^{(n)}$ is a λ-system because

  - $\Omega \in \mathcal{D}^{(n)}$ (easy)
  - $A, \tilde{A} \in \mathcal{D}^{(n)} \Rightarrow \tilde{A} \setminus A \in \mathcal{D}^{(n)}$

$$\mu_1(A \cap \Omega_n) = \mu_2(A \cap \Omega_n) < \infty$$
$$\mu_1(\tilde{A} \cap \Omega_n) = \mu_2(\tilde{A} \cap \Omega_n) < \infty$$

Since finite, $\mu_1(\tilde{A} \cap \Omega_n) - \mu_2(A \cap \Omega_n)$
$= \mu_2(\tilde{A} \cap \Omega_n) - \mu_2(A \cap \Omega_n)$

$\therefore \mu_1((\hat{A}|A)\cap\Omega_n) = \mu_2((\hat{A}|A)\cap\Omega_n)$

$\therefore \hat{A}|A \in \mathcal{D}^{(n)}$

- $\{A_m\}_{m\geq 1} \subset \mathcal{D}^{(n)}$ $\quad A_m \uparrow A$

$\mu_1(A_m\cap\Omega_n) = \mu_2(A_m\cap\Omega_n)$

By letting $m\uparrow\infty$, $\rightarrow \mu_1(A\cap\Omega_n) = \mu_2(A\cap\Omega_n)$

$\therefore A\in\mathcal{D}^{(n)}$

So $\mathcal{P}\subseteq\mathcal{D}^{(n)}$ and $\mathcal{D}^{(n)}$: λ-system

$\Rightarrow \lambda[\mathcal{P}] \subseteq \mathcal{D}^{(n)}$

$\quad\quad \| $

$\quad\quad \sigma[\mathcal{P}]$ $\quad$ ($\because$ π-λ-theorem)

$\therefore \forall A\in\sigma[\mathcal{P}]$ $\quad \mu_1(A\cap\Omega_n) = \mu_2(A\cap\Omega_n)$

Finally $n\uparrow\infty$ we have $\mu_1(A\cap\Omega) = \mu_2(A\cap\Omega)$

$= \mu_1(A) = \mu_2(A)$ $\quad (\forall A\in\sigma[\mathcal{P}])$

$\boxed{9}$ When $\mathcal{S}$ is a semi-algebra,

We define $\mathcal{A} \overset{\text{def}}{=} \{A \mid A = \sum_{k=1}^{n} G_k,\ \{G_k\}_{k=1}^{n} \subset \mathcal{S}\}$

ie $\mathcal{A} = \{A \mid A$ is expressed as a finite disjoint union of sets from $\mathcal{S}\}$

Then $\sigma_0[\mathcal{S}] \overset{\text{equal}}{=} \mathcal{A}$.

(I) $\sigma_0[\mathcal{S}] \supseteq \mathcal{A}$

This is obviously true by the definition of $\mathcal{A}$.

because $\sigma_0(\mathcal{S})$ is an algebra containing $\mathcal{S}$.

(II) $\sigma_0[\mathcal{S}] \subseteq \mathcal{A}$

$\mathcal{S} \subset \mathcal{A}$ is obviously true.

It's enough to show that $\mathcal{A}$ is an algebra.

<claim> $\mathcal{A}$ is an algebra.

- $\phi \in \mathcal{A}$ since $\phi \in \mathcal{S}$

- $A, B \in \mathcal{A} \Rightarrow A \cap B \in \mathcal{A}$

$A = \sum_{j=1}^{J} G_{1,j} \quad B = \sum_{l=1}^{L} G_{2,l}$ (Σ: disjoint union)

$A \cap B = \left(\sum_{j=1}^{J} G_{1,j}\right) \cap \left(\sum_{l=1}^{L} G_{2,l}\right) = \sum_{j=1}^{J} \sum_{l=1}^{L} (G_{1,j} \cap G_{2,l})$

$\{G_{1,j} \cap G_{2,l}\}_{(j,l) \in \{1 \cdots J\} \times \{1 \cdots L\}}$

$\subseteq \mathcal{G}$ ($\because \mathcal{G}$: semi-algebra $\Rightarrow$ π-system)

And $J \cdot L < \infty$: finite disjoint union

$\therefore A \cap B \in \mathcal{A}$

- $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$

$A = \sum_{j=1}^{J} G_j \quad A^c = \bigcap_{j=1}^{J} G_j^c$

Since $\{G_j^c\} \subseteq \mathcal{A}$ ($\because \mathcal{G}$: semi-algebra)

($\forall G \in \mathcal{G}$, G^c: disjoint finite union of sets from $\mathcal{G} \Rightarrow \in \mathcal{A}$)

So $\bigcap_{j=1}^{J} G_j^c \in \mathcal{A}$ ($\because$ we have already shown that $A, B \in \mathcal{A} \Rightarrow A \cap B \in \mathcal{A}$) ∎

The proof is complete. ($\because A, B \in \mathcal{A} \Rightarrow A \cup B = (A^c \cap B^c)^c \in \mathcal{A}$)

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$\boxed{10}$ X is $\mathcal{F}/\mathcal{A}$-measurable

$\overset{\text{def}}{\Leftrightarrow}$ $\forall A \in \mathcal{A}$ $X^{-1}(A) \in \mathcal{F}$

① $\forall A \in \mathcal{A}$ $X^{-1}(A) \in \mathcal{F} \Rightarrow \forall G \in \mathcal{G}$ $X^{-1}(G) \in \mathcal{F}$

Since $\mathcal{G} \subseteq \sigma[\mathcal{G}] = \mathcal{A}$. so it's obviously true

② $\forall G \in \mathcal{G}$ $X^{-1}(G) \in \mathcal{F} \Rightarrow \forall A \in \mathcal{A}$ $X^{-1}(A) \in \mathcal{F}$

Consider $B = \{A \mid X^{-1}(A) \in \mathcal{F}\}$

Since $\mathcal{F}$ is a σ-algebra, thus B is also a σ-algebra. (easy to verify)

By assumption, $\mathcal{G} \subseteq B$. And B is a σ-algebra

Thus $\sigma[\mathcal{G}] \subseteq B \Rightarrow \mathcal{A} \subset B$.

$\therefore \forall A \in \mathcal{A}$ $X^{-1}(A) \in \mathcal{F}$.

$\boxed{11}$ $\forall B \in \mathcal{B}$ $\quad (f \circ X)^{-1}(B) = X^{-1} \circ f^{-1}(B)$

Since f is a measurable map from $(S, \mathcal{A}) \to (T, \mathcal{B})$

thus $f^{-1}(B) \in \mathcal{A}$. And since $f^{-1}(B) \in \mathcal{A}$

$X^{-1}(f^{-1}(B)) \in \mathcal{F}$. ($\because$ X is a measurable map)

$\boxed{12}$

① $\sup_{n \geq 1} X_n(w)$

$\{\omega \in \Omega \mid \sup_{n \geq 1} X_n(w) > a\}$

$= \bigcup_{n \geq 1}^{\infty} \{\omega \in \Omega \mid X_n(w) > a\} \in \mathcal{F}$

$\therefore \{\omega \in \Omega \mid X_n(w) > a\} \in \mathcal{F}$

② $\inf X_n(w) = -\sup_{n \geq 1} (-X_n(w))$

$X_n(w)$: measurable $\Rightarrow -X_n(w)$: measurable

$\Rightarrow \sup_{n \geq 1} (-X_n(w)$: measurable$) \Rightarrow -\sup_{n \geq 1} (-X_n(w))$: measurable

③ $\liminf X_n = \lim_{n\to\infty} \inf_{m\geq n} X_m(w) = \sup_{n\geq 1} \inf_{m\geq n} X_m(w)$

since ②, $\inf_{m\geq n} X_m(w)$ is measurable

since ① $\sup_{n\geq 1}\left(\inf_{m\geq n} X_m(w)\right)$ is measurable

④ $\limsup X_n(w)$ (Similar to ③)

$\boxed{13}$ $D_n = \{x\in\mathbb{R} \mid F(x) - F(x-) > \frac{1}{n}\}$

Since $0 \leq F(x) \leq 1$, thus $\# D_n < \infty$.

(Otherwise, $F(x)$ will be out of $[0,1]$)

$\bigcup_{n=1}^{\infty} \{x\in\mathbb{R} \mid F(x) - F(x-) > \frac{1}{n}\}$

$= \{x\in\mathbb{R} \mid F(x) - F(x-) > 0\}$

will be countable

$\boxed{\text{NOTE}}$ $\#$ ... number of elements in a set (card)

$\boxed{A}$ We show that $P(\{w \mid F(X(w)) \leq y\}) = y$

when $y \in (0,1)$.

Consider $\{x \mid F(x) = y\}$ $(y \in (0,1))$

It will be a closed set.

$\because \{x \in \mathbb{R} \mid F(x) \neq y\} = \{x \in \mathbb{R} \mid F(x) \in (y, \infty)\}$ ①
$\cup \{x \in \mathbb{R} \mid F(x) \in (-\infty, y)\}$ ②

Since $F(x)$ is continuous, ①, ② are open.

$\therefore \{x \in \mathbb{R} \mid F(x) \neq y\}^c = \{x \in \mathbb{R} \mid F(x) = y\}$ is closed

And $F(x)$ is monotone increasing, thus

$\{x \in \mathbb{R} \mid F(x) \neq y\}$ will be a closed interval.

$x_0 \overset{\text{def}}{=} \max\{x \mid F(x) = y\}$

$\therefore P(F(X(w)) \leq y) = P(X \leq x_0) = F(x_0) = y$

by definition of x_0.

Now the proof is complete ∎

$\boxed{15}$

① f is lower-semi continuous : $\forall x_0 \in \mathbb{R}$

$\ulcorner \liminf_{x \to x_0} f(x) \geq f(x_0) \lrcorner \Leftrightarrow \ulcorner \lim_{\delta \to 0} \inf_{y \in B(x_0, \delta)} f(y) \geq f(x_0) \lrcorner$

$\Rightarrow$ We show that $\{x \mid f(x) \leq a\}$ is closed for $\forall a \in \mathbb{R}$.

Take $\{x_n\}_{n \geq 1} \subseteq \{x \mid f(x) \leq a\}$ where $x_n \to x_0$

We prove that $f(x_0) \leq a$.

Let $\varepsilon > 0$. $\lim_{\delta \to 0} \inf_{y \in B(x_0, \delta)} f(y) \geq f(x_0)$

$\Rightarrow \exists \delta_\varepsilon > 0$ s.t. $\inf_{y \in B(x_0, \delta_\varepsilon)} f(y) > f(x_0) - \varepsilon$.

Since $x_n \to x_0$, thus for sufficiently large n,

$x_n \in B(x_0, \delta_\varepsilon)$. Therefore $f(x_n) \geq \inf_{y \in B(x_0, \delta_\varepsilon)} f(y)$

$> f(x_0) - \varepsilon$

$\therefore a \geq f(x_n) > f(x_0) - \varepsilon$. $\Rightarrow f(x_0) \leq a$ (by ε is arbitrary)

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② $\{x \in \mathbb{R} \mid f(x) \leq a\}$ is a closed set for $\forall a \in \mathbb{R}$

$\Rightarrow f(\cdot)$ is lower-semi continuous

We consider its contra position.

「f is NOT lower-semi continuous」 $\Rightarrow$ 「$\exists a \in \mathbb{R}$ s.t. $\{x \mid f(x) \leq a\}$ is not closed」

「f is lower-semi continuous」 $\overset{def}{=}$

「$\forall x_0$, $\forall \varepsilon > 0$ $\exists \delta_\varepsilon > 0$ s.t. $\forall 0 < \delta^* \leq \delta_\varepsilon$ $\inf_{y \in B(x_0, \delta^*)} f(y) > f(x_0) - \varepsilon$」

「f is NOT lower-semi continuous」 $\overset{def}{=}$

「$\exists x_0$ $\exists \varepsilon > 0$ $\forall \delta_n > 0$ s.t. $\exists 0 < \delta_n^* < \delta_n$ s.t. $\inf_{y \in B(x_0, \delta_n^*)} f(y) \leq f(x_0) - \varepsilon$」

We consider $\{\delta_n\}_{n \geq 1}$ $\delta_n \downarrow 0$.

We may find $x_n \in B(x_0, \delta_n^*)$ s.t. $f(x_n) - \frac{\varepsilon}{2} \leq \inf_{y \in B(x_0, \delta_n^*)} f(y)$.

Therefore $f(x_n) \leq f(x_0) - \frac{\varepsilon}{2}$. Let $a = f(x_0) - \frac{\varepsilon}{2}$

$\therefore \{x_n\}_{n \geq 1} \subseteq \{f(x) \leq a\}$ and $x_n \to x_0$. ($\because |x_n - x_0| \leq \delta_n^* \leq \delta_n \to 0$)

But $f(x_0) \leq a$ does not hold. ($\because a = f(x_0) - \frac{\varepsilon}{2}$)

So $\{f(x) \leq a\}$ is not closed. ■

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(1)

[6] We show that $f^{\delta}(x)$ is lower semi continuous

Take $\{x_n\}_{n\geq 1} \subseteq \{f^{\delta}(x) \leq a\}$ and $x_n \to x_0$.
We show that $x_0 \in \{f^{\delta}(x) \leq a\}$.

$\boxed{\text{claim}}$

$$\bigcup_{n\geq 1}^{\infty} B(x_n, \delta) \supseteq B(x_0, \delta)$$

pick $x_0^* \in B(x_0, \delta)$.

$$|x_0^* - x_n| \leq \underbrace{|x_0 - x_0^*|}_{=\delta_0 < \delta} + |x_n - x_0|$$

Since $x_n \to x_0$ we may find sufficiently large n such that $|x_n - x_0| < \delta - \delta_0$.

Thus $|x_0^* - x_n| < \delta$.

$\therefore x_0^* \in B(x_n, \delta) \subseteq \bigcup_{n=1}^{\infty} B(x_n, \delta)$

$\therefore B(x_0, \delta) \subseteq \bigcup_{n=1}^{\infty} B(x_n, \delta)$ ∎

Next, we consider $f^{\delta}(x_0) \overset{(def)}{=} \sup_{z \in B(x_0, \delta)} \{f(z)\}$

We may find $z_0 \in B(x_0, \delta)$ $f^{\delta}(x_0) \leq f(z_0) + \varepsilon$.

where $\varepsilon > 0$ is an arbitrary positive number

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Since $y_0 \in B(x_0, \delta) \subseteq \bigcup_{n=1}^{\infty} B(x_n, \delta)$.

We may find $n \geq 1$ such that $y_0 \in B(x_n, \delta)$ (by $\boxed{\text{claim}}$)

$\therefore f(y_0) \leq \sup_{y \in B(x_n, \delta)} f(y) = f^{\delta}(x_n)$.

So we have $f^{\delta}(x_0) \leq f(y_0) + \varepsilon \leq f^{\delta}(x_n) + \varepsilon \leq a + \varepsilon$

Since $\varepsilon > 0$ is arbitrary, $\varepsilon \downarrow 0$ we have $f^{\delta}(x_0) \leq a$. ∎

$f_{\delta}(x)$ is upper-semi continuous...

$$f_{\delta}(x) = \inf_{y \in B(x, \delta)} \{f(y)\} = \underbrace{-\sup_{y \in B(x, \delta)} \{-f(y)\}}_{\text{lower semi continuous}}$$

$\therefore -(\text{l.s.c}) = \text{u.s.c}$

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(2)

Next we prove $\{f^0 \neq f_0\}$ is measurable

$\{f^0 \neq f_0\} = \{f^0 - f_0 > 0\}$ ($\because f^\delta \geq f_\delta$ for all $\delta > 0$)

$= \bigcup_{m=1}^{\infty} \{f^0 - f_0 \geq \frac{1}{m}\}$.

We prove that $\{f^0 - f_0 \geq a\}$ is measurable for all $a > 0$.

$g_n \overset{\text{def}}{=} f^{\frac{1}{n}} - f_{\frac{1}{n}}$ ... is lower semi continuous.

g_n decreases as $n \nearrow \infty$.

$g_n \downarrow f^0 - f_0$. So $-g_n \nearrow f_0 - f^0$.

$$\left(\bigcup_{n=1}^{\infty} \{-g_n > -a\}\right)^c = \bigcap_{n=1}^{\infty} \{a \leq g_n\}$$

$$\{-g_n > -a\} = \{f_0 - f^0 > -a\}^c = \{f^0 - f_0 < a\}^c = \{f^0 - f_0 \geq a\}$$

(We prove $\{g_n \geq a\}$ is measurable)

Since $\{g_n \leq a\}$ is closed ($\because g_n$ ... l.s.c)

$\{g_n > a\}$ is open ($\Rightarrow$ measurable)

$\{g_n > a - \frac{1}{k}\}$ is open ($\Rightarrow$ measurable)

$\bigcap_{k=1}^{\infty} \{g_n > a - \frac{1}{k}\} = \{g_n \geq a\}$ is measurable

Now the proof is complete.

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[17] $\{M_t\}_{t\in T}$... family of functions

$\forall t\in T$... M_t satisfies :

(· M_t contains $\mathcal{F}$-measurable simple functions
· $\forall f_n \in M_t$ if $f_n \to f$ then $f \in M_t$

Suppose f is a $\mathcal{F}$-measurable function

We prove that $f \in M_t$ for all $t \in T$.

There exists f_n^+, f_n^- : non-negative

$\mathcal{F}$-measurable functions s.t. $f_n^+ \nearrow f^+$, $f_n^- \nearrow f^-$

$f_n \overset{\text{def}}{=} f_n^+ - f_n^-$ f_n is also a $\mathcal{F}$-measurable

simple function. And $f_n \to f^+ - f^- = f$

Since $\{f_n\} \subset M_t$ ($\forall t \in T$) Thus $f \in M_t$ ($\forall t \in T$)

($\because f_n \to f$)

So the proof is complete

· $\boxed{\text{Note}}$ Let f: be a non-negative $\mathcal{F}$-measurable function.

$$f_n \overset{\text{def}}{=} \min\left\{n, \frac{[2^n f(x)]}{2^n}\right\}$$

Then f_n: non-negative measurable simple function

$$f_n \nearrow f.$$

(proof) $g_n(x) \overset{\text{def}}{=} \dfrac{[2^n f(x)]}{2^n}$

$$0 \leq f(x) - g_n(x) \leq \frac{1}{2^n}$$

$$\therefore \lim_{n\to\infty} g_n(x) = f(x)$$

$$\therefore \lim_{n\to\infty} f_n(x) = \min\left\{\infty, \lim_{n\to\infty} \frac{[2^n f(x)]}{2^n}\right\} = f(x)$$

$$f_n \leq f_{n+1} \quad (\because g_n(x) \leq g_{n+1}(x))$$

$$(\because 2[a] \leq [2a])$$

· So $f_n \nearrow f$.

(f_n: measurable $\because$ $f_n = \displaystyle\sum_{k=0}^{2^n-1} \frac{k}{2^n} \cdot \mathbb{1}\left\{\frac{k}{2^n} \leq f(x) < \frac{k+1}{2^n}\right\}$

$+ n \cdot \mathbb{1}\{f(x) \geq n\}$)

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$\boxed{18}$

① $Y \geq 0$.

We may find $Y_n = \sum_{j=1}^{J_n} a_j^{(n)} \cdot I_{A_j^{(n)}}(\omega)$ $\quad Y_n \nearrow Y$.

(non-negative $\sigma(X)$-measurable) $\quad \{a_j^{(n)}\} \subset [0, \infty)$

$\{A_j^{(n)}\} \subseteq \sigma[X]$

(disjoint)

Since $A_j^{(n)}$ are $\sigma(X)$-measurable,

Thus $\exists B_j^{(n)} \in \mathcal{B}(\mathbb{R})$ (Borel-sets) such that

$A_j = X^{-1}(B_j)$

So $Y_n = \sum_{j=1}^{J_n} a_j^{(n)} \cdot I_{B_j^{(n)}}(X(\omega))$

$\{B_j^{(n)}\}_{j=1}^{J_n}$ are not always disjoint, ... ⊛

So we define $C_1^{(n)} = B_1^{(n)}$ $\quad (j=1)$

$C_j^{(n)} = B_j^{(n)} \setminus \bigcup_{k=1}^{j-1} B_k^{(n)}$ $\quad (j \geq 2)$

Then $Y_n = \sum_{j=1}^{J_n} a_j^{(n)} I_{C_j^{(n)}}(X(\omega))$ $\quad \{C_j^{(n)}\}$ are disjoint

(⊛ You may suppose you can choose $\{B_j^{(n)}\}$ so that they are disjoint)

$f_n(x) \overset{\text{def}}{=} \sum_{j=1}^{J_n} a_j^{(n)} \mathbb{1}_{C_j^{(n)}}(x)$ is a borel measurable function

$Y_n(\omega) \leq Y_{n+1}(\omega) \overset{\text{implies}}{\Rightarrow} f_n(X(\omega)) \leq f_{n+1}(X(\omega))$

$\Rightarrow f_n(x) \leq f_{n+1}(x)$ if $x \in \{X(\omega) \mid \omega \in \Omega\}$

$\Rightarrow \lim_{n\to\infty} f_n(x)$ exists if $x \in \{X(\omega) \mid \omega \in \Omega\}$

$$f(x) \overset{\text{def}}{=} \begin{cases} \lim_{n\to\infty} f_n(x) & (\text{if } \lim_{n\to\infty} f_n(x) \text{ exists}) \\ 0 & (\text{if } \lim_{n\to\infty} f_n(x) \text{ does not exist}) \end{cases}$$

Since $\{x \in \mathbb{R} \mid \lim f_n(x) \text{ exists}\}$
$\{x \in \mathbb{R} \mid \lim f_n(x) \text{ does not exists}\}$

are "borel" measurable ($\because$ $\boxed{12}$)

Thus $f(x)$ is a borel measurable function

$$\underset{\overset{\parallel}{Y(\omega)}}{\lim_{n\to\infty} Y_n(\omega)} = \lim_{n\to\infty} f_n(X(\omega)) = f(X(\omega))$$

($\because$ $x \in \{X(\omega) \mid \omega \in \Omega\}$
$\Rightarrow f_n(x) \to f(x)$)

② In a general case

$$Y = Y^+ - Y^- = f^+(X(\omega)) - f^-(X(\omega)) = (f^+ - f^-)(X(\omega))$$

$(f^+ : \mathbb{R} \to [0,\infty),\ f^- : \mathbb{R} \to [0,\infty))$

$\boxed{19}$ We put $Y_n(w) = \sum_{m\in\mathbb{Z}} \frac{m}{2^n} \cdot I_{\{\frac{m}{2^n} \leq Y(w) < \frac{m+1}{2^n}\}}$

$= \sum_{m\in\mathbb{Z}} \frac{m}{2^n} I_{Y^{-1}([\frac{m}{2^n}, \frac{m+1}{2^n}))}(w)$

$= \sum_{m\in\mathbb{Z}} \frac{m}{2^n} I_{B_{n,m}}(X(w)) \quad (B_{n,m} \in \mathcal{B}(\mathbb{R}))$

$(\because Y^{-1}([\frac{m}{2^n}, \frac{m+1}{2^n})) \in \sigma(X) = \{X^{-1}(B) \mid B \in \mathcal{B}(\mathbb{R})\})$

And we may suppose $\{B_{n,m}\}$ are disjoint.

(same as $\boxed{18}$)

$f_n(x) \overset{\text{def}}{=} \sum_{m\in\mathbb{Z}} \frac{m}{2^n} I_{B_{n,m}}(x)$: Borel-Measurable

If $x \in \{X(w) \mid w \in \Omega\}$, $f_n(x) \leq f_{n+1}(x)$ holds.

$(\because Y_n(w) \leq Y_{n+1}(w))$

Thus $f_n \uparrow$.

$f(x) \overset{\text{def}}{=} \begin{cases} \lim_{n\to\infty} f_n(x) & (\text{if } \lim_{n\to\infty} f_n(x) \text{ exists}) \\ 0 & (\text{otherwise}) \end{cases}$

$\{x \in \mathbb{R} \mid \lim f_n(x) \text{ exists}\}$ is borel-measurable.

$(\because \{\lim\sup f_n(x) = \liminf f_n(x)\}$ $\limsup f_n(x)$, $\liminf f_n(x)$ are Borel-Measurable $)$

And $Y(\omega) = \lim_{n\to\infty} Y_n(\omega)$. $(\because 0 \leq Y(\omega) - Y_n(\omega) \leq \frac{1}{2^n})$

$$\| $$

$$\lim_{n\to\infty} f_n(X(\omega)) = f(X(\omega))$$

$(\because \lim_{n\to\infty} f_n(x)$ exists for all $x \in \{X(\omega) \mid \omega \in \Omega\})$

So $Y = f(X(\omega))$ f: Borel-measurable

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$\boxed{20}$ g integrable (we may suppose $g \geq 0$), $\hat{g}$: step function

$$\left| \int_R g \cos nx \, dx \right|$$

$$= \left| \int_R (g \cos nx - \hat{g} \cos nx) \, dx + \int_R \hat{g} \cos nx \, dx \right|$$

$$\leq \int_R |g - \hat{g}| \, dx + \left| \int_R \hat{g} \cos nx \, dx \right|$$

$$\leq \underbrace{\int_R |g - g_k| \, dx}_{①} + \underbrace{\int_R |g_k - \hat{g}| \, dx}_{②} + \underbrace{\left| \int_R \hat{g} \cos nx \, dx \right|}_{③}$$

① We may find a sequence of non-negative simple measurable function $\{g_k\}_{k \geq 1}$ st $0 \leq g_k \nearrow g$.

By Monotone Convergence Theorem

$\int_R g_k \, dx \nearrow \int_R g \, dx < \infty$

$\therefore$ ① $= \int_R (g - g_k) \, dx < \frac{\varepsilon}{3}$ (for sufficiently large k)

② $g_K \overset{\text{def}}{=} \sum_{j=1}^{J} b_j \cdot \mathbb{I}_{B_j}$

We show that there exists $\tilde{g}$: step function

$\int_{\mathbb{R}} |g_K - \tilde{g}| dx < \frac{\varepsilon}{3}$.

$\tilde{g} \overset{\text{def}}{=} \sum_{j=1}^{J} b_j \cdot \mathbb{I}_{A_j}$ $\{A_j\} \subseteq \sigma_0[\mathcal{G}]$

($\sigma_0[\mathcal{G}]$ ∵ smallest algebra containing $\mathcal{G}$)

where $\mathcal{G} \overset{\text{def}}{=} \{ (a,b] \mid a, b \in \mathbb{R} \}$

(semi-algebra)

Then $\int_{\mathbb{R}} |g_K - \tilde{g}| \leq \sum_{j=1}^{J} b_j \, \mu(B_j \triangle A_j)$

(μ: Lebesgue-measure)

$\boxed{\text{Note}}$: $\tilde{g}$: is a step function, ∵ By [9] (Lemma 1.1.7)

$A_j \in \sigma_0[\mathcal{G}] \overset{\text{def}}{=} \{ A \mid A$ is expressed as disjoint union of sets from $\mathcal{G} \}$

($\mathcal{G}$: semi-algebra)

A finite sum of step functions is also a step function.

So we show that we may find $\{A_j\} \subseteq \sigma_0[\mathcal{G}]$

st $\mu(B_j \triangle A_j) < \frac{\varepsilon}{3 b_j J}$.

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It's enough to show that $\forall B \in \mathcal{B}(\mathbb{R})$ $\exists A \in G_0[\mathcal{G}]$ s.t. $\mu(B \triangle A) < \varepsilon_0$ where ε_0 is an arbitrary positive number.

(*) Generally, let $(\mathcal{G}, \mu_0)$ be a semi algebra and a set function with finite additivity.

Let μ be an outer measure derived from $(\mathcal{G}, \mu_0)$

Then $\forall B \in \sigma[\mathcal{G}]$ $\exists A \in G_0[\mathcal{G}]$ s.t. $\mu(A \triangle B) < \varepsilon_0$

(proof) By the definition of an outer measure

$\exists \{G_n\} \subseteq \mathcal{G}$ $\mu(B) \leq \sum_{n=1}^{\infty} \mu_0(G) < \mu(B) + \frac{\varepsilon_0}{2}$

$(B \subseteq \bigcup_{n=1}^{\infty} G_n)$

$\tilde{B} \overset{\text{def}}{=} \bigcup_{n=1}^{\infty} G_n$

$A_n \overset{\text{def}}{=} \bigcup_{m=1}^{n} G_m$ $A_n \nearrow B$. $\Rightarrow B \setminus A_n \searrow \phi$.

$\mu(B \setminus A_1) < \infty$ $(\because \mu(B) < \infty)$ $\mu(B \setminus A_n) \downarrow 0$ (as $n \to \infty$)

$\therefore \mu(B \setminus A_n) < \frac{\varepsilon_0}{2}$ when n is large.

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$$\mu(B \triangle A_n) = \mu(B \setminus A_n) + \mu(A_n \setminus B)$$
$$\leq \frac{\varepsilon_0}{2} + \mu(A_n \setminus B) \leq \frac{\varepsilon_0}{2} + \mu(\tilde{B} \setminus B)$$
$$= \frac{\varepsilon_0}{2} + \mu(\tilde{B}) - \mu(B) \leq \frac{\varepsilon_0}{2} + \sum_{n=1}^{\infty} \mu_0(G_n) - \mu(B)$$
$$< \varepsilon_0$$

So proof of ② is complete.

$\boxed{\text{NOTE}}$ Since $\mathcal{G} = \{ (a,b) \mid a, b \in \mathbb{R} \} \Rightarrow \sigma[\mathcal{G}] = \mathcal{B}(\mathbb{R})$

Finally ③ $= \left| \int_{\mathbb{R}} \hat{g} \cos nx \right|$ $\quad \left(\hat{g} = \sum_{j=1}^{J} b_j \cdot I_{(s_j, t_j]} \right)$

$$= \left| \sum_{j=1}^{J} \int_{s_j}^{t_j} b_j \cos nx \, dx \right|$$
$$\leq \sum_{j=1}^{J} \frac{|b_j|(t_j - s_j)}{n} \to 0 \quad (\text{as } n \to \infty)$$

So the proof is now complete.

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[2] Jensen's inequality

(1) Convexity: $\forall \lambda \in (0,1)$. $\forall x, y \in \mathbb{R}$

$$\varphi(\lambda x + (1-\lambda) y) \leq \lambda \varphi(x) + (1-\lambda) \varphi(y)$$

(2) f and $\varphi(f)$ is μ-integrable and φ is convex

Then $\varphi(\int f d\mu) \leq \int \varphi(f) d\mu$.

(proof) $c \overset{\text{put}}{=} \int f d\mu$ and consider $l(x) = ax+b$

where $\varphi(x) \geq l(x)$ and $\varphi(c) = l(c)$

$$\int \varphi(f) d\mu \geq \int l(f) d\mu = \int (af+b) d\mu = ac+b$$

$$= l(c) = \varphi(c) = \varphi(\int f d\mu)$$

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$\boxed{22}$ $\frac{1}{p}+\frac{1}{q}=1$ f, g measurable $(p, q>1)$

$$\int |f||g|\,d\mu \leq \|f\|_p \|g\|_q$$

$$\left(\begin{array}{l} \|f\|_p = \left(\int |f|^p d\mu\right)^{\frac{1}{p}} \\ \|g\|_q = \left(\int |g|^q d\mu\right)^{\frac{1}{q}} \end{array} \right)$$

When $\|f\|_p = \infty$ or 0 $(\|g\|_q = \infty$ or $0)$

it's easy. So we suppose $\|f\|_p, \|g\|_q \in (0, \infty)$

We use Young's inequality. $\frac{1}{p}a^p + \frac{1}{q}b^q - ab \geq 0$

$(a, b \geq 0)$

put $a = \frac{|f|}{\|f\|_p}$ $b = \frac{|g|}{\|g\|_q}$

then $\frac{1}{p}\frac{|f|^p}{\|f\|_p^p} + \frac{1}{q}\frac{|g|^q}{\|g\|_q^q} \geq \frac{|f||g|}{\|f\|_p \|g\|_q}$

By integrating both sides we have

$$1 = \frac{1}{p}+\frac{1}{q} \geq \frac{1}{\|f\|_p \|g\|_q} \int |f||g|\,d\mu$$

Now we have the desired result.

(Note) Young's inequality's proof

$$h(a) = \frac{1}{p}a^p + \frac{1}{q}b^q - ab$$

$$h'(a) = a^{p-1} - b = 0 \Rightarrow a = b^{\frac{1}{p-1}} = b^{\frac{q}{p}}$$

a	0	$b^{\frac{q}{p}}$	
$h'(a)$	$-$	0	$+$
$h(a)$	$\searrow$	0	$\nearrow$

So $h(a) \geq 0$ when $a = b^{\frac{q}{p}}$ equality holds

$(\Omega, \mathcal{F}, \mu)$

$\boxed{23}$ $f_n \geq 0$: non-negative measurable function

$f_n \nearrow f$. Then $\lim_{n\to\infty} \int_\Omega f_n d\mu = \int_\Omega f d\mu$

$\boxed{\text{Step 1}}$ $\{f_n\}_{n\geq 1} \cup \{g\}$: non-negative simple functions

Suppose $\lim_{n\to\infty} f_n \geq g$ Then $\lim_{n\to\infty} \int_\Omega f_n d\mu \geq \int_\Omega g d\mu$

$A_n^{(c)} = \{\omega \in \Omega \mid f_n \geq cg\}$ $(0 < c < 1)$

Then $A_n^{(c)} \nearrow \Omega$. (for all $c \in (0,1)$ as $n \nearrow \infty$)

$\circledast$ $\int_\Omega f_n d\mu \geq \int_\Omega f_n \cdot I_{A_n^{(c)}} d\mu = \int_{A_n^{(c)}} f_n d\mu$

$$\geq \int_{A_n^{(c)}} cg d\mu = c \int_{A_n^{(c)}} g d\mu$$

$$= c \sum_{k=1}^{K} a_k \mu(A_n^{(c)} \cap B_k)$$

$$g \overset{df}{=} \sum_{k=1}^{K} a_k I_{B_k} \quad \left(\begin{array}{l} \{a_k\} \subset [0,\infty) \\ \{B_k\} \subseteq \mathcal{F} \end{array} \right)$$

$$\lim_{n\to\infty} \int_\Omega f_n d\mu \geq \lim_{n\to\infty} c \sum_{k=1}^{K} a_k \mu(A_n^{(c)} \cap B_k)$$

$$= c \sum_{k=1}^{K} a_k \mu(B_k) = c \int_\Omega g d\mu$$

$c \nearrow 1$ we have $\lim_n \int_\Omega f_n d\mu \geq \int_\Omega g d\mu$

(NOTE) If f, g are non-negative simple functions
it's easy to verify $\int_\Omega f \cdot \mathbb{I}_A \, d\mu = \int_A f \, d\mu$
$\int_\Omega f \cdot \mathbb{I}_A \, d\mu = \int_A f \, d\mu$, $f \le g \Rightarrow \int f \, d\mu \le \int g \, d\mu$.

$\boxed{\text{Step 2}}$ $\{f_n\}_{n\ge1}$: non-negative simple functions
f non-negative measurable
$f_n \uparrow f \Rightarrow \lim \int f_n \, d\mu = \int f \, d\mu$.

① $\lim \int f_n \, d\mu \le \int f \, d\mu$
② Since $\int f = \sup\{\int g \, d\mu \mid g \le f,\ g$: non-negative, simple$\}$
it's enough to pick $g \le f$: g : non-negative simple function and show $\lim \int f_n \, d\mu \ge \int g \, d\mu$.
We can verify this by $\boxed{\text{Step 1}}$.

$\boxed{\text{Step 3}}$ f : non-negative measurable $\int_A f \, d\mu = \int_\Omega f \cdot \mathbb{I}_A \, d\mu$

There exists $\{f_n\}_{n\ge1}$: non-negative simple s.t. $f_n \uparrow f$.
And apply $\boxed{\text{Step 2}}$.

$\boxed{\text{Step 4}}$ f_1, f_2 non-negative $f_1 \le f_2 \Rightarrow \int f_1 \, d\mu \le \int f_2 \, d\mu$
$\left(\because \{\int g \, d\mu \mid g \le f_1\} \subseteq \{\int g \, d\mu \mid g \le f_2\}\right)$

$\boxed{\text{Step 5}}$ In $\boxed{\text{Step 1}}$, we may rewrite $\{f_n\}_{n\ge1}$: non-negative
g : non-negative simple.

The strategy is the same as $\boxed{\text{Step 1}}$. But we need to use $\boxed{\text{Step 3}}$ and $\boxed{\text{Step 4}}$ in ⊛.

$\boxed{\text{Step 6}}$ In $\boxed{\text{Step 2}}$, we may rewrite $\{f_n\}_{n\ge1}$: non-negative.
(Use $\boxed{\text{Step 5}}$)

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$\boxed{24}$ Fatou's lemma $\{f_n\}_{n\geq 1}$ ... non-negative measurable

Then $\int_\Omega \liminf f_n \, d\mu \leq \liminf \int_\Omega f_n \, d\mu$.

(proof) $g_n \overset{\text{def}}{=} \inf_{m\geq n} f_m$

$g_n \nearrow g = \liminf f_n$

By Monotone Convergence Theorem

$\lim_{n\to\infty} \int_\Omega g_n \, d\mu = \int_\Omega g \, d\mu = \int_\Omega \liminf f_n \, d\mu$

And $g_n \leq f_n$ (for all $n \geq 1$)

$\therefore \int_\Omega g_n \, d\mu \leq \int_\Omega f_n \, d\mu$

Thus $\lim \int_\Omega g_n \, d\mu \leq \liminf \int_\Omega f_n \, d\mu$

$\boxed{25}$ $\{f_n\}_{n\geq 1}$ $|f_n| \leq g$ $(\forall n \geq 1)$ g: integrable

$f_n \to f$ (a.e) Then $\lim_{n\to\infty} \int f_n d\mu = \int f d\mu$

(proof) $\limsup \left| \int_\Omega f_n - \int_\Omega f \right| \leq \limsup \int_\Omega |f_n - f|$

$\because$ $|f_n - f| \leq |f_n| + |f| \leq 2 \sup_{n\geq 1} |f_n| \leq 2g$

$\therefore h(x) = 2g - |f_n - f| \geq 0$

We apply Fatou's lemma to $h(x)$

$$\int_\Omega \liminf h(x) \leq \liminf \int_\Omega h(x)$$

$$\int_\Omega 2g \qquad \liminf \int_\Omega (2g - |f_n - f|)$$

$$= \liminf \left(\int_\Omega 2g - \int_\Omega |f_n - f| \right)$$

$$(\because \int_\Omega 2g < \infty)$$

$$= \int_\Omega 2g - \limsup \int_\Omega |f_n - f| d\mu$$

$\therefore \limsup \int_\Omega |f_n - f| = 0$

$\therefore \lim \int_\Omega f_n = \lim \int_\Omega f$

$\boxed{26}$ $f_n \overset{\text{def}}{=} \min\{n, f(x)\}$

$f_n \nearrow f$.

Thus by Monotone Convergence Theorem

$\lim_{n\to\infty} \int f_n d\mu = \int f d\mu < \infty$

For sufficiently large $n^{(\varepsilon)}$

$\int_\Omega f d\mu - \int_\Omega f_n d\mu < \frac{\varepsilon}{2}$

$\therefore$ So for any $A \in \mathcal{F}$

$\int_A f d\mu - \int_A f_n d\mu < \frac{\varepsilon}{2}$

$\therefore \int_A f d\mu < \frac{\varepsilon}{2} + \int_A f_n d\mu \leq \frac{\varepsilon}{2} + n^{(\varepsilon)}\mu(A)$

$\therefore$ if $\mu(A) < \frac{\varepsilon}{2n^{(\varepsilon)}}$

then $\int_A f d\mu < \varepsilon$.

$\left(\therefore \delta = \frac{\varepsilon}{2n^{(\varepsilon)}}\right)$

(Another method)

$$\int_\Omega f d\mu = \int_{\{x\in\Omega \mid f(x)>M\}} f d\mu \quad \cdots ①$$
$$+ \int_{\{x\in\Omega \mid f(x)\leq M\}} f d\mu \quad \cdots ②$$

By Lebesgue Dominated Convergence Theorem

$$① = \int_\Omega f \cdot \mathbb{I}_{\{f(x)>M\}} \downarrow 0 \quad \text{as } M\uparrow\infty$$

$$\left(\because\ 0 \leq f \cdot \mathbb{I}_{\{f>M\}} \leq f : \text{integrable}\right.$$
$$\left. M\uparrow\infty \cdots f\cdot\mathbb{I}_{\{f>M\}} \downarrow f\cdot\mathbb{I}_\phi = 0 \right)$$

② Take $M(\varepsilon)$ st $① < \frac{\varepsilon}{2}$

Then $\int_A f d\mu \leq \underbrace{①}_{<\frac{\varepsilon}{2}} + M\mu(A)$

If $\mu(A) < \frac{\varepsilon}{2M(\varepsilon)}$ then $\int_A f d\mu < \varepsilon$ for all $A\in\mathcal{F}$
$\mu(A) < \frac{\varepsilon}{2M\varepsilon}$

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$\boxed{27}$ Chebyshev's inequality (Probability Space Version)

Let $\varphi(\cdot)$ be a Borel-measurable function and be non-negative

Then $E[\varphi(X) \cdot \mathbb{I}\{\varphi(X) \geq a\}] \geq a \cdot P(\{\varphi(X) \geq a\})$

(proof)

$\int_\Omega \varphi(X)\, \mathbb{I}\{\varphi(X) \geq a\}\, dP$

$\geq \int_\Omega a\, \mathbb{I}\{\varphi(X) \geq a\}\, dP = a P(\{\varphi(X) \geq a\})$

$\boxed{\text{Sol}}$ $P(X \le x)$ is discontinuous at only countable points

We may take infinitely large $M > 0$ s.t

$P(|X| = M) = 0$.

$\tilde{X}_n \overset{\text{def}}{=} X_n \cdot I_{\{|X_n| \le M\}}$ $\quad \tilde{X} = X \cdot I_{\{|X| \le M\}}$

We may suppose $h(0) = 0$. $\left(\tilde{h}(x) = h(x) - h(0)\right)$

$$\left|E[h(X_n)] - E[h(X)]\right| \le \left|E[h(X_n)] - E[h(\tilde{X}_n)] + E[h(\tilde{X}_n)] - E[h(\tilde{X})] + E[h(\tilde{X})] - E[h(X)]\right|$$

$$\le E\left|h(X_n) - h(\tilde{X}_n)\right| + E\left|h(\tilde{X}_n) - h(\tilde{X})\right| + E\left|h(\tilde{X}) - h(X)\right|$$

$$= \underbrace{E\left[|h(X_n)| \cdot I_{\{|X_n| > M\}}\right]}_{①} + \underbrace{E\left|h(\tilde{X}_n) - h(\tilde{X})\right|}_{②} + \underbrace{E\left[|h(X)| \cdot I_{\{|X| > M\}}\right]}_{③}$$

$$① \le E\left[\frac{|h(X_n)|}{g(X_n)} g(X_n) \cdot I_{\{|X_n| > M\}}\right]$$

$$\le \varepsilon_M \cdot E[g(X_n)] \le K \varepsilon_M$$

$$\left(\varepsilon_M = \sup\left\{\frac{|h(x)|}{g(x)} \cdot \middle| x > M\right\}\right)$$

$M \uparrow \infty$. $\varepsilon_M \downarrow 0$ $(K < \infty)$ $\therefore ① \downarrow 0$

$① < \frac{\varepsilon}{3}$

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② $\tilde{X}_n = X_n \cdot \mathbb{I}\{|X_n| \le M\}$ $\quad \tilde{X} = X \cdot \mathbb{I}\{|X| \le M\}$

$\tilde{X}_n \to \tilde{X}$ (as) $\quad (\because X_n \to X$ (a.s) and

$(\Rightarrow h(\tilde{X}_n) \to h(\tilde{X}))$ $\quad P(|X| = M) = 0)$

$\therefore$ By bounded convergence theorem,

we have ② $\to 0$ (as $n \to \infty$)

$(\therefore ② < \frac{\varepsilon}{3})$ $\quad (\because h(x)$ is continuous on $[-M, M]$. So it's bounded$)$

③ $E[|h(X)| \cdot \mathbb{I}\{|h(X)| > M\}]$

$E[\frac{|h(X)|}{g(X)} \cdot g(X) \cdot \mathbb{I}\{|h(X)| > M\}] \le$

$\le \varepsilon_M \cdot E[g(X)] = \varepsilon_M E[\liminf g(X_n)]$

$\le \varepsilon_M \sup_n E[g(X_n)]$

$\le K \cdot \varepsilon_M$

$\therefore M \to \infty$ $\quad ③ < \frac{\varepsilon}{3}$

$(\varepsilon_M = \sup\{\frac{|h(x)|}{g(x)} \mid |x| > M\})$

$\boxed{29}$ Consider $f \geq 0$ (In a general case, $f = f^+ - f^-$)

We have a sequence of nonnegative $\mathcal{S}$-measurable simple function $f_n \nearrow f$.

$f_n(X(\omega)) \nearrow f(X(\omega))$

By Monotone Convergence Theorem

$$\lim_{n\to\infty} \int_\Omega f_n(X(\omega))\,dP = \int_\Omega f(X)\,dP$$

Since f_n is simple $f_n = \sum_{j=1}^{J_n} a_j^{(n)} \cdot \mathbb{I}_{A_j^{(n)}}$

$$\{a_j^{(n)}\} \subseteq [0, \infty)$$
$$\{A_j^{(n)}\} \subseteq \mathcal{S}$$

$$\begin{aligned}
\int_\Omega f_n(X(\omega))\,dP &= \int_\Omega \sum_{j=1}^{J_n} a_j^{(n)} \cdot \mathbb{I}_{A_j^{(n)}}(X(\omega))\,dP \\
&= \int_\Omega \sum_{j=1}^{J_n} a_j^{(n)} \cdot \mathbb{I}_{X^{-1}(A_j^{(n)})}(\omega)\,dP \\
&= \sum_{j=1}^{J_n} a_j^{(n)} P(X \in A_j^{(n)}) \\
&= \sum_{j=1}^{J_n} a_j^{(n)} \mu(A_j^{(n)}) = \sum_{j=1}^{J_n} \int_{\mathbb{R}} a_j^{(n)} \cdot \mathbb{I}_{A_j^{(n)}}(x)\,d\mu \\
&= \int_{\mathbb{R}} \sum_{j=1}^{J_n} a_j^{(n)} \mathbb{I}_{A_j^{(n)}}(x)\,d\mu = \int_{\mathbb{R}} f_n(x)\,d\mu
\end{aligned}$$

$$\int_\Omega f(X)\,dP = \lim_{n\to\infty} \int_\Omega f_n(X)\,dP = \lim_{n\to\infty} \int_{\mathbb{R}} f_n(x)\,d\mu = \int_{\mathbb{R}} f(x)\,d\mu$$

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$\boxed{30}$ We prove $P(Y>0) \cdot E[Y^2] \geq E[Y]^2$

By Cauchy Schwartz's inequality,

$$E[Y \cdot I_{\{Y>0\}}]^2 \leq E[Y^2] \cdot E[I_{\{Y>0\}}^2]$$

$$= E[Y^2] \cdot E[I_{\{Y>0\}}]$$

$$= E[Y^2] \cdot P(Y>0)$$

And $L.H.S = E[Y]^2$

$$\left(\because \int Y dP = \int_{\{Y>0\}} Y dP + \int_{\{Y=0\}} Y dP \right.$$

$$= \int_{\{Y>0\}} Y dP.$$

$\boxed{3}$ (Inclusion Exclusion Formula)

$$\mathbb{I}_A = \mathbb{I}_{\bigcup_{i=1}^{n} A_i} = \mathbb{I}_{\left(\bigcap_{i=1}^{n} A_i^c\right)^c} = 1 - \mathbb{I}_{\bigcap_{i=1}^{n} A_i^c}$$

$$= 1 - \prod_{i=1}^{n} \mathbb{I}_{A_i^c} = 1 - \prod_{i=1}^{n} (1 - \mathbb{I}_{A_i})$$

$$= 1 - (1 - \mathbb{I}_{A_1})(1 - \mathbb{I}_{A_2}) \cdots (1 - \mathbb{I}_{A_n})$$

So $$P(A) = 1 - \sum_{k=0}^{n} \sum_{\substack{I \subseteq \{1 \cdots n\} \\ \#I = k}} (-1)^k \cdot P\left(\bigcap_{i \in I} A_i\right)$$

$$= \sum_{k=1}^{n} \sum_{\substack{I \subseteq \{1 \cdots n\} \\ \#I = k}} (-1)^{k+1} \cdot P\left(\bigcap_{i \in I} A_i\right)$$

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$\boxed{32}$ We prove the general case

「When k is odd $(0 \leq k \leq n)$

$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{j=1}^{k} \sum_{\substack{\#I=j \\ I \subseteq \{1,2,\dots,n\}}} (-1)^{j+1} P\left(\bigcap_{i \in I} A_i\right)$$

and

When k is even $(0 \leq k \leq n)$

$$P\left(\bigcup_{i=1}^{n} A_i\right) \geq \sum_{j=1}^{k} \sum_{\substack{\#I=j \\ I \subseteq \{1,2,\dots,n\}}} (-1)^{j+1} P\left(\bigcap_{i \in I} A_i\right)」 \quad (*)$$

We prove by induction about n.

- $n=1$.. It's obviously true.
- $n=m$... We suppose the statement $(*)$ is true.
- $n=m+1$.. We prove the statement $(*)$ is also true

(case) k is odd

$$P\left(\bigcup_{i=1}^{m+1} A_i\right) = P\left(\bigcup_{i=1}^{m} A_i \cup A_{m+1}\right)$$

$$= \underbrace{P\left(\bigcup_{i=1}^{m} A_i\right)}_{\text{apply } n=m} + P(A_{m+1}) - P\left(\bigcup_{i=1}^{m} A_i \cap A_{m+1}\right)$$

$$\leq \sum_{j=1}^{k} \sum_{\substack{\#I=j \\ I\subseteq\{1\cdots m\}}} (-1)^{j+1} P\left(\bigcap_{i\in I} A_i\right) + P(A_{m+1}) - \underbrace{P\left(\bigcup_{i=1}^{m} A_i \cap A_{m+1}\right)}_{\text{since } k-1 \text{ is even}}$$

$$= \sum_{j=1}^{k} \sum_{\substack{\#I=j \\ I\subseteq\{1\cdots m\}}} (-1)^{j+1} P\left(\bigcap_{i\in I} A_i\right) + P(A_{m+1})$$

$$- \sum_{j=1}^{k-1} \sum_{\substack{\#I=j \\ I\subseteq\{1\cdots m\}}} (-1)^{j+1} P\left(\bigcap_{i\in I} A_i \cap A_{m+1}\right)$$

And we may rewrite this as:

$$= \sum_{j=1}^{k} \sum_{\substack{\#I=j \\ I\subseteq\{1\cdots m+1\} \\ m+1\notin I}} (-1)^{j+1} P\left(\bigcap_{i\in I} A_i\right) + P(A_{m+1})$$

$$- \sum_{j=2}^{k} \sum_{\substack{\#I=j \\ I\subseteq\{1\cdots m+1\} \\ m+1\in I}} (-1)^{j} P\left(\bigcap_{i\in I} A_i\right)$$

$$= \sum_{j=1}^{k} \sum_{\substack{\#I=j \\ I\subseteq\{1\cdots m+1\} \\ m+1\notin I}} (-1)^{j+1} P\left(\bigcap_{i\in I} A_i\right) + P(A_{m+1}) + \sum_{j=2}^{k} \sum_{\substack{\#I=j \\ I\subseteq\{1\cdots m+1\} \\ m+1\in I}} (-1)^{j+1} P\left(\bigcap_{i\in I} A_i\right)$$

we can first merge these two terms

$$= \sum_{j=1}^{k} \sum_{\substack{\#I=j \\ I\subseteq\{1\cdots m+1\}}} (-1)^{j+1} P\left(\bigcap_{i\in I} A_i\right) \quad \blacksquare$$

(when k is even, we can prove in a similar way)

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$\boxed{33}$

(1) $\lim_{y\to\infty} E\left[\frac{y}{X}\cdot \mathbb{I}\{1>\frac{y}{X}\}\right]$

$\sup_{y>0}\left|\frac{y}{X}\mathbb{I}\{1>\frac{y}{X}\}\right| \le 1$ ... integrable

By L.D.C.T (or bounded convergence theorem)

$E\left[\lim_{y\to\infty}\frac{y}{X}\cdot\mathbb{I}\{1>\frac{y}{X}\}\right] = E\left[\frac{\infty}{X}\cdot\mathbb{I}_{\phi}\right]$

$= E[0] = 0$

(2) $\lim_{y\to 0} E\left[\frac{y}{X}\mathbb{I}\{1>\frac{y}{X}\}\right]$

$\left|\frac{y}{X}\mathbb{I}\{1>\frac{y}{X}\}\right| \le 1$ ... integrable

In the same way, $E\left[\lim_{y\to 0}\frac{y}{X}\mathbb{I}\{1>\frac{y}{X}\}\right]$

$= E[0\cdot\mathbb{I}_{\phi}] = E[0] = 0$

$\boxed{34}$

① Uniqueness: $\mathcal{G} = \mathcal{A} \times \mathcal{B}$ is a π-system.

And $(\mathcal{G}, \mu)$ is also σ-finite

If $\mu_1 = \mu_2$ on $\mathcal{G}$, Then $\mu_1 = \mu_2$ on $\sigma(\mathcal{G})$

② Existence:

We show that $(\mathcal{G}, \mu)$ has countable additivity.

i.e. $\{G_n\}_{n \geq 1} \cup \{G\} \subseteq \mathcal{G}$ and

$$G = \sum_{n=1}^{\infty} G_n \quad (\text{disjoint union})$$

Then $\mu(G) = \sum_{n=1}^{\infty} \mu(G_n)$ $\quad (G = A \times B,\ G_n = A_n \times B_n)$

$$\mu(G) = \mu_1(A)\mu_2(B) = \int_X \int_Y \mathbb{I}_A \mathbb{I}_B \, d\mu_2 d\mu_1$$

$$= \int_X \int_Y \mathbb{I}_{A \times B}(x,y) \, d\mu_2 d\mu_1 \qquad A \times B = \sum_{n=1}^{\infty} A_n \times B_n.$$

$$= \int_X \int_Y \sum_{n=1}^{\infty} \mathbb{I}_{A_n \times B_n}(x,y) \, d\mu_2 d\mu_1$$

$$= \int_X \int_Y \sum_{n=1}^{\infty} \mathbb{I}_{A_n} \cdot \mathbb{I}_{B_n} \, d\mu_2 d\mu_1 \quad \downarrow \text{MCT}$$

$$= \int_X \sum_{n=1}^{\infty} \int_Y \mathbb{I}_{A_n} \mathbb{I}_{B_n} \, d\mu_2 d\mu_1$$

$$= \int_X \sum_{n=1}^{\infty} \mathbb{I}_{A_n} \cdot \mu_2(B_n) \, d\mu_1 \quad \downarrow \text{M.CT}$$

$$= \sum_{n=1}^{\infty} \int_X \mathbb{I}_{A_n} \cdot \mu_2(B_n) \, d\mu_1 = \sum_{n=1}^{\infty} \mu_1(A_n)\mu_2(B_n) = \sum_{n=1}^{\infty} \mu(G_n)$$

$\boxed{35}$ Fubini's Theorem. , $\mu = \mu_1 \times \mu_2$

Let $(X \times Y, \mathcal{A} \otimes \mathcal{B}, \mu)$ be a product measure space

$\int_{X \times Y} f d\mu = \int_X \int_Y f \, d\mu_2 d\mu_1$ $f \geq 0$ or integrable ;

First we suppose $f \geq 0$.

We define a measure $\tilde{\mu}(A) \overset{\text{def}}{=} \int_X \int_Y \mathbb{I}_A \, d\mu_2 d\mu_1$

(It's easy to verify $(\mathcal{A} \otimes \mathcal{B}, \tilde{\mu})$ is a measure space)

And $\int_X \int_Y f d\mu_2 d\mu_1 = \int_{X \times Y} f d\tilde{\mu}$.

(It's also easy to verify) -- ⊛

So we prove that $\mu = \tilde{\mu}$ on $\mathcal{A} \otimes \mathcal{B}$

Since $\mu = \tilde{\mu}$ on $\mathcal{G} = \mathcal{A} \times \mathcal{B}$ (rectangle)

$\because \tilde{\mu}(A \times B) = \int_X \int_Y \mathbb{I}_{A \times B} \, d\mu_2 d\mu_1 = \mu_1(A) \mu_2(B)$

$\mathcal{G}$ is a π-system and $(\mathcal{G}, \mu)$ is σ finite

So $\mu = \tilde{\mu}$ on $\sigma(\mathcal{G}) = \mathcal{A} \otimes \mathcal{B}$ ■

$\Downarrow$ f_n: simple $f_n \nearrow f$.

(proof of ⊗) $\int_{X\times Y} f\,d\tilde{\mu} = \lim_{n\to\infty} \int_{X\times Y} f_n\,d\tilde{\mu}$

$= \lim_{n\to\infty} \sum_{j=1}^{J_n} a_j^{(n)} \cdot \tilde{\mu}(A_j^{(n)}) = \lim_{n\to\infty} \int_X \int_Y a_j^{(n)} \mathbb{I}_{A_j^{(n)}}\,d\mu_2\,d\mu_1$

$= \lim_{n\to\infty} \int_X \int_Y f_n\,d\mu = \int_X \lim_{n\to\infty} \int_Y f_n\,d\mu = \int_X \int_Y \lim_n f_n\,d\mu =$

(Monotone Convergence Theorem)

$= \int_X \int_Y f\,d\mu.$

Next, consider f: integrable.

$\int_{X\times Y} f^+\,d\mu = \int_X \int_Y f^+\,d\mu_2\,d\mu_1 < \infty$
$\int_{X\times Y} f^-\,d\mu = \int_X \int_Y f^-\,d\mu_2\,d\mu_1 < \infty$

$\Rightarrow \int_{X\times Y} f^+\,d\mu - \int_{X\times Y} f^-\,d\mu = \int_X \int_Y f^+\,d\mu_2\,d\mu_1 - \int_X \int_Y f^-\,d\mu_2\,d\mu_1$

Since $-\int_{X\times Y} f^+\,d\mu < \infty$ and $\int_{X\times Y} f\,d\mu < \infty$.
we may merge $= \int_{X\times Y} f\,d\mu$

In the same way, RHS $= \int_X \left(\int_Y f^+\,d\mu_2 - \int_Y f^-\,d\mu_2 \right) d\mu_1$

$\int_Y f^+\,d\mu_2 < \infty$ (μ_1-ae) and $\int_Y f^-\,d\mu_2 < \infty$ μ_1-ae

Thus $= \int_Y f\,d\mu_2$ (μ_1-ae).

So we may rewrite as $\int_X \int_Y f\,d\mu_2\,d\mu_1$. ∎

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NOTE

In the proof, we supposed the following two facts.

(I) $f(x,y)$: $\mathcal{A}\otimes\mathcal{B}$-measurable. Then for fixed $x\in X$
$y \mapsto f(x,y)$ is $\mathcal{B}$-measurable.

(II) $\int_Y f(x,y)\, d\mu_2$ is $\mathcal{A}$-measurable

We also prove (I) and (II) in order to complete the proof.

(I). There exists $\{f_n\}_{n\geq 1}$ simple s.t. $f_n \nearrow f$.
If the statement is true for $\{f_n\}_{n\geq 1}$,
it's also true for f.

It's enough to show for $f = \mathbb{I}_A$ $(A \in \mathcal{A}\otimes\mathcal{B})$.

Consider $\mathcal{D} = \{A \in \mathcal{A}\otimes\mathcal{B} \mid \forall x\in X,\ y \mapsto \mathbb{I}_A$ is $\mathcal{B}$-measurable.$\}$

Then $\mathcal{D}$ is a Dynkin-system. (λ-system)

- $\Omega \in \mathcal{D}$. $\quad y \mapsto \mathbb{I}_\Omega = 1$.
- $A, \hat{A} \in \mathcal{D}$ $(A \subset \hat{A})$. $\mathbb{I}_{\hat{A}} - \mathbb{I}_A$ is $\mathcal{B}$-measurable
$(= \mathbb{I}_{\hat{A}\setminus A})$
- $\{A_n\} \subset \mathcal{D}$, $A_n \nearrow A$. $\quad \lim_{n\to\infty} \mathbb{I}_{A_n} = \mathbb{I}_A$: $\mathcal{B}$-measurable

($\lambda[\cdot]$: smallest Dynkin system (λ-system) containing $\cdot$)

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$\mathcal{D} \supseteq \mathcal{A} \times \mathcal{B}$ ... and $\mathcal{A} \times \mathcal{B}$ is a π-system.

So $\mathcal{D} \supseteq \lambda[\mathcal{A} \times \mathcal{B}] = \sigma[\mathcal{A} \times \mathcal{B}] \quad \mathcal{A} \otimes \mathcal{B}$ (π-λ theorem) ✓ ∎

(II) In the same way, if the statement is true for $\{f_n\}_{n \geq 1}$ simple functions, it's true for f.

It's enough to show $f = \mathbb{I}_A \quad (A \in \mathcal{A} \otimes \mathcal{B})$

$\mathcal{D}_n^* \overset{\text{def}}{=} \{A \in \mathcal{A} \otimes \mathcal{B} \mid \int_Y \mathbb{I}_{A \cap S_n} \, d\mu_2 \text{ is } \mathcal{A}\text{-measurable}\}$

where $S_n = A_n \times B_n \quad \{A_n\} \subseteq \mathcal{A} \quad A_n \uparrow X \quad \mu_1(A_n) < \infty$
$\{B_n\} \subseteq \mathcal{B} \quad B_n \uparrow Y \quad \mu_2(B_n) < \infty$

$\mathcal{D}_n^*$ is a Dynkin-system (λ-system) (proof is similar to (I))

(note) $A, \tilde{A} \in \mathcal{D}_n^* \quad (A \subseteq \tilde{A}) \quad \int_Y \mathbb{I}_{\tilde{A} \cap S_n} d\mu_2 \leq \int_Y \mathbb{I}_{S_n} d\mu_2$
$= \mathbb{I}_{A_n} \cdot \mu_2(B_n) < \infty$

Thus $\int_Y \mathbb{I}_{\tilde{A} \cap S_n} d\mu_2 - \int_Y \mathbb{I}_{A \cap S_n} d\mu_2$
$= \int_Y \mathbb{I}_{(\tilde{A} \setminus A) \cap S_n} d\mu_2$ is also $\mathcal{A}$-measurable.

And $\mathcal{A} \times \mathcal{B} \subseteq \mathcal{D}_n^* \Rightarrow \lambda[\mathcal{A} \times \mathcal{B}] = \sigma[\mathcal{A} \times \mathcal{B}] = \mathcal{A} \otimes \mathcal{B} \subseteq \mathcal{D}_n^*$ (π-λ theorem)

So $\forall A \in \mathcal{A} \times \mathcal{B} \quad \int_Y \mathbb{I}_{A \cap S_n} d\mu_2$ is $\mathcal{A}$-measurable.

$n \to \infty$. $\lim_n \int_Y \mathbb{I}_{A \cap S_n} d\mu_2 = \int_Y \mathbb{I}_A d\mu_2$ $\mathcal{A}$-measurable ∎

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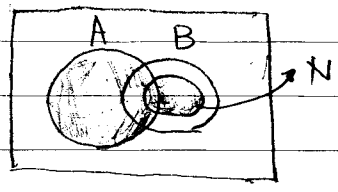
(1) $\tilde{\mathcal{F}} = \{ A \cup N \mid A \in \mathcal{F},\ \exists B \in \mathcal{F}\ \ N \subseteq B,\ \mu(B) = 0 \}$

$\tilde{\mu}(A \cup N) \overset{def}{=} \mu(A)$

(2) $\phi \in \tilde{\mathcal{F}}$. $\quad A = \phi \quad B = N = \phi$

$A \cup N \in \tilde{\mathcal{F}}$



$$(A \cup N)^c = \underbrace{(A \cup B)^c}_{①} \cup \underbrace{B \setminus (A \cup N)}_{②}$$

① $\in \mathcal{F}$ $(\because A, B \in \mathcal{F})$

② $\subseteq B$.

$\therefore (A \cup N)^c \in \tilde{\mathcal{F}}$

$\{A_n \cup N_n\} \subseteq \tilde{\mathcal{F}}$

$$\bigcup_{n=1}^{\infty} A_n \cup N_n = \left(\bigcup_{n=1}^{\infty} A_n\right) \cup \left(\bigcup_{n=1}^{\infty} N_n\right)$$

$\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$

$\bigcup_{n=1}^{\infty} N_n \subseteq \bigcup_{n=1}^{\infty} B_n \in \mathcal{F}$. $\quad \mu\left(\bigcup_{n=1}^{\infty} B_n\right) \leq \sum_{n=1}^{\infty} \mu(B_n) = 0$

$\therefore \bigcup_{n=1}^{\infty} A_n \cup N_n \in \tilde{\mathcal{F}}$

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So $\tilde{\mathcal{F}}$ is a σ-algebra

(3) $\tilde{\mu}$ is well-defined.

We ~~must~~ verify if $A_1 \cup N_1 = A_2 \cup N_2$

Then $\tilde{\mu}(A_1 \cup N_1) = \tilde{\mu}(A_2 \cup N_2)$.

$N_1 \subseteq B_1 \in \mathcal{F}$, $N_2 \in B_2 \in \mathcal{F}$ $\mu(B_1) = 0$, $\mu(B_2) = 0$

$A_1 \subseteq A_1 \cup N_1 = A_2 \cup N_2 \subseteq A_2 \cup B_2$

$$\therefore \mu(A_1) \leq \mu(A_2 \cup B_2) = \mu(A_2) + \mu(B_2 \setminus A_2)$$
$$\leq \mu(A_2) + \mu(B_2) = \mu(A_2)$$

In the same way $A_2 \subseteq A_2 \cup N_2 = A_1 \cup N_1 \subseteq A_1 \cup B_1$

and we have $\mu(A_2) \leq \mu(A_1)$ $\therefore \mu(A_1) = \mu(A_2)$

So $\tilde{\mu}(A_1 \cup N_1) = \tilde{\mu}(A_2 \cup N_2)$

(4) $\tilde{\mu}$ is a measure.

$\{A_n \cup N_n\}$: disjoint

$$\tilde{\mu}\left(\bigcup_{n=1}^{\infty} A_n \cup N_n\right) = \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n) = \sum_{n=1}^{\infty} \tilde{\mu}(A_n \cup N_n)$$

$(\because (2), (3))$

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(5) $(\Omega, \tilde{\mathcal{F}}, \tilde{\mu})$: complete

Complete ... $\forall B \in \mathcal{F}$ $\mu(B) = 0$.
$\forall N \subseteq B$ $N \in \tilde{\mathcal{F}}$

(proof) $A = \phi$, $\forall N \subseteq B$. $(\mu(B) = 0 \; B \in \mathcal{F})$

$\therefore A \cup N = N \in \tilde{\mathcal{F}}$

So $(\Omega, \tilde{\mathcal{F}}, \tilde{\mu})$ is complete.