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chapter 2. Solution by Toshinari Morimoto

$\boxed{1}$

(1) $P(A \cap B) = P(A)P(B)$

(2) $^\forall B_1, B_2 \in \mathcal{B}(\mathbb{R})$ $P(X \in B_1, Y \in B_2)$
$= P(X \in B_1)\, P(Y \in B_2)$

(3) $^\forall A \in \mathcal{F}, \ ^\forall B \in \mathcal{G}$ $P(A \cap B) = P(A)P(B)$

(4) $^\forall A_1 \in \mathcal{F}_1 \cdots A_n \in \mathcal{F}_n$ $P\left(\bigcap_{j=1}^{n} A_j\right) = \prod_{j=1}^{n} P(A_j)$

(5) $^\forall B_1 \in \mathcal{B}(\mathbb{R}) \cdots B_n \in \mathcal{B}(\mathbb{R})$

$$P\left(\bigcap_{i=1}^{n} X_i \in B_i\right) = \prod_{i=1}^{n} P(X_i \in B_i)$$

(6) $^\forall I \subseteq \{1, 2 \ldots n\}$ $P\left(\bigcap_{i \in I} A_i\right) = \prod_{i=1}^{n} P(A_i)$

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$\boxed{2}$.

(1) $\sigma[X] \overset{\text{def}}{=} \{ X^{-1}(B) \mid B \in \mathcal{B}(\mathbb{R}) \}$
$\sigma[Y] = \{ Y^{-1}(B) \mid B \in \mathcal{B}(\mathbb{R}) \}$

pick $A_1 \in \sigma[X]$ $A_2 \in \sigma[Y]$

There exists $B_1 \in \mathcal{B}(\mathbb{R})$ and $B_2 \in \mathcal{B}(\mathbb{R})$

s.t $A_1 = X^{-1}(B_1)$ $A_2 = Y^{-1}(B_2)$

$$P(A_1 \cap A_2) = P(X^{-1}(B_1) \cap Y^{-1}(B_2))$$
$$= P(X \in B_1 \cap Y \in B_2)$$
$$= P(X \in B_1) P(Y \in B_2)$$
$$= P(A_1) P(A_2)$$

(2) Let $B_1, B_2 \in \mathcal{B}(\mathbb{R})$

$$P(X \in B_1, Y \in B_2) = P(X^{-1}(B_1) \cap Y^{-1}(B_2))$$

$X^{-1}(B_1) \in \mathcal{F}$ $Y^{-1}(B_2) \in \mathcal{G}$

$$= P(X^{-1}(B_1)) P(Y^{-1}(B_2)) = P(X \in B_1) P(Y \in B_2)$$

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$\boxed{3}$

(1) $P(A^c \cap B) = P(B) - P(A \cap B) = P(B) - P(A)P(B)$
$= P(B)(1 - P(A)) = P(B)P(A^c)$

$P(A \cap B^c)$ ... same as above

$P(A^c \cap B^c) = 1 - P(A \cup B) = 1 - P(A) - P(B) + P(A \cap B)$
$= 1 - P(A) - P(B) + P(A)P(B)$
$= (1 - P(A))(1 - P(B))$
$= P(A^c)P(B^c)$

(2) $P(\mathbb{I}_A \in G_1, \mathbb{I}_B \in G_2)$

$G_1, G_2 \in \mathcal{B}(\mathbb{R})$

$\{\mathbb{I}_A \in G_1\} = \phi, A, A^c, \Omega$
$\{\mathbb{I}_B \in G_2\} = \phi, B, B^c, \Omega$

Since ϕ and Ω are independent of any (measurable) sets, we just have to consider A, A^c, B, B^c

$P(\mathbb{I}_A \in G_1, \mathbb{I}_B \in G_2) = P(A \cap B)$ or $P(A^c \cap B)$ or $P(A \cap B^c)$ or $P(A^c \cap B^c)$

By (1) they are all independent of each other.
So the proof is complete.

[4] Consider $P(I_{A_{i_1}} \in G_1, \cdots I_{A_{i_k}} \in G_k)$

$1 \leq i_1 < i_2 < \cdots < i_k \leq n$

Since $\{I_{A_{i_j}} \in G_j\} = \phi, \Omega, A_{i_j}, A_{i_j}^c$

and ϕ, Ω are independent of any measurable

sets, we just consider independence of

$\{A_{i_1}, A_{i_1}^c\} \times \cdots \times \{A_{i_k}, A_{i_k}^c\}$.

$\{A_1 \cdots A_n\}$: independent $\Rightarrow$ $\{A_{i_1} \cdots A_{i_k}\}$: independent

Without loss of generality, we just have to

prove that if $\{A_1 \cdots A_n\}$ are independent

then $\{A_1, A_1^c\} \times \cdots \times \{A_n, A_n^c\}$: are independent.

(step 1) $A_1^c, A_2 \cdots A_n$ are independent

$P(A_1^c \cap A_2 \cdots A_n) = P(A_2 \cap \cdots \cap A_n) - P(A_1 \cap \cdots A_n)$

$= (1 - P(A_1)) P(A_2) \cdots P(A_n)$

$= P(A_1^c) P(A_2) \cdots P(A_n)$

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(Step 2) By repeating the similar discussion (at most 2^n-times)

we have $\forall A_1^* \in \{A_1, A_1^c\} \dots A_n^* \in \{A_n, A_n^c\}$

$A_1^*, \dots, A_n^*$ are independent

$\boxed{5}$ Pairwise independence does not imply independence.

Suppose $P(X_i=0)=\frac{1}{2}$ $P(X_i=1)=\frac{1}{2}$ and

$X_1 \sim X_3$ are iid.

Consider events $A_1 = \{X_1 = X_2\}$
$A_2 = \{X_2 = X_3\}$
$A_3 = \{X_3 = X_1\}$

Then $\{A_1, A_2, A_3\}$ are pairwise independent

but not independent

$P(A_1 = A_2) = P(X_1 = X_2 = X_3) = \left(\frac{1}{2}\right)^3 \times 2 = \frac{1}{4}$

$P(A_1)\,P(A_2) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

So A_1 A_2 are independent (So are $\{A_2, A_3\}$, $\{A_3, A_1\}$)

However, $P(A_1 \cap A_2 \cap A_3) = P(X_1 = X_2 = X_3) = \frac{1}{4}$

$P(A_1)\,P(A_2)\,P(A_3) = \frac{1}{8}$

So $\{A_1, A_2, A_3\}$ are not independent

$\boxed{6}$ $\mathcal{A}_1, \dots, \mathcal{A}_n$: independent and π-system

$\boxed{\text{step 1}}$ $\sigma[\mathcal{A}_1], \mathcal{A}_2 \cdots \mathcal{A}_n$ are independent

(proof) pick arbitrary sets from $\mathcal{A}_2 \sim \mathcal{A}_n$ and fix them

(Let $A_2 \in \mathcal{A}_2, \dots, A_n \in \mathcal{A}_n$)

Consider $\mathcal{D}_1 = \{ A_1 \in \sigma[\mathcal{A}_1] \mid P(\bigcap_{i=1}^{n} A_i) = \prod_{i=1}^{n} P(A_i) \}$

$\mathcal{D}_1$ contains $\mathcal{A}_1$ because $\forall A_1 \in \mathcal{A}_1$, $A_1 \sim A_n$ are independent.

$\therefore \mathcal{A}_1 \subseteq \mathcal{D}_1$

And $\mathcal{D}_1$ is a Dynkin-system (λ-system)

- $\Omega \in \mathcal{D}_1$ … trivial
- $A_1, \tilde{A}_1 \in \mathcal{D}_1$ $\quad P(\tilde{A}_1 \cap A_2 \cdots \cap A_n) = P(\tilde{A}_1) P(A_2) \cdots P(A_n)$
  $P(A_1 \cap \cdots \cap A_n) = P(A_1) P(A_2) \cdots P(A_n)$
  We have $P((\tilde{A}_1 \setminus A_1) \cap A_2 \cdots A_n) = P(\tilde{A}_1 \setminus A_1) \cdots P(A_n)$
- $\{A_{1k}\}_{k \ge 1} \subseteq \mathcal{D}_1$ $\quad A_{1,k} \nearrow A_1$ $\quad P(A_{1k} \cap \cdots \cap A_n)$
  $= P(A_{1k}) \cdots P(A_n)$
  $k \to \infty$ we have $P(A_1 \cap \cdots \cap A_n) = P(A_1) \cdots P(A_n)$

So $\mathcal{D}_1$ is a Dynkin-system

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Since $\mathcal{A}_1 \subseteq \mathcal{D}_1$ and $\mathcal{D}_1$ is a Dynkin-system

$\lambda[\mathcal{A}_1] \subseteq \mathcal{D}_1$. And by π-λ theorem, we have

$\lambda[\mathcal{A}_1] = \sigma[\mathcal{A}_1] \subseteq \mathcal{D}_1$ ($\because \mathcal{A}_1$: π-system)

So we find out that $\sigma[\mathcal{A}_1], \mathcal{A}_2 \cdots \mathcal{A}_n$ are independent.

$\boxed{\text{step2}}$ In the same way, chose arbitrary sets from $\sigma[\mathcal{A}_1], \mathcal{A}_3, \mathcal{A}_4 \cdots \mathcal{A}_n$, and fix them

Let $A_1 \in \sigma[\mathcal{A}_1]$, $A_3 \in \mathcal{A}_3 \cdots A_n \in \mathcal{A}_n$

Consider $\mathcal{D}_2 = \{ A_2 \in \sigma(\mathcal{A}_2) \mid P(\bigcap_{i=1}^{n} A_i) = \prod_{i=1}^{n} P(A_i) \}$

Similary $\mathcal{A}_2 \subseteq \mathcal{D}_2$. $\mathcal{D}_2$: Dynkin-system

So $\lambda[\mathcal{A}_2] = \sigma(\mathcal{A}_2) \subseteq \mathcal{D}_2$
(π-λ system)

And we have $\sigma[\mathcal{A}_1]\ \sigma[\mathcal{A}_2] \cdots \mathcal{A}_3 \cdots \mathcal{A}_n$

are independent.

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$\boxed{\text{step 3}}$ Repeating the similar arguments again and finally we will prove that $\sigma(c_1) \cdots \sigma(c_n)$ are independent.

$\boxed{7}$ We apply theorem 21.7

$\mathcal{G}_i = \{ X_i^{-1}((-\infty, a]) \mid a \in \mathbb{R} \}$ $(i = 1 \sim n)$

$\mathcal{G}_1 \sim \mathcal{G}_n$ are independent and π-systems

$\left(X_i^{-1}((-\infty, a]) \cap X_i^{-1}((-\infty, b]) = X_i^{-1}((-\infty, a \wedge b]) \right.$

$a \wedge b = \min\{a, b\}$)

So by theorem 21.7, we have $\sigma[\mathcal{G}_1] \cdots \sigma[\mathcal{G}_n]$

are independent.

We still have to show that

$\sigma[\mathcal{G}_i] = \sigma[X_i] \overset{\text{def}}{=} \{ X_i^{-1}(B) \mid B \in \mathcal{B}(\mathbb{R}) \}$

We prove the following fact. Let f: measurable function,

$\mathcal{E}$: family of measurable sets.

$f^{-1}(\mathcal{E}) \overset{\text{def}}{=} \{ f^{-1}(E) \mid E \in \mathcal{E} \}$

$f^{-1}(\sigma[\mathcal{E}]) \overset{\text{def}}{=} \{ f^{-1}(B) \mid B \in \sigma[\mathcal{E}] \}$

Then $\sigma[f^{-1}(\mathcal{E})] = f^{-1}(\sigma[\mathcal{E}])$, (proof) Since $\sigma[\mathcal{E}]$ is a σ-algebra, thus $f^{-1}(\sigma[\mathcal{E}])$ is also a σ-algebra. (easy to verify)

And $f^{-1}(\mathcal{E}) \subseteq f^{-1}(\sigma[\mathcal{E}])$ therefore we have

$\sigma[f^{-1}(\mathcal{E})] \subseteq f^{-1}(\sigma[\mathcal{E}])$.

Next we prove $\sigma[f^{-1}(\mathcal{E})] \supseteq f^{-1}(\sigma[\mathcal{E}])$

We consider $\{B \mid f^{-1}(B) \in \sigma[f^{-1}(\mathcal{E})]\}$

This is also a σ-algebra ($\because \sigma[f^{-1}(\mathcal{E})]$ is a σ-algebra)

This contains $\mathcal{E}$, thus it also contains $\sigma[\mathcal{E}]$.

So $\forall B \in \sigma[\mathcal{E}]$ $f^{-1}(B) \in \sigma[f^{-1}(\mathcal{E})]$

This implies $f^{-1}(\sigma[\mathcal{E}]) \subseteq \sigma[f^{-1}(\mathcal{E})]$. ■

Using this fact. $\sigma[\mathcal{G}_i] \overset{\text{equal}}{=} \{X_i^{-1}(B) \mid B \in \sigma[\mathcal{E}]\}$ $\quad \mathcal{E} = \{(-\infty, a] \mid a \in \mathbb{R}\}$

$= \{X_i^{-1}(B) \mid B \in \mathcal{B}(\mathbb{R})\}$

$= \sigma(X_i)$

Now the proof is complete.

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$\boxed{8}$ Consider $\mathcal{A}_i = \{ \bigcap_{j=1}^{m_i} A_{ij} \mid A_{ij} \in \mathcal{F}_{ij} \}$ $(i=1\sim n)$

Then $\{\mathcal{A}_i\}_{i=1}^{n}$ are π-systems and independent

(I) $\mathcal{A}_i$ is a π-system.:

$$\underbrace{\left(\bigcap_{j=1}^{m_i} A_{ij}\right)}_{\in \mathcal{A}_i} \cap \underbrace{\left(\bigcap_{j=1}^{m_i} B_{ij}\right)}_{\in \mathcal{A}_i}$$

$$= \bigcap_{j=1}^{m_i} (A_{ij} \cap B_{ij})$$

Since $\mathcal{F}_{ij}$ is a σ algebra for each (i,j)

So $A_{ij} \cap B_{ij} \in \mathcal{F}_{ij}$. So $\bigcap_{j=1}^{m_i} A_{ij} \cap B_{ij} \in \mathcal{A}_i$

(II) $\mathcal{A}_1 \sim \mathcal{A}_n$ are independent

Let $I \subset \{1 \cdots n\}$ and $A_{ij} \in \mathcal{F}_{ij}$.

$$P\left(\bigcap_{i\in I} \bigcap_{j=1}^{m_i} A_{ij}\right) = \prod_{i\in I} \prod_{j=1}^{m_i} P(A_{ij}) \quad (\because \{\mathcal{F}_{ij}\} \text{ are indep})$$

$$= \prod_{i\in I} P\left(\bigcap_{j=1}^{m_i} A_{ij}\right)$$

So $\mathcal{A}_1 \sim \mathcal{A}_n$ are independent

By theorem 2.1.7 $\sigma[\mathcal{A}_1) \cdots \sigma(\mathcal{A}_n)$ are independent.

$\boxed{\text{claim}}$ $\mathcal{G}_i = \sigma[\mathcal{A}_i]$ $(i=1\sim n)$

① $\mathcal{G}_i \subseteq \sigma[\mathcal{A}_i]$

We prove $\bigcup_{j=1}^{m_i} \mathcal{F}_{ij} \subseteq \sigma[\mathcal{A}_i]$

If $A \in \bigcup_{j=1}^{m_i} \mathcal{F}_{ij}$, there exists j_0 s.t.

$A \in \mathcal{F}_{ij_0}$. We rewrite A as A_{ij_0}.

Since $A_{ij_0} = \Omega \cap \Omega \cap \cdots \underbrace{A_{ij_0}}_{\in \mathcal{F}_{ij_0}} \cap \cdots \Omega \in \mathcal{A}_i$

$\Rightarrow A_{ij_0} \in \sigma[\mathcal{A}_i]$

Thus $\bigcup_{j=1}^{m_i} \mathcal{F}_{ij} \subseteq \sigma(\mathcal{A}_i)$ $\left(\Rightarrow \underbrace{\sigma[\bigcup_{j=1}^{m_i} \mathcal{F}_{ij})}_{\mathcal{G}_i} \subseteq \sigma(\mathcal{A}_i)\right)$

② $\mathcal{G}_i \supseteq \sigma[\mathcal{A}_i]$

It's enough to show that $\mathcal{A}_i \subseteq \mathcal{G}_i$

$A_i = A_{i1} \cap A_{i2} \cdots \cap A_{i,m_i}$

$\{A_{i1} \cdots A_{i,m_i}\} \subseteq \mathcal{G}_i$ and $\mathcal{G}_i$ is a σ-algebra

Thus $\bigcap_{j=1}^{m_i} A_{ij} \in \mathcal{G}_i$. $\therefore \mathcal{A}_i \subseteq \mathcal{G}_i$ $(\Rightarrow \sigma(\mathcal{A}_i) \subseteq \mathcal{G}_i)$

Now the proof is complete ■

$\boxed{9}$ $\mathcal{F}_{ij} \overset{\text{def}}{=} \sigma[X_{ij}] = \{X_{ij}^{-1}(B) \mid B \in \mathcal{B}(\mathbb{R})\}$

$\mathcal{G}_i \overset{\text{def}}{=} \sigma[\bigcup_{j=1}^{m_i} \mathcal{F}_{ij}]$

By theorem 2.1.9. $\mathcal{G}_1 \sim \mathcal{G}_n$ are independent

We prove that $f_i(X_{i,1}, \ldots X_{i,m_i})$

is a $\mathcal{G}_i$-measurable random variable

claim) $f_i(X_{i,1} \cdots X_{i,m_i})$ is $\mathcal{G}_i$-measurable

Let $B \in \mathcal{B}(\mathbb{R})$. $(f_i(X_{i,1} \ldots X_{i,m_i}))^{-1}(B)$

$= (X_{i,1}, \ldots, X_{i,m_i})^{-1} \circ f_i^{-1}(B)$

Since f_i is $\mathbb{R}^{m_i}/\mathbb{R}$ measurable
thus $f_i^{-1}(B) \in \mathcal{B}(\mathbb{R}^{m_i})$.

So we prove that $\forall B \in \mathcal{B}(\mathbb{R}^{m_i})$,

$(X_{i,1} \cdots X_{i,m_i})^{-1}(B) \in \mathcal{G}_i$.

However. $\{\prod_{i=1}^{m_i} (-\infty, a_i] \mid a_i \in \mathbb{R}\}$ generates $\mathcal{B}(\mathbb{R}^{m_i})$,

it's enough to show that $\forall \prod_{i=1}^{m_i} (-\infty, a_i]$.

$(X_{i,1} \cdots X_{i,m_i})^{-1}\left(\prod_{j=1}^{m_i}(-\infty, a_{ij}]\right) \in \mathcal{G}_i$

(see, $\boxed{\text{NOTE}}$)

And $(X_{i,1} \cdots X_{i,m_i})^{-1}\left(\prod_{j=1}^{m_i}(-\infty, a_{ij}]\right)$

$= \bigcap_{j=1}^{m_i} \underbrace{X_{i,j}^{-1}((-\infty, a_{ij}])}_{\in \mathcal{F}_{ij}} \in \mathcal{A}_i = \left\{ \bigcap_{j=1}^{m_i} A_{ij} \;\middle|\; A_{ij} \in \mathcal{F}_{ij} \right\}$

$\subseteq \sigma[\mathcal{A}_i] = \mathcal{G}_i$

(see $\boxed{8}$)

So the proof is complete.

$\boxed{\text{NOTE}}$ $(S, \mathcal{A}), (T, \mathcal{B})$ -- measurable spaces.

$f: S \to T$ is a $\mathcal{A}/\mathcal{B}$-measurable function

$\mathcal{B} = \sigma[\mathcal{G}]$.

Then $\forall G \in \mathcal{G}$ $f^{-1}(G) \in \mathcal{A} \Rightarrow \forall B \in \mathcal{B}$ $f^{-1}(B) \in \mathcal{A}$.

(Proof) $\{B \mid f^{-1}(B) \in \mathcal{A}\}$ is a σ-algebra containing
($\because \mathcal{A}$: σ-algebra)
$\mathcal{G}$. So $\sigma[\mathcal{G}]$ is also contained by it.
(by assumption).
Hence $\forall B \in \sigma[\mathcal{G}]$ $f^{-1}(B) \in \mathcal{A}$.

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$\boxed{10}$ Consider a measure $\hat{\mu}$ on $(\mathbb{R}, \mathbb{B}(\mathbb{R}^n))$

which satisfies $\hat{\mu}\left(\prod_{i=1}^{n}(-\infty, x_i]\right) = P\left(\bigcap_{i=1}^{n}\{X_i \le x_i\}\right)$

Let $\mathcal{g} = \{(-\infty, x_1] \times \cdots \times (-\infty, x_n] \mid x_1 \sim x_n \in \mathbb{R}\}$

$\mathcal{g}$ is a π-system and $\sigma[\mathcal{g}] = \mathbb{B}(\mathbb{R}^n)$

By assumption, $\hat{\mu}\left(\prod_{i=1}^{n}(-\infty, x_i]\right)$

$= P\left(\bigcap_{i=1}^{n}\{X_i \le x_i\}\right)$ & (by independence)

$= \prod_{i=1}^{n} P(X_i \le x_i) = \prod_{i=1}^{n} \mu_i((-\infty, x_i])$

Thus on $\mathcal{g}$, $\hat{\mu} = \mu_1 \times \mu_2 \times \cdots \mu_n$.

And $(\mathcal{g}, \hat{\mu})$ is σ-finite (finite) and $\mathcal{g}$ is a π-system.

By uniqueness of measure, $\hat{\mu} = \mu_1 \times \cdots \times \mu_n$

$\boxed{\text{NOTE}}$

(The definition of $\mu_1 \times \cdots \times \mu_n$ is a measure on $\mathbb{B}(\mathbb{R}^n)$

which satisfies $(\mu_1 \times \cdots \mu_n)\left(\prod_{i=1}^{n}(-\infty, x_i]\right) = \prod_{i=1}^{n} \mu_i((-\infty, x_i])$)

Here $\hat{\mu}$ also satisfies this condition and

we have proved such a measure is unique

So $\hat{\mu} = \mu_1 \times \cdots \times \mu_n$

[1] Suppose $h \geq 0$.

$$E[h(X,Y)] = \int_\Omega h(X,Y)\, dP$$

By change Variable Formula. (We will show it again)

$$= \int_{\mathbb{R}^2} h(x,y)\, (\mu \times \nu)(dx\, dy)$$

By Tonelli's Theorem.

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} h(x,y)\, \mu(dx)\, \nu(dy) \quad \blacksquare$$

NOTE We show the change variable formula for two random variables.

Let $h : \mathbb{R}^2 \to [0,\infty)$ $\quad \exists \{h_n\}_{n \geq 1}$: nonnegative simple measurable s.t $h_n \uparrow h$

$$E[h(X,Y)] = \lim_{n\to\infty} E[h_n(X,Y)] \quad (\text{By MCT})$$

$$E[h_n(X,Y)] = \int_\Omega h_n(X,Y)\, dP$$

$$= \int_\Omega \sum_{j=1}^{J_n} a_j^{(n)} I_{A_j^{(n)}}(X(\omega), Y(\omega))\, dP \qquad \{a_j\} \subseteq [0,\infty) \quad \{A_j\} \subseteq \mathcal{B}(\mathbb{R}^2)$$

$$= \sum_{j=1}^{J_n} a_j^{(n)} \cdot P((X,Y) \in A_j^{(n)})$$

Let $\mu \sim X$, $\nu \sim Y$. By the previous exercise,

$$P \circ (X,Y)^{-1}(\cdot) = (\mu \times \nu)(\cdot)$$

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We can rewrite it as

$$= \sum_{j=1}^{J_n} a_j^{(n)} (\mu \times \nu)(A_j^{(n)})$$

$$= \int_{\mathbb{R}^2} h_n(x,y) \, (\mu \times \nu)(dx\,dy)$$

So $\lim_{n\to\infty} (\cdots) = \int_{\mathbb{R}^2} h(x,y) \, (\mu \times \nu)(dx\,dy)$

Finally by Tonelli's theorem

$$\int_{\mathbb{R}} \int_{\mathbb{R}} h(x,y) \, \mu(dx) \, \nu(dy)$$

Now the proof is complete. ∎

$\boxed{12}$ Consider (X, Y) below.

$X \backslash Y$	-1	0	1
-1	0	b	0
0	a	c	a
1	0	b	0

$2a + 2b + c = 1$ (probability) $(a>0.\ b>0.\ c>0)$

$E[X]E[Y] = 0 \cdot 0 = 0$

$\left(E[X] = \sum_{x=-1,0,1} x\, P(X=x) = (-1) \cdot b + 0(2a+c) + 1 \cdot (b) = 0 \right.$

$E[XY] = 0$

$\left(E[XY] = \sum_{(x,y) \in \{(-1,1).(1,-1).(-1,-1)(1,1)\}} xy\, P(X=x.\ Y=y) = 0 \right)$

However $P(X=1.\ Y=1) = 0$

$P(X=1)\, P(Y=1) = ab > 0$

So they are not independent

[3] $X \sim \mu,\ Y \sim \nu$ (i.e. $P(X \leq \lambda) = \mu((-\infty, \lambda])$
$= F(x)$)

$$P(X+Y \leq Z) = \int_\Omega \mathbb{I}\{X+Y \leq Z\}\, dP$$

By Change Variable Formula

$$= \int_{\mathbb{R}^2} \mathbb{I}\{x+y \leq Z\}\ \mu\times\nu(dx\,dy)$$

By Tonelli's Theorem ($\because \mathbb{I}\{\sim\} \geq 0$: non-negative)

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{I}\{x \leq Z-y\}\ \mu(dx)\, \nu(dy)$$

$$= \int_{\mathbb{R}} \mu((-\infty, Z-y])\, \nu(dy)$$

$$= \int_{\mathbb{R}} F(Z-y)\, \nu(dy)$$

$\downarrow$ (The same meaning)

$$= \int_{\mathbb{R}} F(Z-y)\, dG$$

$\boxed{14}$ By $\boxed{13}$. $P(X+Y \le Z) = \int F(Z-y)\, dG(y)$

Since X has a probability density function f,

$= \int_{\mathbb{R}} \int_{-\infty}^{Z-y} f(x)\, dx\, dG(y)$ ↓ $x \to x-y$

$= \int_{\mathbb{R}} \int_{-\infty}^{Z} f(x-y)\, dx\, dG(y)$ ↓ Tonelli's theorem

$= \int_{-\infty}^{Z} \int_{\mathbb{R}} f(x-y)\, dG(y)\, dx$

So. $g(x) \stackrel{\text{def}}{=} \int_{\mathbb{R}} f(x-y)\, dG(y)$

then $P(X+Y \le Z) = \int_{-\infty}^{Z} g(x)\, dx$

Hence $g(\cdot)$ is a pdf of Z

$\boxed{15}$ $\mathcal{g} \overset{def}{=} \{ \prod_{i=1}^{n} (a_i, b_i] \times \mathbb{R} \times \mathbb{R} \cdots \mid n \in \mathbb{N}.\ a_i, b_i \in \mathbb{R} \}$
$\mathcal{g}$ is a semi-algebra

$G_0[\cdot]$ is a smallest algebra containing $\cdot$.

We define a set function $P_0 : \mathcal{g} \to [0, \infty]$.
as $P_0(G) = \mu_n(\prod_{i=1}^{n} (a_i, b_i])$
where $G \in \mathcal{g}$, $G = \prod_{i=1}^{n} (a_i, b_i] \times \mathbb{R} \times \mathbb{R} \times \cdots$.

$\boxed{\text{Step 1}}$ First we extend $(\mathcal{g}, P_0)$ on an algebra $G_0[\mathcal{g}]$. : $(G_0[\mathcal{g}], P_0)$ which has finite additivity.

As we have proved in chapter 1, when $\mathcal{g}$ is a semi-algebra. $G_0[\mathcal{g}] = \mathcal{A} \overset{def}{=} \{ A \mid A$ is a finite disjoint union of sets from $\mathcal{g} \}$

And we define $P_0(A)$ $(A \in G_0[\mathcal{g}] \overset{eq}{=} \mathcal{A})$

as $P_0(A) \overset{def}{=} \sum_{j=1}^{J} P_0(G_j)$ $(A = \sum_{j=1}^{J} G_j)$

Then $(G_0[\mathcal{g}]\ P_0)$ has finite additivity and $P_0(\cdot)$ is well-defined.

(proof:) $A_1, A_2 \in \mathcal{g}$ and $A_1 \cap A_2 = \phi$
$A_1 = \sum_{j=1}^{J_1} G_{1j}$ $A_2 = \sum_{j=1}^{J_2} G_{2j}$

Σ .. disjoint union

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$$P_0(A_1 \cup A_2) = P_0\left(\sum_{i=1}^{2}\sum_{j=1}^{J_i} G_{(ij)}\right) = \sum_{i=1}^{2}\sum_{j=1}^{J_i} P_0(G_{(ij)})$$
$$= \sum_{i=1}^{2} P_0(A_i)$$

So it has finite additivity.

If $A = \sum_{j=1}^{J} G_j = \sum_{l=1}^{L} H_l$ $\quad \left(\{G_j \cap H_l\}_{\substack{j=1\sim J \\ l=1\sim L}} \subset \mathcal{G}\right.$

$$= \sum_{j=1}^{J}\sum_{l=1}^{L} G_j \cap H_l \qquad \left.(\because \mathcal{G} : \pi\text{-system})\right)$$

So $P_0(A) = \sum_{j=1}^{J}\sum_{l=1}^{L} P_0(G_j \cap H_l) = \sum_{l=1}^{L}\sum_{j=1}^{J} P_0(G_j \cap H_l)$

$$\sum_{j=1}^{J} P_0(G_j) \qquad\qquad \sum_{l=1}^{L} P_0(H_l)$$

$$\left(\because G_j = \sum_{l=1}^{L} G_j \cap H_l \ , \ H_l = \sum_{j=1}^{J} G_j \cap H_l\right.$$

So it's well-defined

$\boxed{\text{Step2}}$ Next we want to show that $(G_0[\mathcal{G}], P_0)$ can be extended $(G[\mathcal{G}], P)$

We show that 「$(G_0[\mathcal{G}], P_0)$ has countable additivity」

$\Leftrightarrow$ 「$\forall \{B_n\}_{n\ge 1} \subseteq G_0[\mathcal{G}]$ $\quad B_n \downarrow \phi \Rightarrow P_0(B_n) \downarrow 0$.」

(We will show this in the next exercise)

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Let $\mathcal{F}_n$ be a sub-σ-algebra of $\mathcal{B}(\mathbb{R}^{\mathbb{N}})$ containing $\{B \mid B = B^* \times \mathbb{R} \times \mathbb{R} \cdots, \ B^* \in \mathcal{B}(\mathbb{R}^n),$
$n = 1, 2, 3, \cdots \}$

And if $B \in \mathcal{B}(\mathbb{R}^n)$, then $B^* \in \mathcal{B}(\mathbb{R}^n)$ is a set such that $B = B^* \times \mathbb{R} \times \mathbb{R} \cdots$

in $\boxed{\text{Step 2}}$ We first prove that without loss of generality we may suppose $B_n \in \mathcal{F}_n$ for each $n = 1, 2, \ldots$

(proof) Since $B_n \in G_0[\mathcal{g}]$, B_n can be written as
$= \sum_{j=1}^{J_n} G_j^{(n)} \quad \{G_j^{(n)}\}_{j=1}^{J_n} \subset \mathcal{g}$

And each $\{G_j^{(n)}\}$ can be written as $G_j^{(n)} =$
$\underbrace{\left(\prod_{i=1}^{k_n} (a_{j,i}^{(n)}, b_{j,i}^{(n)}] \right)}_{k_n\text{-dim}} \times \mathbb{R} \times \mathbb{R} \cdots$

Thus $\{G_j^{(n)}\}_{j=1}^{J_n} \subseteq \mathcal{F}_{k_n}$

Be careful of the fact that $\mathcal{F}_1 \subset \mathcal{F}_2 \subseteq \cdots$

Hence without loss of generality we may assume

$k_1 < k_2 < \cdots$. (increasing)

Now we consider $\{\widetilde{B}_n\}_{n\ge1}$

$\widetilde{B}_1 = \mathbb{R}^N \quad \widetilde{B}_2 = \mathbb{R}^N \cdots \quad \widetilde{B}_{k_1-1} = \mathbb{R}^N$ and
$\widetilde{B}_{k_1} = B_1 \quad \widetilde{B}_{k_1+1} = B_1 \cdots \quad \widetilde{B}_{k_2-1} = B_1$ and
$\widetilde{B}_{k_2} = B_2, \quad \widetilde{B}_{k_2+1} = B_2 \cdots \quad \widetilde{B}_{k_3-1} = B_2 \cdots$

Then $\widetilde{B}_n \in \mathcal{F}_n$ for all $n \ge 1$ $\quad \widetilde{B}_n \downarrow \phi$ as $n \to \infty$.

So $\boxed{\text{step2}}$ is complete ■

$\boxed{\text{step3}}$ We try to derive a contradiction by assuming that $B_n \downarrow \phi$ but $\exists \delta > 0$ s.t. $P_0(B_n) \ge \delta > 0$ for all $n \ge 1$.

(As $\boxed{\text{step2}}$ shows, we assume $B_n \in \mathcal{F}_n$)

We may find $E_n \subseteq B_n$: $E_n = \bigcup_{j=1}^{J_n} [\tilde{a}_{j1}^{(n)}, \tilde{b}_{j1}^{(n)}] \times \cdots \times [\tilde{a}_{jn}^{(n)}, \tilde{b}_{jn}^{(n)}] \times \mathbb{R} \times \mathbb{R} \cdots$

($\because B_n \in \mathcal{F}_n$)

s.t. $\mu_n(B_n^* \setminus E_n^*) \le \frac{\delta}{2^{n+1}}$.

$F_n \overset{\text{def}}{=} \bigcap_{m=1}^{n} E_m$.

Then $B_n \setminus F_n = B_n \setminus \left(\bigcup_{m=1}^{n} E_m^c\right) = \bigcup_{m=1}^{n} (B_n \setminus E_m) \subset \bigcup_{m=1}^{n} (B_m \setminus E_m)$

($\because B_{n+1} \subseteq B_n \subseteq \cdots$)

So $\mu_n(B_n^* \setminus F_n^*) \leq \sum_{m=1}^{n} \mu_m(B_m^* \setminus F_m^*) \leq \sum_{n=1}^{\infty} \frac{\delta}{2^{n+1}} = \frac{\delta}{2}$

$\because \mu_n(B_n^*) = P_0(B_n) \geq \delta$

$\therefore \mu_n(F_n^*) \geq \frac{\delta}{2}$. F_n^* is not empty for all $n \geq 1$.

Moreover $F_n^* = F_n^* \cap (F_{n-1}^* \times \mathbb{R}) \cap \cdots (F_1^* \times \mathbb{R} \cdots \times \mathbb{R})$

is a compact set.

$\boxed{\text{Step 4}}$ Now we take a sequence of points from F_m.

(i.e $\omega^m \in F_m$ and $\omega^m = (\omega_1^m, \omega_2^m, \ldots)$)

Since $\omega^m \in F_m \subseteq F_1$ thus $\omega_1^m \in F_1^*$ and

F_1^* is a compact set (by $\boxed{\text{Step 3}}$), by Bolzano-Weierstrass

Theorem. we may find a subsequence $\{m_1(j)\}_{j \geq 1} \subseteq \mathbb{N}$

s.t $\omega_1^{m_1(j)} \to \theta_1 \in F_1^*$.

In the same way, when $m \geq 2$.

$(\omega_1^m, \omega_2^m) \in F_2^*$ $(F_2 \subseteq F_1)$ and F_2^* is a compact set.

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Thus we may take a subsequence of $\{m_1(j)\}_{j\geq 2}$ (say $m_2(j)$) st (w_1^m, w_2^m) converges.

So $(w_1^{m_2(j)}, w_2^{m_2(j)}) \to (\theta_1, \theta_2) \in F_2^*$.

By repeating the similar argument, $\theta = (\theta_1, \theta_2, \dots)$ $\in F_n$ for all $n \geq 1$. So $\theta \in \bigcap_{n=1}^{\infty} F_n$.

Thus F_n is not empty. $\bigcap_{n=1}^{\infty} F_n \subseteq \bigcap_{n=1}^{\infty} E_n \subseteq \bigcap_{n=1}^{\infty} B_n$

So $B_n \neq \phi$. It contradicts our assumption. ∎

$\boxed{16}$ Let $\mathcal{R}$ be a ring. (i.e. $\forall A, B \in \mathcal{R}$: $A \cup B, A \setminus B \in \mathcal{R}$)
Let $\mu : \mathcal{R} \to [0, \infty]$ and have finite additivity.

First, we show that the following are equivalent.

① $(\mathcal{R}, \mu)$ has countable additivity.
② $(\mathcal{R}, \mu)$ has sub-countable additivity
③ $(\mathcal{R}, \mu)$ is continuous from below.

① ⇒ ② trivial

① ⇒ ③ Suppose $A_n \uparrow A$ ($\{A_n\} \cup \{A\} \subseteq \mathcal{R}$)

Now let $B_1 = A_1$ $B_n = A_n \setminus A_{n-1}$ $(\in \mathcal{R})$ Then $A = \sum_{n=1}^{\infty} B_n$

$\mu(A) = \sum_{n=1}^{\infty} \mu(B_n) = \sum_{n=1}^{\infty} (\mu(A_n) - \mu(A_{n-1}))$ $(A_0 = \phi)$
$= \lim_{n\to\infty} \mu(A_n)$

② ⇒ ① $\{A_n\}_{n\geq 1} \cup \{A\} \subseteq \mathcal{R}$ and $A = \sum_{n=1}^{\infty} A_n$

$\therefore \mu(A) \leq \sum_{n=1}^{\infty} \mu(A_n)$ (by sub-additivity)

$\therefore \forall N \geq 1$ $\bigcup_{n=1}^{N} A_n \subseteq A$. $(\bigcup_{n=1}^{N} A_n \in \mathcal{R})$

$\therefore \mu(\bigcup_{n=1}^{N} A_n) = \sum_{n=1}^{N} \mu(A_n) \leq \mu(A)$ $N \uparrow \infty$.
(finite additive)

(finite additivity implies monotonicity) ∎

③ ⇒ ① $\{A_n\} \cup \{A\} \subseteq \mathcal{R}$: $A = \sum_{n=1}^{\infty} A_n$

$B_n \stackrel{def}{=} \sum_{m=1}^{n} A_m$ ($\in \mathcal{R}$) $B_n \nearrow A$

$\mu(A) = \lim_{n\to\infty} \mu(B_n) = \lim_{n\to\infty} \sum_{m=1}^{n} \mu(A_m) = \sum_{n=1}^{\infty} \mu(A_n)$

(by ③'s assumption) (finite additivity of μ) (So ① ~ ③ are equivalent)

Now we prove ① (⇔ ②, ③) ⇒ ④ ⇒ ⑤

④ μ is continuous from above.
⑤ μ is continuous from above at ϕ.

(④ : $\{A_n\} \cup \{A\} \subseteq \mathcal{R}$ $A_n \downarrow A$, $\mu(A_1) < \infty$
⇒ $\lim_{n\to\infty} \mu(A_n) = \mu(A)$)

(⑤ $\{A_n\} \subseteq \mathcal{R}$ $A_n \downarrow \phi$ $\mu(A_1) < \infty$
⇒ $\lim_{n\to\infty} \mu(A_n) = 0$)

We show ③ ⇒ ④

Suppose $A_n \downarrow A$ $\mu(A_1) < \infty$ ($\{A_n\} \cup \{A\} \subseteq \mathcal{R}$)

Let $B_n = A_1 \setminus A_n$ then $B_n \nearrow A_1 \setminus A$

So $\lim_{n\to\infty} \mu(B_n) = \mu(A_1 \setminus A) = \underset{<\infty}{\mu(A_1)} - \mu(A)$

$\parallel$

$\lim_{n\to\infty} (\underset{<\infty}{\mu(A_1)} - \mu(A_n))$ Hence we have $\lim_{n\to\infty} \mu(A_n) = \mu(A)$

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④ $\Rightarrow$ ⑤ trivial $(A=\phi)$

Finally we prove that if $(\mathcal{R}, \mu)$ is finite

(i.e $\forall A \in \mathcal{R}.\ \mu(A) < \infty$) then ⑤ $\Rightarrow$ ①.

Suppose $\{A_n\}_{n=1} \cup \{A\} \subseteq \mathcal{R}$. $A = \sum_{n=1}^{\infty} A_n$

$B_n = A \setminus \sum_{m=1}^{n} A_m \quad B_n \downarrow \phi \quad \mu(B_1) < \infty$

$\therefore \lim_{n\to\infty} \mu(B_n) = 0 \quad (\because ⑤)$

$= \lim_{n\to\infty} \left(\mu(A) - \mu\left(\sum_{m=1}^{n} A_m\right)\right) = 0$

$= \lim_{n\to\infty} \left(\mu(A) - \sum_{m=1}^{n} \mu(A_m)\right) = 0$

$\therefore \sum_{n=1}^{\infty} \mu(A_n) = \mu(A)$ ■

$\boxed{17}$ (1) "$(S, \mathcal{S})$ is nice" means that

there exists function $\varphi : S \mapsto \mathbb{R}$

s.t φ and φ^{-1} are both measurable

(i.e $\forall B \in \mathcal{B}(\mathbb{R})$ $\varphi^{-1}(B) \in \mathcal{S}$,
$\forall A \in \mathcal{S}$ $\varphi(A) \in \mathcal{B}(\mathbb{R})$)

(2) $\boxed{\text{step 1}}$ Let $S = [0,1)^{\mathbb{N}}$ $\mathcal{S} = \mathcal{B}([0,1)^{\mathbb{N}})$

We show that there exists a desired function φ_0

$x \in S$. $x = (x^1, x^2, x^3, \cdots)$ ($x^j \in [0,1)$ $\boxed{\forall j \geq 1}$)

We consider binary extension of $x^j = 0.x^j_1 x^j_2 \cdots$.

We also define a metric ρ on S as

$\rho(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} |x^n - y^n|$

Then we define $\varphi_0(x) = 0.x^1_1 \; x^1_2 x^2_1 \; x^1_3 x^2_2 x^3_1 \cdots$

$\varphi_0(x)$ and $\varphi_0^{-1}(x)$ is continuous (hence measurable)

($\because$ $\rho(x,y) \to 0 \Rightarrow |\varphi_0(x) - \varphi_0(y)| \to 0$ and $|\varphi_0(x) - \varphi(y)| \to 0$
(I) $\Rightarrow \rho(x,y) \to 0$)
(II)

(I) 「$\rho(x,y) \to 0 \Rightarrow |\varphi_0(x) - \varphi_0(y)| \to 0$」

We show if x is fixed and $^{\exists}\delta > 0$ $|\varphi_0(x) - \varphi_0(y)| \geq \delta > 0$.

then $^{\exists}\varepsilon > 0$ st $\rho(x,y) \geq \varepsilon > 0$

Since $|\varphi_0(x) - \varphi_0(y)| > \delta > 0$ we may find $n, k \in \mathbb{N}$

st $|x_k^n - y_k^n| = 1$. thus we may find $\varepsilon > 0$ st

$\rho(x,y) \geq \varepsilon$.

(II) 「$|\varphi_0(x) - \varphi_0(y)| \to 0$ then $\rho(x,y) \to 0$」

The argument is similar to (I).

$\boxed{\text{step 2}}$ Next suppose that a metric space

$(S, \tilde{d})$ is given.

When $\tilde{d}$ is a metric, $d = \dfrac{\tilde{d}}{1+\tilde{d}}$ is also a metric.

So we define $d(x,y) = \dfrac{\tilde{d}(x,y)}{1+\tilde{d}(x,y)}$

$0 \leq d(x,y) < 1$.

By assumption, we can take a countable dense set $Q = \{q_1, q_2 \cdots q_n \cdots\} \subseteq S$.

We define $\psi(x) : S \to [0,1]^N$ as

$\psi(x) = (d(x, q_1), d(x, q_2), d(x, q_3) \cdots)$

Now we show that ψ, ψ^{-1} are both continuous (thus measurable)

ρ is a metric on $[0,1]^N$ defined at $\boxed{\text{Step 1}}$..

(I) ψ is continuous

$$\rho(\psi(x), \psi(y)) = \sum_{n=1}^{\infty} \frac{1}{2^n} |d(x, q_n) - d(y, q_n)| \quad \text{triangle inequality}$$
$$\leq \sum_{n=1}^{\infty} \frac{1}{2^n} d(x,y) = d(x,y)$$

So $y \to x$ (in d) then $\psi(y) \to \psi(x)$ (in ρ)

(II) ψ^{-1} is continuous

If $\psi(x_n) \to \psi(x)$ (in ρ) (as $n \to \infty$)

Then for each $m \geq 1$, $|d(x_n, q_m) - d(x, q_m)| \to 0$ (as $n \to \infty$)

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Since $Q = \{q_1, q_2, \ldots\}$ is a dense set

$\forall \varepsilon > 0$, we may find $m \geq 1$ s.t. $d(x, q_m) < \varepsilon$.

$$d(x_n, q_m) = |d(x_n, q_m) - d(x, q_m) + d(x, q_m)|$$

$$\leq \underbrace{|d(x_n, q_m) - d(x, q_m)|}_{< \varepsilon} + \underbrace{d(x, q_m)}_{< \varepsilon} < 2\varepsilon$$

So $d(x_n, q_m) < 2\varepsilon$

Also by the triangle inequality,

$$d(x_n, x) \leq \underbrace{d(x_n, q_m)}_{< 2\varepsilon} + \underbrace{d(q_m, x)}_{< \varepsilon} < 3\varepsilon$$

So $d(x_n, x) \to 0$ (as $n \to \infty$) ∎

(So $\varphi_0 \circ \psi$ will be a desired function)

18. We show that

$$\begin{cases} ① \ h\circ\rho(x,y) = h\circ\rho(y,x) \\ ② \ h\circ\rho(x,y) \geq 0 \quad \text{and} \quad h\circ\rho(x,y)=0 \Leftrightarrow x=y \\ ③ \ h\circ\rho(x,y) \leq h\circ\rho(x,z) + h\circ\rho(z,y) \end{cases}$$

① ∴ Since $\rho(x,y)=\rho(y,x)$, So $h\circ\rho(x,y) = h\circ\rho(y,x)$

② $\rho(x,y) \geq 0$ and $h(x) \geq 0$ $(x \geq 0)$ thus

$h\circ\rho(x,y) \geq 0$. Moreover $h(x)$ is strictly increasing $(x>0)$

hence $h(x)=0 \Leftrightarrow x=0$

∴ $h\circ\rho(x,y) = 0 \Leftrightarrow \rho(x,y) \Leftrightarrow x=y$ (∵ ρ is a metric)

③ First we show that $h(x+y) \leq h(x)+h(y)$

$\int_a^b h'(t)\,dt = h(b)-h(a)$

$b = x+y \quad a=y$

$\int_y^{x+y} h'(t)\,dt = h(x+y)-h(y)$

LHS : $\int_y^{x+y} h'(t)\,dt = \int_0^x h'(u+y)\,du$ & h': decreasing

$\leq \int_0^x h'(u)\,du = h(x)-h(0) = h(x)$

So $h(x+y)-h(y) \leq h(x)$

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$\therefore\ h(x+y) \leq h(x) + h(y)$ ... (*1)

Now $\rho(x,y) \leq \rho(x,z) + \rho(y,z)$ ... (*2) ($\because$ ρ: metric)

and $h'(x) > 0 \quad (x > 0)$ ... (*3)

We have $h \circ \rho(x,y) \leq h(\rho(x,z) + \rho(y,z)) \leq h \circ \rho(x,z) + h \circ \rho(y,z)$

($\because$ (*2) & (*3)) ($\because$ (*1))

Now the proof is complete. ■

19 μ: Lebesgue measure

① $\{X_n(w)\}_{n\ge 1}$ are uncorrelated.

(proof) Let $n>m$. $(n, m \in \mathbb{N})$

$$E[X_n(w)] = \int_\Omega \sin 2\pi n w \, P(dw) = \int_0^1 \sin 2\pi w \, dw$$

$$= \left[\frac{-1}{2\pi} \cos 2\pi w \right]_0^1 = \frac{-1}{2\pi}(1-1) = 0$$

$\therefore E[X_m(w)] = 0$.

$$E[X_n(w) X_m(w)] = \int_0^1 \sin 2\pi n w \sin 2\pi m w \, dw$$

$$= \int_0^1 \frac{1}{2}\left(\cos(2\pi n w - 2\pi m w) - \cos(2\pi n w + 2\pi m w) \right) dw$$

$$= \frac{1}{2}\int_0^1 \cos 2\pi(n-m)w - \frac{1}{2}\int_0^1 \cos 2\pi(n+m)w \, dw$$

$$= \frac{1}{2}\left[\frac{\sin 2\pi(n-m)w}{2\pi(n-m)} \right]_0^1 - \frac{1}{2}\left[\frac{\sin 2\pi(n+m)w}{2\pi(n+m)} \right]_0^1$$

$$= 0$$

So. $\underbrace{E[X_n(w) X_m(w)] - E[X_n(w)] E[X_m(w)]}_{\mathrm{cov}[X_n(w),\, X_m(w)]} = 0$

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② $\{X_n(w)\}_{n\ge 1}$ are not independent

Take $n=1$ and 3

· $P(X_1(w)\in[0,1],\ X_3(w)\in[0,1])$

$= P(\{w\in\Omega \mid 0\le w\le \frac{1}{2},\ 0\le w\le \frac{1}{6},\ \frac{1}{3}\le w\le \frac{1}{2},\ \frac{2}{3}\le w\le \frac{5}{6}\})$

$= \frac{1}{3}$

· $P(X_1(w)\in[0,1])\,P(X_3(w)\in[0,1])$

$= \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$

So $P(X_1\in[0,1],\ X_3\in[0,1]) \ne P(X_1\in[0,1])\,P(X_3\in[0,1])$

$\boxed{20}$

① $P(X+Y=0) = \sum_{y \in \mathbb{R}} \mu(\{-y\}) \nu(\{y\})$

(proof) We use change variable formula

$$P(X+Y=0) = \int_\Omega \mathbb{I}\{X+Y=0\} \, dP$$

↓ change Variable Formula

$$= \int_{\mathbb{R}^2} \mathbb{I}\{x+y=0\} \, \mu(dx)\, \nu(dy)$$

↓ Tonelli

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{I}_{\{-y\}}(x) \, \mu(dx) \, \nu(dy)$$

$$= \int_{\mathbb{R}} \mu(\{-y\}) \, \nu(dy)$$

Since μ is a probability measure,

there are at most countable points of discontinuity.

$A \overset{\text{def}}{=} \{y \in \mathbb{R} \mid \mu(\{-y\}) > 0\}$. A is at most countable

Then $\int_{\mathbb{R}} \mu(\{-y\}) \, \nu(dy) = \int_{\mathbb{R}} \mu(\{-y\}) \cdot \mathbb{I}_A(y) \, d\nu$

Let $A = \{a_1, a_2, \dots a_n, \dots\}$ ($\because$ countable)

$$= \int_{\mathbb{R}} \sum_{n=1}^{\infty} \mu(\{-a_n\}) \cdot \mathbb{I}_{\{a_n\}}(y) \, d\nu$$

↓ Monotone Convergence Theorem

$$= \sum_{n=1}^{\infty} \mu(\{-a_n\}) \, \nu(\{a_n\})$$

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$$\therefore \sum_{n=1}^{\infty} \mu(\{-a_n\})\, \nu(\{a_n\})$$

$$= \sum_{y \in A} \mu(\{-y\})\, \nu(\{y\}) \quad \cdots \circledast$$

when $y \in \mathbb{R} \setminus A$ then $\mu(\{-y\}) = 0$.

Hence we may rewrite $\circledast$ as $\sum_{y \in \mathbb{R}} \mu(\{-y\})\, \nu(\{y\})$

Now the proof is complete ■

② X is continuous $\Rightarrow$ $P(X=Y)=0$

(proof) $P(X=Y) = P(X+(-Y)=0)$

By ①, $= \sum_{y \in \mathbb{R}} \mu(\{y\})\, \nu(\{y\})$

However $\mu(\{y\}) = 0$ for all $y \in \mathbb{R}$.

Therefore $=0$.

$\boxed{2}$ We need to show that for all $n \in \mathbb{N}$
$\{Y_1, Y_2, \dots Y_n\}$ are independent and $P(Y_k=0) = P(Y_k=1) = \frac{1}{2}$

(I) $P(Y_k=1) = P(\{[2^k w] \text{ is odd}\})$

$\left(I_{k\ell} \overset{\text{def}}{=} \left[\frac{\ell}{2^k}, \frac{\ell+1}{2^k}\right) ; \ \ell = 0, 1 \dots 2^k - 1 \right)$

$= P\left(\left\{ w \in \bigcup_{\ell = 1, 3, 5, \dots, 2^k-1} I_{k\ell} \right\}\right) = \mu(I_{k\ell}) \times 2^k \times \frac{1}{2}$

$= \frac{1}{2^k} \times 2^k \times \frac{1}{2} = \frac{1}{2}$

(μ: Lebesgue measure)

So $P(Y_k=0) = P(Y_k=1) = \frac{1}{2}$ for all $k \geq 1$ ■

(II) Independence of $\{Y_1, Y_2 \dots Y_n\}$

Let $\{y_1, y_2 \dots y_n\} \subseteq \{0, 1\}$

$P(Y_1 = y_1, Y_2 = y_2, \dots, Y_n = y_n)$

$= \mu\left(\left[\sum_{k=1}^{n} \frac{y_k}{2^k}, \sum_{k=1}^{n} \frac{y_k}{2^k} + \frac{1}{2^n}\right)\right) = \frac{1}{2^n}$ ⊛

$= P(Y_1 = y_1) \cdot P(Y_2 = y_2) \cdots P(Y_n = y_n)$ ■

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(*)

We show that $\{\omega \in \Omega \mid Y_1 = y_1, \dots, Y_n = y_n\}$

$$= \left[\sum_{k=1}^{n} \frac{y_k}{2^k}, \sum_{k=1}^{n} \frac{y_k}{2^k} + \frac{1}{2^n} \right)$$

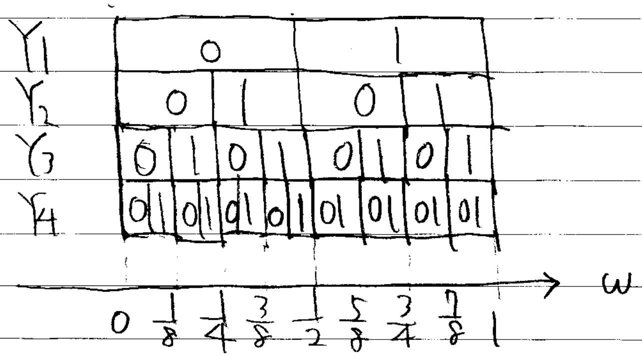
by induction.

$\underline{n=1}$ … $\{Y_1 = y_1\}$ … $\{Y_1 = 0\} = [0, \frac{1}{2})$
$\{Y_1 = 1\} = [\frac{1}{2}, 1)$

(*) is true

$\underline{n=n_0}$ We suppose (*) is true.

$\underline{n=n_0+1}$ $\{Y_1 = y_1, \dots Y_{n_0} = y_{n_0}, Y_{n_0+1} = y_{n_0+1}\}$



As the figure shows, when $\{Y_1 = y_1 \dots Y_{n_0} = y_{n_0}\}$,

if $y_{n_0+1} = 0$ … the left edge of interval does not change
$y_{n_0+1} = 1$ … the left edge of interval moves $+\frac{1}{2^{n_0+1}}$

and the width of the interval becomes half anyway.

So we find out that (*) is true.

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$\boxed{22}$ $V[X_1 + \cdots X_n] = E\{(X_1 - \mu_1) + \cdots (X_n - \mu_n)\}^2$

$= \sum_{i=1}^{n} E[(X_i - \mu_i)^2] + \sum_{i \neq j} E[(X_i - \mu_i)(X_j - \mu_j)]$

$= \sum_{i=1}^{n} V(X_i)$ $\qquad\qquad = 0$ $\quad$ ($\because$ uncorrelated)

■

$\boxed{23}$ If $Z_n \xrightarrow{L^p} Z$ $(p>0)$ $\Rightarrow$ $Z_n \xrightarrow{p} Z$

(proof) $E[|Z_n - Z|^p] = \int_\Omega |Z_n - Z|^p dp$

$\geq \int_{\{w \in \Omega \mid |Z_n - Z| > \varepsilon\}} |Z_n - Z|^p dp$

$\geq \varepsilon^p \cdot p(\{|Z_n - Z| > \varepsilon\})$

Thus $p(|Z_n - Z| > \varepsilon) \leq \frac{1}{\varepsilon^p} E|Z_n - Z|^p \to 0$

(as $n \to \infty$)

So $Z_n \xrightarrow{L^p} Z \Rightarrow Z_n \xrightarrow{p} Z$

$\boxed{24}$ Suppose $X_1, X_2, \cdots X_n \overset{iid}{\sim}$ Bernoulli(x)

$S_n \overset{def}{=} X_1 + X_2 + \cdots + X_n \sim Bin(n, x)$

$\bar{X} \overset{def}{=} (X_1 + X_2 + \cdots + X_n)/n$

$E[f(\bar{X})] \overset{eq}{=} \sum_{k=0}^{n} f\left(\frac{k}{n}\right) x^k (1-x)^{k} \cdot {}_nC_k$

$\left(\because \sum_{k=0}^{n} P(S_n = k) \cdot f\left(\frac{k}{n}\right) \right)$

So $f_n(x) = E[f(\bar{X})]$.

$|f(x) - f_n(x)| \overset{eq}{=} |f(x) - E[f(\bar{X})]| = |E[f(x)] - E[f(\bar{X})]|$
$\leq E|f(x) - f(\bar{X})|$ (triangle inequality)

$(\because f(x)$ is a constant thus $f(x) = E[f(x)])$

By assumption $f(x)$ is continuous at $[0,1]$

which is a compact set (bounded closed set)

therefore $f(x)$ is uniformly continuous and bounded.

$\Rightarrow \forall \varepsilon > 0 \ \exists \delta(\varepsilon) > 0$ s.t. $\forall x_1, x_2 \in [0,1]$: $|x_1 - x_2| < \delta(\varepsilon)$

$|f(x_1) - f(x_2)| < \varepsilon$

$$E|f(x)-f(\bar{X})| = \int_{\{\omega \mid |\bar{X}-x| > \delta_\varepsilon\}} |f(x)-f(\bar{X})|\,dP \;\cdots\; ①$$

$$+ \int_{\{\omega \mid |\bar{X}-x| \le \delta_\varepsilon\}} |f(x)-f(\bar{X})|\,dP \;-\; ②$$

$$① \le 2M\,P(\{\omega\in\Omega \mid |\bar{X}(\omega)-x| > \delta_\varepsilon\})$$

$$② \le \varepsilon \cdot P(\{\omega\in\Omega \mid |\bar{X}(\omega)-x| \le \delta_\varepsilon\}) \le \varepsilon$$ (not related to x)

As for ①, by Markov's inequality,

$$\le 2M \cdot \frac{1}{\delta_\varepsilon^2} \cdot V[\bar{X}(\omega)] = \frac{2M}{\delta_\varepsilon^2} \cdot \frac{x(1-x)}{n}$$

Since $x(1-x) \le \frac{1}{4}$

$$\therefore \sup_{x\in[0,1]} ① \le \frac{M}{2\delta_\varepsilon^2} \cdot \frac{1}{n}$$

Hence we have $\sup_{x\in[0,1]} E|f(x)-f(\bar{X})| \le \varepsilon + \frac{M}{2n\cdot\delta_\varepsilon^2}$

$$\overline{\lim_{n\to\infty}} \sup_{x\in[0,1]} E|f(x)-f(\bar{X})| \le \varepsilon$$

$$\therefore \overline{\lim_{n\to\infty}} \sup_{x\in[0,1]} |f(x)-f_n(x)| \le \overline{\lim_{n\to\infty}} \sup_{x\in[0,1]} E|f(x)-f(\bar{X})| \le \varepsilon. \quad (\varepsilon\downarrow 0)$$

Now the proof is complete. ∎

$\boxed{25}$ By using the conclusion of $\boxed{23}$,

$$E\left[\left\{\frac{(S_n - \mu_n)}{b_n}\right\}^2\right] = \frac{1}{b_n^2} V[S_n] = \frac{\sigma_n^2}{b_n^2} \to 0 \quad (\text{as } n \to \infty)$$

So $\frac{S_n - \mu_n}{b_n} \xrightarrow{L^2} 0 \Rightarrow \frac{S_n - \mu_n}{b_n} \xrightarrow{P} 0$

$\boxed{26}$ Coupon Collector's problem.

There are n- different items

We want to collect all kinds of items

In each trial, the probability of obtaining each item is $\frac{1}{n}$.

When we have already collected $k-1$, items the number of trial to obtain the new item follows geometric distribution $\sim Geo\left(\frac{n-k+1}{n}\right)$

Let $T_k \sim Geo\left(\frac{n-k+1}{n}\right)$

(Here we define T_k is the number trials to succeed)

(※ Geometric distribution has two different definition)

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So $T_1+T_2+\dots+T_n$ is the number of trials to collect all kinds of items ($T_1 \sim T_n$: independent)

When $X \sim Geo(p)$, $E[X]=\frac{1}{p}$ $V[X]=\frac{1-p}{p^2}=\frac{q}{p^2}$ $(q=1-p)$

Let $T=T_1+T_2+\dots+T_n$

$$E[T]=\sum_{k=1}^{n}\frac{n}{n-k+1}=\sum_{k=1}^{n}\frac{n}{k}=n\sum_{k=1}^{n}\frac{1}{k}$$

$$V[T]=\sum_{k=1}^{n}\left(\frac{n}{n-k+1}\right)^2\cdot\left(\frac{k-1}{n}\right)=\sum_{k=1}^{n}\frac{(k-1)n}{(n-k+1)^2}$$

$$V[T]\leq\sum_{k=1}^{n}\frac{n^2}{(n-k+1)^2}=\sum_{k=1}^{n}\frac{n^2}{k^2}\leq\frac{\pi^2}{6}n^2$$

$$E\left[\left(\frac{T-E[T]}{n\log n}\right)^2\right]=\frac{V[T]}{n^2(\log n)^2}\leq\frac{\pi^2}{6}\cdot\frac{1}{(\log n)^2}\to 0 \quad (\text{as } n\to\infty)$$

By $\boxed{2.5}$ (theorem 2.2.6) $\frac{T-E[T]}{n\log n}\xrightarrow{P}0$ $\left(\frac{T}{n\log n}-\frac{E[T]}{n\log n}\xrightarrow{P}0 \cdots (*_1)\right)$

$$\log n\leq\sum_{k=1}^{n}\frac{1}{k}\leq\log n+1 \Rightarrow n\log n\leq E[T]\leq n\log n+n$$

$\therefore$ $\frac{E[T]}{n\log n}\xrightarrow{P}1 \cdots (*_2)$
(a.s)

$(*_1)+(*_2)$ We have the desired result.

(NOTE, $X_n\xrightarrow{P}X$ $Y_n\xrightarrow{P}Y$ $\Rightarrow$ $X_n+Y_n\xrightarrow{P}X+Y$)

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27 Consider the conditions below.

① $\sum_{k=1}^{n} P(|X_{nk}| > b_n) \to 0$ (as $n \to \infty$)

② $\frac{1}{b_n^2} \sum_{k=1}^{n} E[\widehat{X_{nk}}^2] \to 0$ (as $n \to \infty$)

Let $\widetilde{S_n} = \sum_{k=1}^{n} \widetilde{X_{nk}}$

$$P\left(\left|\frac{S_n - a_n}{b_n}\right| > \varepsilon\right) = P\left(\left|\frac{S_n - a_n}{b_n}\right| > \varepsilon,\ S_n = \widetilde{S_n}\right)$$

$$+ P\left(\left|\frac{S_n - a_n}{b_n}\right| > \varepsilon,\ S_n \neq \widetilde{S_n}\right)$$

$$\leq P\left(\left|\frac{\widetilde{S_n} - a_n}{b_n}\right| > \varepsilon\right) + P(S_n \neq \widetilde{S_n})$$ ($\because A \subseteq B \Rightarrow P(A) \leq P(B)$)

$$\leq \frac{1}{\varepsilon^2} \frac{1}{b_n^2} V[\widetilde{S_n}] + P(\exists k \text{ s.t. } X_{nk} \neq \widetilde{X_{nk}})$$ ⊛

$$\leq \frac{1}{\varepsilon^2 b_n^2} \sum_{k=1}^{n} E[\widehat{X_{nk}}^2] + P\left(\bigcup_{k=1}^{n} \{|X_{nk}| > b_n\}\right)$$ ($V[X] \leq E[X^2]$)

$$\leq \frac{1}{\varepsilon^2 b_n^2} \sum_{k=1}^{n} E[\widehat{X_{nk}}^2] + \sum_{k=1}^{n} P(|X_{nk}| > b_n)$$

by ② $\to 0$ by ①, $\to 0$

(⊛ Markov's inequality, $X_{nk} \neq \widehat{X_{nk}} \Leftarrow |X_{nk}| > b_n$)

$\boxed{28}$ Let $X_{nk} = X_k \quad b_n = n.$

$S_n = X_1 + \cdots + X_n = \sum_{k=1}^{n} X_{nk}$

$\tilde{S}_n = \sum_{k=1}^{n} X_{nk} \cdot \mathbb{I}\{|X_{nk}| \leq b_n\} \qquad a_n = E[\tilde{S}_n]$

We verify the condition in $\boxed{27}$

① $\sum_{k=1}^{n} P(|X_{nk}| > b_n) = \sum_{k=1}^{n} P(|X_1| > n) = n P(|X_1| > n) \to 0$

$(\because$ identically distributed$)$ (by assumption)

② $\frac{1}{b_n^2} \sum_{k=1}^{n} E[|\tilde{X}_{nk}|^2]$

$= \frac{1}{n^2} \sum_{k=1}^{n} E[X_{nk}^2 \cdot \mathbb{I}\{|X_{nk}| \leq n\}]$ identically distributed

$= \frac{1}{n^2} \sum_{k=1}^{n} E[X_1^2 \cdot \mathbb{I}\{|X_1| \leq n\}]$

$= \frac{1}{n} E[X_1^2 \cdot \mathbb{I}\{|X_1| \leq n\}]$ ⊛ when $X \geq 0$ (a.s) $E[X^p] = \int_0^\infty p x^{p-1} P(X > x)\,dx$

$= \frac{1}{n} \int_0^\infty 2x\, P(|X_1| \cdot \mathbb{I}\{|X_1| \leq n\} > x)\,dx$

$\leq \frac{1}{n} \int_0^n 2x\, P(|X_1| > x)\,dx$

$\leq \frac{1}{n} \int_0^M \underbrace{2x\, P(|X_1| > x)}_{(\leq 2M)}\,dx + \frac{1}{n} \int_M^n \underbrace{2x\, P(|X_1| > x)}_{< \varepsilon}\,dx$

$\leq \frac{2M^2}{n} + \frac{(n-M)}{n} \varepsilon$

$\because$ $xP(|X_i|>x) \to 0$ as $x \to \infty$

So we may find $M>0$ s.t. $2x\,P(|X_i|>x) < \varepsilon$

when $x > M$.

$$\therefore \liminf_{n\to\infty} \left(\frac{2M^2}{n} + \frac{(n-M)}{n}\varepsilon \right) = \varepsilon.$$

By taking $\varepsilon \downarrow 0$, we have the desired result.

(Satisfy ① & ②)

$$\therefore \frac{S_n - E[S_n]}{b_n} = \frac{S_n}{n} - \mu_n \xrightarrow{P} 0 \quad \blacksquare$$

(L.D.C.T = Lebesgue's Dominated Convergence Theorem)

$\boxed{29}$ We use the conclusion of $\boxed{28}$.

When $E|X_i| < \infty$, by L.D.C.T $\left(|X_i| \geq |X_i| \, \mathbb{I}\{|X_i| > \lambda\} \right)$, we have

$\lambda P(|X_i| > \lambda) \to 0$ (as $\lambda \to \infty$)

$$\because E[|X_i|] \geq E[|X_i| \cdot \mathbb{I}\{|X_i| > \lambda\}] \geq E[\lambda \cdot \mathbb{I}\{|X_i| > \lambda\}]$$

$$= \lambda P(|X_i| > \lambda)$$

$$\lim_{\lambda \to \infty} E[|X_i| \cdot \mathbb{I}\{|X_i| > \lambda\}] \overset{\text{L.D.C.T}}{=} E[\lim_{\lambda \to \infty} |X_i| \cdot \mathbb{I}\{|X_i| > \lambda\}]$$

$$= E[\infty \cdot \mathbb{I}_\phi] = \infty \cdot P(\phi)$$

$$= \infty \cdot 0 = 0$$

Therefore, $\dfrac{S_n - E[\tilde{S}_n]}{n} \xrightarrow{P} 0$ (by $\boxed{28}$)

$$E[\tilde{S}_n] = \sum_{k=1}^{n} E[X_k \cdot \mathbb{I}\{|X_k| \leq n\}] = n E[X_1 \mathbb{I}\{|X_1| \leq n\}]$$

$$\Rightarrow \hat{\mu}_n \overset{\text{def}}{=} \frac{1}{n} E[\tilde{S}_n] = E[X_1 \cdot \mathbb{I}\{|X_1| \leq n\}]$$

By L.D.C.T $\lim_{n \to \infty} \hat{\mu}_n = \lim_{n \to \infty} E[X_1 \cdot \mathbb{I}\{|X_1| \leq n\}]$ ↓ L.D.C.T

$$= E[\lim_{n \to \infty} X_1 \cdot \mathbb{I}\{|X_1| \leq n\}]$$

$$= E[X_1] = \mu$$

Hence

$\cdot$ $\dfrac{S_n}{n} - \hat{\mu}_n \xrightarrow{P} 0$

$\cdot$ $\hat{\mu}_n \to \mu$ ($\Rightarrow \hat{\mu}_n \xrightarrow{P} \mu$) $\Rightarrow \dfrac{S_n}{n} - \mu \xrightarrow{P} 0$ ∎

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$\boxed{30}$ $E\left[\left(\frac{S_n}{n}-\nu_n\right)^2\right] = \frac{1}{n^2}V[S_n] = \frac{1}{n^2}\sum_{m=1}^{n}V[X_m]$

($\because$ uncorrelated)

Since $\frac{V[X_m]}{m} \to 0$, $\forall \varepsilon>0$ $\exists M^{(\varepsilon)}$ s.t. $m > M^{(\varepsilon)} \Rightarrow \frac{V[X_m]}{m} < \varepsilon$

Thus $\frac{1}{n^2}\sum_{m=1}^{n}V[X_m] \leq \frac{1}{n^2}\sum_{m=1}^{M}V[X_m] + \frac{1}{n}\sum_{m=M+1}^{n}\frac{V[X_m]}{m}$ $\left(\because \frac{1}{m} \geq \frac{1}{n}\right)$

$\leq \frac{1}{n^2} M^{(\varepsilon)} \max_{1\leq m \leq M(\varepsilon)} V[X_m] + \frac{1}{n}(n-M)\varepsilon$

$\liminf_{n\to\infty}\left(\frac{1}{n^2} M^{(\varepsilon)} \max_{1\leq m\leq M_\varepsilon} V[X_m] + \frac{1}{n}(n-M)\varepsilon\right)$

$= \varepsilon.$

$\therefore$ $\varepsilon \downarrow 0$ we have the desired result ■

[31] We show that $S_n/n \xrightarrow{L_2} 0$

$$E\left[\left(\frac{X_1+\cdots X_n}{n}\right)^2\right] = \frac{1}{n^2}\Big(\underbrace{\sum_{k=1}^{n} E[X_k^2]}_{①} + \underbrace{\sum_{k\neq l} E[X_k X_l]}_{②}\Big)$$

① Since $E[X_k X_l] \leq r(k-l)$, $E[X_k^2] \leq r(0)$ $(<\infty)$

$$\therefore \frac{1}{n^2}\sum_{k=1}^{n} E[X_k^2] \leq \frac{1}{n} r(0) \to 0 \quad (\text{as } n\to\infty)$$

② Next we consider $\frac{1}{n^2}\sum_{k\neq l} E[X_k X_l]$

Since $r(m)\to 0$ (as $k\to\infty$), $\forall \varepsilon>0$ $\exists M$ s.t. $\forall m>M$

$r(m)<\varepsilon$.

$$\frac{1}{n^2}\sum_{k\neq l} E[X_k X_l] = \frac{1}{n^2}\sum_{\substack{(k,l)\\ k\neq l\\ |k-l|\leq M}} E[X_k X_l] + \frac{1}{n^2}\sum_{\substack{(k,l)\\ k\neq l\\ |k-l|>M}} E[X_k X_l]$$

(We may suppose that $\sup_{m\geq 1} r(m) = R < \infty$)

$$\leq \frac{1}{n^2} M(M-1)\cdot R + \frac{1}{n^2} n(n-1)\cdot\varepsilon.$$

$(\because \#\{(k,l) \mid k\neq l,\ |k-l|\leq M\} \leq M(M-1))$

$$\therefore \liminf_{n\to\infty}\left(\frac{1}{n^2} M(M-1)R + \frac{1}{n^2} n(n-1)\varepsilon\right) = \varepsilon$$

Now the proof is complete.

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① We use W.L.L.N

$f(U_1), f(U_2) \cdots$ are independently identically distributed

$$E[f(U_1)] = \int_0^1 f(u) \cdot 1 \cdot du = I$$

Hence by W.L.L.N, $\frac{1}{n}(f(U_1) + \dots + f(U_n)) \xrightarrow{P} I$

$(= I_n)$

② Estimate $P(|I_n - I| > a\sqrt{n})$

by Chebyshev's inequality $\frac{1}{a^2 n} E[(I_n - I)^2]$

$$= \frac{1}{a^2 n} V[I_n]$$

$$V[I_n] = \frac{1}{n} V[f(U_1)] = \frac{1}{n}\left(E[f(U_1)^2] - E[f(U_1)]^2\right)$$

$$\therefore \frac{1}{a^2 n^2}\left(E[f(U_1)^2] - E[f(U_1)]^2\right)$$

$$\left(= \frac{1}{a^2 n^2}\left(\int_0^1 f(x)^2 dx - \left(\int_0^1 f(x) dx\right)^2\right)\right.$$

$\boxed{33}$ We use the conclusion of $\boxed{28}$.

$$x P(|X_1|>x) = x P(|X_1| \geq [x]+1)$$
$$\leq ([x]+1)\, P(|X_1| \geq [x]+1)$$

We show that $\quad n P(|X_1| \geq n) \to 0 \quad (\text{as } n\to\infty)$

$$= n \cdot \sum_{k=n}^{\infty} \frac{c}{k^2 \log k} \leq \frac{c \cdot n}{\log n} \sum_{k\geq n}^{\infty} \frac{1}{k^2} \leq \frac{c \cdot n}{\log n} \cdot \frac{1}{n-1} \to 0$$
(as $n\to\infty$)

$\boxed{\text{NOTE}}$
$$\int_{n-1}^{\infty} \frac{1}{x^2}\,dx = \sum_{k\geq n}^{\infty} \int_{k-1}^{k} \frac{1}{x^2}\,dx \geq \sum_{k\geq n}^{\infty} \frac{1}{k^2}.$$
$$\left(\frac{1}{n-1}\right)$$

Hence $\quad n P(|X_1| \geq n) \to 0.$

Therefore $\quad \dfrac{S_n - E[\tilde{S}_n]}{n} \xrightarrow{P} 0.$

$$E[\tilde{S}_n] = \sum_{k=1}^{n} E[X_k \cdot I\{|X_k| \leq n\}] \overset{(\text{iid})}{=} n\, E[X_1 \cdot I\{|X_1| \leq n\}]$$

$$\hat{\mu}_n \overset{\text{def}}{=} \frac{1}{n} E[\tilde{S}_n] = E[X_1 \cdot I\{|X_1| \leq n\}]$$
$$= \sum_{k=2}^{n} \frac{c \cdot (-1)^k \cdot k}{k^2 \log k} = \sum_{k=2}^{n} \frac{(-1)^k \cdot c}{k \log k} \to \mu \in (0,\infty) \quad (\text{as } n\to\infty)$$

$\therefore \quad \dfrac{S_n}{n} - \hat{\mu}_n \xrightarrow{P} 0 \quad \hat{\mu}_n \xrightarrow{P} \mu \Rightarrow \dfrac{S_n}{n} \xrightarrow{P} \mu$ ■

$\boxed{\text{NOTE}}$ $\displaystyle\sum_{k=2}^{n} \frac{(-1)^k \cdot C}{k\log k}$ converges as $n \to \infty$.

$$\left|\frac{1}{k\log k} - \frac{1}{(k+1)\log(k+1)}\right| = \left|\frac{(k+1)\log(k+1) - k\log k}{k(k+1)\log k \cdot \log(k+1)}\right|$$

$$= \frac{\left|(k+1)\log(k+1) - (k+1)\log k + (k+1)\log k - k\log k\right|}{k(k+1)\log k \cdot \log(k+1)}$$

$$\leq \frac{\left\{(k+1)\log\left(1+\frac{1}{k}\right) + \log k\right\}}{k(k+1)\log k \cdot \log(k+1)} \leq \frac{\log 2}{k(\log k)^2} + \frac{1}{k(k+1)\log(k+1)}$$

$$\leq \frac{1}{k(k+1)} + \frac{\log 2}{k(\log k)^2}$$

$$\sum_{k\geq 2}^{\infty} \frac{1}{k(\log k)^2} < \infty \qquad \sum_{k\geq 2}^{\infty} \frac{1}{k(k+1)} < \infty$$

$$\int_2^{\infty} \frac{1}{x(\log x)^2}\,dx = \sum_{k\geq 2}^{\infty} \int_k^{k+1} \frac{1}{x(\log x)^2}\,dx \geq \sum_{k\geq 3}^{\infty} \frac{1}{k(\log k)^2}$$

$$\parallel$$

$$\int_{\log 2}^{\infty} \frac{1}{t^2}\,dt < \infty \qquad (\log x = t)$$

$\boxed{\text{NOTE}}$ $E[|X_1|] = \infty$

Show $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k\log k} = \infty$. $\displaystyle\infty = \int_2^{\infty} \frac{1}{x\log x}\,dx = \sum_{k\geq 2}^{\infty} \int_k^{k+1} \frac{1}{x\log x}\,dx$

$$\leq \sum_{k\geq 2} \frac{1}{k\log k} \quad \left(\int_2^{\infty} \frac{1}{x\log x}\,dx = \int_{\log 2}^{\infty} \frac{1}{t}\,dt\right)$$

$\boxed{34}$

① $E|X_1| = \infty \quad (= E[X_1] \ \because \ X_1 \geq 0 \ (a.s))$

$$E[|X_1|] = \int_0^\infty P(|X_1| > x)\,dx = \int_0^e P(|X_1| > x)\,dx + \int_e^\infty P(|X_1| > x)\,dx$$

$$> \int_e^\infty P(X_1 > x)\,dx = \int_e^\infty \frac{e}{x\log x}\,dx$$

$$= \int_1^\infty \frac{e}{t}\,dt = \infty$$

② We show that $x\,P(|X_1| > x) \to 0$

$$x\,P(|X_1| > x) \underset{(x>e)}{=} x \cdot \frac{e}{x\log x} = \frac{e}{\log x} \to 0 \quad (\text{as } x \uparrow \infty)$$

Hence $\mu_n \overset{def}{=} \frac{1}{n} E[\hat{S}_n] = E[X_1 \cdot I\{|X_1| \leq n\}]$

$$\Rightarrow \frac{S_n}{n} - \mu_n \xrightarrow{P} 0$$

$\mu_n \to \infty$ as $n \to \infty$ ($\because$ by Monotone Convergence Theorem)

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35 $E[X] = \int_0^\infty P(X>x)\,dx$ ($\because$ X is non-negative)

$= \sum_{n=1}^{\infty} \int_{[n-1,n)} P(X>x)\,dx$

when $x \in [n-1, n)$ $\quad P(X>x) = P(X \geq n)$ ... ⊛

hence $= \sum_{n=1}^{\infty} \int_{[n-1,n)} P(X \geq n)\,dx$

$= \sum_{n=1}^{\infty} P(X \geq n)$

Similary we consider $E[X^2]$.

$E[X^2] = \int_0^\infty 2x P(X>x)\,dx$

$= \sum_{n=1}^{\infty} \int_{[n-1,n)} 2x P(X>x)\,dx$

$\downarrow$ by ⊛

$= \sum_{n=1}^{\infty} \int_{[n-1,n)} 2x P(X \geq n)\,dx$

$= \sum_{n=1}^{\infty} P(X \geq n) \int_{[n-1,n)} 2X\,dx$

$= \sum_{n=1}^{\infty} P(X \geq n) \left\{ \left[x^2\right]_{n-1}^{n} \right\}$

$= \sum_{n=1}^{\infty} (2n-1) P(X \geq n)$ ∎

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$\boxed{36}$

$$\mathbb{E}[H(X)] = \mathbb{E}\left[\int_{(-\infty, X]} h(y)\,dy\right] \quad \text{(by definition)}$$

$$= \int_\Omega \int_{(-\infty, X]} h(y)\,dy\,dP$$

$$= \int_\Omega \int_\mathbb{R} h(y) \cdot \mathbb{1}\{y \leq X\}\,dy\,dP$$

Since non-negative, apply Tonelli's theorem.

$$= \int_\mathbb{R} \int_\Omega h(y)\,\mathbb{1}\{X \geq y\}\,dP\,dy$$

$$= \int_\mathbb{R} h(y)\,P(X \geq y)\,dy \quad \blacksquare$$

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[37] $X_n \xrightarrow{as} 0 \Leftrightarrow \forall \varepsilon > 0 \quad P(\{|X_n| > \varepsilon \text{ i.o}\}) = 0$

① $X_n \xrightarrow{as} 0 \Rightarrow P(\{|X_n| > \varepsilon \text{ i.o}\}) = 0$ (for all $\varepsilon > 0$)

$P(\{|X_n| > \varepsilon \text{ i.o}\}) = P(\bigcap_{n=1}^{\infty} \bigcup_{m \geq n}^{\infty} \{|X_m| > \varepsilon\})$ (by definition)

$= P(\bigcap_{n=1}^{\infty} \{\sup_{m \geq n} |X_m| > \varepsilon\})$ $\left(\because \bigcup_{n \geq 1} \{f_n > a\} = \{\sup_{n \geq 1} f_n > a\}\right)$

Now we let $Y_n \overset{def}{=} \sup_{m \geq n} |X_m|$ $(Y_n \downarrow \limsup X_n)$

$= P(\bigcap_{n=1}^{\infty} \{Y_n > \varepsilon\})$

Since $\varepsilon > 0$, we may take $\varepsilon^* \in (0, \varepsilon)$

then $\leq P(\bigcap_{n=1}^{\infty} \{Y_n \geq \varepsilon^*\})$ ↓ see [NOTE]

$= P(\lim_{n \to \infty} Y_n \geq \varepsilon^*) = P(\limsup_{n \to \infty} |X_n| \geq \varepsilon^*) = 0$

$\boxed{\text{NOTE}}$ $f_n \uparrow f \Rightarrow \{\lim_{n \to \infty} f_n > a\} = \{f > a\} = \bigcup_{n=1}^{\infty} \{f_n > a\}$.

$\Leftrightarrow \bigcap_{n=1}^{\infty} \{f_n \leq a\} = \{f \leq a\}$

So if $g_n \downarrow g \Rightarrow -g_n \uparrow -g$ and $b = -a$

$\therefore \bigcap_{n=1}^{\infty} \{-g_n \leq a\} = \{-g \leq a\}$

$\Leftrightarrow \bigcap_{n=1}^{\infty} \{g_n \geq b\} = \{g \geq b\}$

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Hence $g_n \downarrow g \Rightarrow \bigcap_{n=1}^{\infty}\{g_n \geq b\} = \{g \geq b\}$

$\llcorner\ Y_n \downarrow \limsup_{n\to\infty} X_n \quad \therefore \bigcap_{n=1}^{\infty}\{Y_n \geq \varepsilon^*\} = \{\limsup|X_n| \geq \varepsilon^*\} \lrcorner$

In conclusion, $p(\{X_n > \varepsilon \text{ i.o}\}) \leq p(\limsup|X_n| \geq \varepsilon^*) = 0$

$(\because X_n \to 0 \text{ (as)} \Leftrightarrow |X_n| \to 0 \text{ (as)} \Leftrightarrow p(\limsup|X_n| = 0) = 1)$

② $p(\{|X_n| > \varepsilon \text{ i.o}\}) = 0 \Rightarrow X_n \xrightarrow{a.s} 0$

Take $\varepsilon^* > 0$: an arbitrary positive number.

$p(\limsup|X_n| \geq \varepsilon^*) = p(\bigcap_{n=1}^{\infty}\{Y_n \geq \varepsilon^*\}) \quad (\because ①)$

(We can take $\varepsilon \in (0, \varepsilon^*)$.)

$\leq p(\bigcap_{n=1}^{\infty}\{Y_n > \varepsilon\}) = p(\bigcap_{n=1}^{\infty}\bigcup_{m\geq n}^{\infty}\{|X_m| > \varepsilon\}) \quad (\because ①)$

$= p(\{|X_n| > \varepsilon \text{ i.o}\}) = 0$

Hence $p(\limsup|X_n| \geq \varepsilon^*) = 0$ for all $\varepsilon^* > 0$

$\Rightarrow p(\bigcup_{k=1}^{\infty}\{\limsup|X_n| \geq \frac{1}{k}\}) = 0 \quad (\varepsilon^* \leftarrow \frac{1}{k},\ k = 1, 2, 3 \cdots)$

$\Rightarrow p(\limsup|X_n| > 0) = 0 \quad \therefore p(\limsup|X_n| = 0) = 1$

Now the proof is complete. ∎

$\boxed{38}$ $\{A_n\}_{n\geq1} \subseteq \mathcal{F}$ $(\Omega, \mathcal{F}, P)$ .. Probability Space

If $\sum_{n=1}^{\infty} P(A_n) < \infty \Rightarrow$ Then $P(A_n \text{ i.o}) = 0$

(proof I)

$P(A_n \text{ i.o}) = P\left(\bigcap_{n=1}^{\infty} \bigcup_{m\geq n} A_m\right) = \lim_{n\to\infty} P\left(\bigcup_{m\geq n} A_m\right)$

$\leq \lim_{n\to\infty} \sum_{m\geq n}^{\infty} P(A_m)$

Since $\sum_{m=1}^{\infty} P(A_m) < \infty \Rightarrow \lim_{n\to\infty} \sum_{m\geq n}^{\infty} P(A_m) = 0$

■

(proof II)

$\sum_{n=1}^{\infty} P(A_n) = \sum_{n=1}^{\infty} \int_{\Omega} \mathbb{I}_{A_n} dP = \int_{\Omega} \sum_{n=1}^{\infty} \mathbb{I}_{A_n} dP < \infty$

(by Monoton Convergence Theorem)

$\Rightarrow \sum_{n=1}^{\infty} \mathbb{I}_{A_n}(w) < \infty$ (a.s)

$\Rightarrow \#\{n \mid w \in A_n\} < \infty$ (a.s)

$\Rightarrow P(\{w \mid$ there are infinitely many A_n including $w\}) = 0$

$P(A_n \text{ i.o}) \overset{\shortparallel}{=} 0$ ■

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$\boxed{39}$ First we show that $X_n \xrightarrow{P} X \Leftrightarrow \{n_k\}_{k\geq1} \subseteq \mathbb{N}$

$\exists \{n_{k_l}\}_{l\geq1} \subseteq \{n_k\}_{k\geq1}$ s.t $X_{n_{k_l}} \xrightarrow{a.s} X$.

① $(\Rightarrow)$

$X_n \xrightarrow{P} X \Rightarrow \forall n_k \quad X_{n_k} \xrightarrow{P} X$.

hence it is enough to show that $\exists \{n_k\} \subseteq \mathbb{N}$

$X_{n_k} \xrightarrow{a.s} X$,

Since $\lim_{n\to\infty} P(|X_n - X| > \varepsilon) = 0$

we may take sub-sequence $\{n_k\}$ s.t

$P\left(|X_{n_k} - X| > \frac{1}{k}\right) \leq \frac{1}{2^k}$.

And $\sum_{k=1}^{\infty} P\left(|X_{n_k} - X| > \frac{1}{k}\right) < \infty$ ↓ Borel-Cantelli's lemma

$\therefore P\left(\left\{|X_{n_k} - X| > \frac{1}{k} \text{ i.o}\right\}\right) = 0$

$\#\left\{k \,\middle|\, |X_{n_k} - X| > \frac{1}{k}\right\} < \infty$ (a.s)

$\therefore \limsup |X_{n_k} - X| \leq \liminf_{k\to\infty} \frac{1}{k} = 0$ (a.s)

$\therefore X_{n_k} \to X$ (a.s)

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② ($\Leftarrow$) Consider contraposition

$\ulcorner X_n \overset{P}{\not\to} X \lrcorner \Rightarrow \ulcorner \exists \{n_k\} \subseteq \mathbb{N}$ st $\forall \{n_{k_\ell}\}_{\ell \geq 1} \subseteq \{n_k\}_{k \geq 1}$

$$X_{n_{k_\ell}} \overset{(a.s)}{\not\to} X \lrcorner$$

$\ulcorner X_n \overset{P}{\to} X \lrcorner \cdots \forall \varepsilon > 0 \;\; \forall \delta > 0 \;\; \exists N(\varepsilon, \delta)$ st $\forall n > N(\varepsilon, \delta)$

$$P(|X_n - X| > \varepsilon) < \delta$$

$\ulcorner X_n \overset{P}{\not\to} X \lrcorner \cdots \exists \varepsilon > 0. \;\; \exists \delta > 0 \;\; \forall N$ st $\exists n > N$

$$P(|X_n - X| > \varepsilon) \geq \delta.$$

We may take a sub-sequence $\{n_k\}_{k \geq 1} \subseteq \mathbb{N}$

st $P(|X_{n_k} - X| > \varepsilon) \geq \delta > 0$

Now consider a sub-sequence of $\{n_k\}_{k \geq 1} : \{n_{k_\ell}\}_{\ell \geq 1}$

Recall that

$X_{n_{k_\ell}} \overset{a.s}{\to} X \Leftrightarrow P(|X_{n_{k_\ell}} - X| > \varepsilon \text{ i.o}) = 0 \quad (\forall \varepsilon > 0)$

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$$P(|X_{n_{k_\ell}} - X| > \varepsilon \text{ i.o.}) = P\left(\bigcap_{m=1}^{\infty} \bigcup_{\ell \ge m} |X_{n_{k_\ell}} - X| > \varepsilon\right)$$

$$= \lim_{m \to \infty} P\left(\bigcup_{\ell \ge m} |X_{n_{k_\ell}} - X| > \varepsilon\right)$$

$$\ge \liminf_{m \to \infty} P\left(|X_{n_{k_m}} - X| > \varepsilon\right) \quad \downarrow \circledast$$

$$\ge \liminf_{k \to \infty} P\left(|X_{n_k} - X| > \varepsilon\right) \ge \delta > 0$$

Hence $\exists \varepsilon > 0$ s.t. $P(|X_{n_{k_\ell}} - X| > \varepsilon \text{ i.o.}) > 0$.

So the proof is complete. ■

$\circledast$ $\{P(|X_{n_{k_m}} - X| > \varepsilon)\}_{m \ge 1} \subseteq \{P(|X_{n_k} - X| > \varepsilon)\}_{k \ge 1}$.

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$\boxed{39}$* $\boxed{\text{NOTE}}$

Consider the generalized version of the theorem

Let $(S, \mathcal{A}, \mu)$ be a measure space. ($\mu(S)$ may be ∞)

Then $f_n \xrightarrow{\mu} f \Leftrightarrow {}^{\forall}\{n_k\}_{k\geq 1} \subseteq \mathbb{N}$ ${}^{\exists}\{n_{k_\ell}\}_{\ell\geq 1} \subseteq \{n_k\}_{k\geq 1}$.

$f_{n_{k_\ell}} \xrightarrow{au} f$.

⊛ $\xrightarrow{a\mu}$: ${}^{\forall}\varepsilon>0$ ${}^{\exists}A^{(\varepsilon)} \in \mathcal{A}$ s.t $\mu(S \setminus A) < \varepsilon$

and $\lim_{n\to\infty} \sup_{x\in A} |f_n(x) - f| = 0$.

We use the fact that $\ulcorner f_n \xrightarrow{au} f \Leftrightarrow$

${}^{\forall}\varepsilon>0$ $\lim_{m\to\infty} \mu\left(\bigcup_{n\geq m} \{|f_n - f| > \varepsilon\}\right) = 0$ $\lrcorner$

① $(\Rightarrow)$ $f_n \xrightarrow{\mu} f \Rightarrow {}^{\forall}n_k$ $f_{n_k} \xrightarrow{\mu} f$

thus we prove that we may take $\{n_k\}_{k\geq 1}$

$f_{n_k} \xrightarrow{au} f$.

Since $\lim_{n\to\infty} \mu(\{|f_n - f| > \varepsilon\}) = 0$

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We may take a subsequence $\{n_k\}_{k\geq 1}$

$\mu(\{|f_{n_k}-f| > \frac{1}{k}\}) \leq \frac{1}{2^k}$

Then $\mu(\bigcup_{k\geq m}\{|f_{n_k}-f| > \frac{1}{k}\}) \leq \sum_{k\geq m}\mu(\{|f_{n_k}-f| > \frac{1}{k}\})$

$\leq \frac{1}{2^m}$

$\therefore \lim_{m\to\infty} \mu(\bigcup_{k\geq m}\{|f_{n_k}-f| > \frac{1}{k}\}) = 0$

For suffiently large m, $\mu(\bigcup_{k\geq m}\{|f_{n_k}-f| > \frac{1}{k}\})$

$\geq \mu(\bigcup_{k\geq m}\{|f_{n_k}-f| > \varepsilon\})$

Hence $\lim_{m\to\infty} \mu(\bigcup_{k\geq m}\{|f_{n_k}-f| > \varepsilon\}) = 0$

$\therefore f_{n_k} \to f$ (a.u)

② ($\Leftarrow$) In the same way, we suppose $f_n \overset{\mu}{\not\to} f$

$\exists \varepsilon>0$ $\exists \delta>0$ $\forall N$ $\exists n>N$ s.t $\mu(\{|f_n-f| > \varepsilon\}) \geq \delta > 0$

So we may take $\{n_k\}_{k\geq 1} \subseteq \mathbb{N}$

$\mu(\{|f_{n_k}-f| > \varepsilon\}) \geq \delta > 0$ (for all k)

Then $\forall \{n_{k_\ell}\}_{\ell\geq 1} \subseteq \{n_k\}_{k\geq 1}$

$$\mu\left(\bigcup_{\ell\geq m}\{|f_{n_{k_\ell}}-f|>\varepsilon\}\right) \geq \mu\left(\{|f_{n_{k_m}}-f|>\varepsilon\}\right)$$

Therefore

$$\liminf \mu\left(\bigcup_{\ell\geq m}\{|f_{n_{k_\ell}}-f|>\varepsilon\}\right) \geq \liminf \mu\left(\{|f_{n_{k_m}}-f|>\varepsilon\}\right)$$

$$\geq \liminf_{k\to\infty} \mu\left(\{|f_{n_k}-f|>\varepsilon\}\right) \geq \delta>0$$

So there exists $\{n_k\}_{k\geq 1}$ $\forall \{n_{k_\ell}\}_{\ell\geq 1} \subseteq \{n_k\}_{k\geq 1}$

$f_{n_{k_\ell}} \overset{(a.e)}{\not\to} f$.

Hence we have proved that

$\ulcorner f_n \overset{\mu}{\not\to} f \lrcorner \Rightarrow \ulcorner \exists \{n_k\}_{k\geq 1}\ \forall \{n_{k_\ell}\}\ \ f_{n_{k_\ell}} \overset{a.e}{\not\to} f \lrcorner$

$\therefore \ulcorner \forall \{n_k\}_{k\geq 1}\ \exists \{n_{k_\ell}\}_{\ell\geq 1}\ f_{n_{k_\ell}} \overset{a.e}{\to} f \lrcorner \to \ulcorner f_n \overset{\mu}{\to} f \lrcorner$

$\boxed{\text{NOTE}}$ We also prove that $f_n \xrightarrow{a.u.} f \Leftrightarrow \lim_{m\to\infty} \mu\left(\bigcup_{n\geq m}\{|f_n-f|\geq\varepsilon\}\right)=0$ (for all $\varepsilon>0$)

① ($\Rightarrow$) $\forall\delta>0$ $\exists A_{(\delta)}\in\mathcal{A}$ s.t. $\mu(A)<\delta$

and $\lim_{n\to\infty} \sup_{x\in A^c}|f_n(x)-f(x)|=0.$

$\therefore x\in A^c \Rightarrow \forall\varepsilon>0$ $\exists m_\varepsilon$ $|f_n(x)-f(x)|<\varepsilon$ $\forall n\geq m$

Hence $A^c\subseteq \bigcap_{n\geq m}\{|f_n(x)-f(x)|<\varepsilon\}$

This implies $\bigcup_{n\geq m}\{|f_n-f|\geq\varepsilon\}\subseteq A$

$\therefore \mu\left(\bigcup_{n\geq m}\{|f_n-f|\geq\varepsilon\}\right)\leq\mu(A)<\delta$

This implies $\lim_{m\to\infty}\mu\left(\bigcup_{n\geq m}\{|f_n-f|\geq\varepsilon\}\right)=0$

② ($\Leftarrow$) $\lim_{m\to\infty}\mu\left(\bigcup_{n\geq m}\{|f_n-f|\geq\varepsilon\}\right)=0$ ($\forall\varepsilon>0$)

Then we may take $\{m_k\}$ s.t.

$$\mu\left(\bigcup_{n\geq m_k}\{|f_n-f|\geq\tfrac{1}{k}\}\right)\leq\frac{\delta}{2^k}\quad(\delta>0)$$

$$\Rightarrow \mu\left(\bigcup_{k=1}^{\infty}\bigcup_{n\geq m_k}\{|f_n-f|\geq\tfrac{1}{k}\}\right)\leq\delta$$

Let $A=\bigcup_{k=1}^{\infty}\bigcup_{n\geq m_k}\{|f_n-f|\geq\frac{1}{k}\}$. $A^c=\bigcap_{k=1}\bigcap_{n\geq m_k}\{|f_n-f|<\frac{1}{k}\}$

This implies $\sup_{x\in A^c}|f_n-f|<\frac{1}{k}$ $(n\geq m_k)$ $\Rightarrow \lim_{n\to\infty}\sup_{x\in A^c}|f_n-f|=0.$

$\boxed{40}$ $y_n \to y \Leftrightarrow \forall \{n_k\}_{k=1} \subseteq \mathbb{N} \ \exists \{n_{k_l}\}_{l=1} \ y_{n_{k_l}} \to y$

① ($\Rightarrow$) trivial

② ($\Leftarrow$) We show $y_n \not\to y \Rightarrow \exists \{n_k\}_{k=1} \subseteq \mathbb{N} \ \forall \{n_{k_l}\}$ $y_{n_{k_l}} \not\to y$

$\left(\ulcorner y_n \to y \lrcorner : \forall \varepsilon > 0 \ \exists N_\varepsilon > 0 \ \forall n > N_\varepsilon \ |y_n - y| < \varepsilon \right)$

$\ulcorner y_n \not\to y \lrcorner \ \exists \varepsilon > 0 \ \forall N > 0 \ \exists n > N \ |y_n - y| \geq \varepsilon$

Hence we may take a subsequence $\{n_k\}$

$|y_{n_k} - y| \geq \varepsilon \ (>0)$

For any subsubsequence $\{n_{k_l}\}$

$\liminf_{l \to \infty} |y_{n_{k_l}} - y| \geq \liminf_{k \to \infty} |y_{n_k} - y| \geq \varepsilon > 0$

So $y_{n_{k_l}} \not\to y$.

Now the proof is complete ∎

$\boxed{41}$ (Bounded Convergence Theorem with converge in Probability)

We show that if $X_n \xrightarrow{P} X$, $\sup_{n\geq 1}|X_n| \leq M < \infty$
then $\lim_{n\to\infty} E[X_n] = E[X]$.

(proof) by $\boxed{39}$ $X_n \xrightarrow{P} X \Rightarrow {}^{\forall}\{n_k\}_{k\geq 1} \subseteq \mathbb{N}$

${}^{\exists}\{n_{k_\ell}\}_{\ell\geq 1}$ $X_{n_{k_\ell}} \xrightarrow{a.s.} X$.

Since $X_{n_{k_\ell}} \xrightarrow{a.s} X$ and $|X_{n_{k_\ell}}| \leq M$ (integrable)

by bounded convergence theorem $E[X_{n_{k_\ell}}] \to E[X]$.

By $\boxed{40}$: $E[X_n] \to E[X] \Leftrightarrow {}^{\forall}\{n_k\}_{k\geq 1} \subseteq \mathbb{N}$

${}^{\exists}\{n_{k_\ell}\}_{\ell\geq 1}$ $E[X_{n_{k_\ell}}] \to E[X]$. $\circledast$

Now $\{X_n\}_{n\geq 1}$ satisfies $\circledast$, hence $E[X_n] \to E[X]$

$\boxed{\#2}$

Without loss of generality, we may suppose $\mu=0$.

$\left(\because \tilde{X}_i \overset{\text{def}}{=} X_i - \mu \qquad \frac{S_n}{n} \xrightarrow{a.s} \mu \iff \frac{\tilde{S}_n}{n} \xrightarrow{a.s} 0 \right)$

(Proof I)

$$P\left(\left|\frac{S_n}{n}\right| > \varepsilon\right) \le \frac{1}{\varepsilon^4}\frac{1}{n^4} E[S_n^4] \quad \text{(Markov's inequality)}$$

$$E[S_n^4] = E[(X_1+\dots+X_n)(X_1+\dots+X_n)(X_1+\dots+X_n)(X_1+\dots+X_n)]$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} E[X_i X_j X_k X_l]$$

$$= \sum_{i=j=k=l} (\cdot) + \sum_{\substack{i=j\,(\neq)\,k=l \\ i=k\,(\neq)\,j=l \\ i=l\,(\neq)\,j=k}} (\cdot) + \underbrace{\sum_{\substack{i\,(\neq)\,j=k=l \\ j\neq i=k=l \\ k\neq i=j=l \\ l\neq i=j=k}} (\cdot)}_{=0} + \underbrace{\cdots}_{=0\,\dots}$$

$$= n E[X_i^4] + 3n(n-1) E[X_i^2]$$

So there exists $C>0$ s.t.

$$E[S_n^4] \le Cn^2$$

$$P\left(\left|\frac{S_n}{n}\right|>\varepsilon\right) \le \frac{C}{\varepsilon^4 n^2}$$

$$\therefore \sum_{n=1}^{\infty} P\left(\left|\frac{S_n}{n}\right|>\varepsilon\right) < \infty$$

By Borel-Cantelli's lemma $\quad P\left(\left|\frac{S_n}{n}\right|>\varepsilon \text{ i.o.}\right) = 0$

$$\Rightarrow \frac{S_n}{n} \xrightarrow{a.s} 0 \quad (\because \boxed{3'H}) \qquad ∎$$

(proof II)

$$E\left[\sum_{n=1}^{\infty} \frac{S_n^4}{n^4}\right] = \sum_{n=1}^{\infty} E\left[\frac{S_n^4}{n^4}\right] \le \sum_{n=1}^{\infty} \frac{C}{n^2} < \infty \quad (\because \text{ same as proof I})$$

(Monotone Convergence Theorem)

$$\therefore \sum_{n=1}^{\infty} \frac{S_n^4}{n^4} < \infty \ (a.s) \qquad (\text{converge } a.s)$$

$$\therefore \lim_{n\to\infty} \frac{S_n^4}{n^4} = 0 \ (a.s) \qquad \left(\because \sum_{n=1}^{\infty} a_n : \text{converge} \Rightarrow a_n \to 0\right)$$

$$\Rightarrow \lim_{n\to\infty} \frac{S_n}{n} = 0 \ (a.s)$$

 $(\Omega, \mathcal{F}, P)$: probability Space

$\boxed{13}$ Borel-Cantelli's lemma (II)

$\{A_n\}_{n\geq 1} \subseteq \mathcal{F}$: independent and $\sum_{n=1}^{\infty} P(A_n) = \infty$

$\Rightarrow P(A_n \text{ i.o}) = 1$

(proof) $\{A_n\}_{n\geq 1}$: independent $\Rightarrow$ $\{A_n^c\}_{n\geq 1}$: independent

$B_{n,k} \overset{\text{def}}{=} \bigcap_{m=n}^{n+k} A_m^c$

$(B_{n,k+1} \subseteq B_{n,k})$

$P(B_{n,k}) = P\left(\bigcap_{m=n}^{n+k} A_m^c\right) = \prod_{m=n}^{n+k} P(A_m^c) = \prod_{m=n}^{n+k} (1 - P(A_m))$

(when $x \geq 0$ ... $1 - x \leq e^{-x}$)

$\therefore \leq \prod_{m=n}^{n+k} \exp(-P(A_m)) = \exp\left(-\sum_{m=n}^{n+k} P(A_m)\right)$

$\therefore \lim_{k\to\infty} P(B_{n,k}) = \lim_{k\to\infty} \exp\left(-\sum_{m=n}^{n+k} P(A_m)\right)$

$= \exp\left(-\sum_{m=n}^{\infty} P(A_m)\right) = \exp(-\infty) = 0$

$\therefore \lim_{k\to\infty} P(B_{n,k}) = 0$

And $P\left(\bigcap_{k=1}^{\infty} B_{n,k}\right) = \lim_{k\to\infty} P(B_{n,k}) = 0$

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$B_n \overset{\text{def}}{=} \bigcap_{k=1}^{\infty} B_{n,k} \quad (B_{n,k} \downarrow B_n)$

$P(B_n) = 0 \quad (\text{for all } n \geq 1)$

$\therefore \; P(B_n^c) = 1 \quad (\text{for all } n \geq 1)$

$= P\left(\bigcup_{k=1}^{\infty} B_{n,k}^c\right) = P\left(\bigcup_{k=1}^{\infty} \bigcup_{m=n}^{n+k} A_m\right) = P\left(\bigcup_{m \geq n}^{\infty} A_m\right) = 1 \quad (\text{for all } n \geq 1)$

$\bigcup_{m=n}^{\infty} A_m \downarrow \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m \quad (= \{A_n \text{ i.o.}\})$

$\lim_{n\to\infty} P\left(\bigcup_{m=n}^{\infty} A_m\right) = 1 = P\left(\bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m\right) = P(A_n \text{ i.o.}) \quad \blacksquare$

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$\boxed{11}$

(1) $E|X_1| = \infty \Rightarrow P(|X_n| \geq n \text{ i.o}) = 1$

(proof) Use Borel-Cantelli's lemma (II)

$\{X_n \geq n\}$: independent

If $\sum_{n=1}^{\infty} P(\{|X_n| \geq n\}) = \infty \Rightarrow P(|X_n| \geq n \text{ i.o}) = 1$

(by Borel-Cantelli's lemma (II))

$E|X_1| = \infty = \int_0^{\infty} P(|X_1| > x)dx$ (See Durrett's chapter 1)

$= \sum_{n=0}^{\infty} \int_n^{n+1} P(|X_1| > x)dx$

$\leq \sum_{n=0}^{\infty} \int_n^{n+1} P(|X_1| > n)dx$

$\leq \sum_{n=0}^{\infty} \int_n^{n+1} P(|X_1| \geq n)dx$

$= \sum_{n=0}^{\infty} \int_n^{n+1} P(|X_n| \geq n)dx$ ($\because$ iid)

$= \sum_{n=0}^{\infty} P(|X_n| \geq n)$

$\Rightarrow \sum_{n=1}^{\infty} P(|X_n| \geq n) = \infty$

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(2) We show that $\{|X_n| \geq n \text{ i.o}\}$ and $\{\lim \frac{S_n}{n} \text{ exists in } \mathbb{R}\}$ are disjoint.

$(\Rightarrow \{\lim \frac{S_n}{n} \text{ exists in } \mathbb{R}\} \subseteq \{|X_n| \geq n \text{ i.o}\}^c)$

We prove by deriving a contradiction.

Let $w \in \{|X_n| \geq n \text{ i.o}\} = \{\frac{|X_n|}{n} \geq 1 \text{ i.o}\}$

there are infinitely many n s.t $\frac{|X_n|}{n} \geq 1$

And $\left|\frac{S_n}{n} - \frac{S_{n+1}}{n+1}\right| = \left|\frac{(n+1)S_n - nS_{n+1}}{n(n+1)}\right| = \left|\frac{(n+1)S_n - nS_n - nX_{n+1}}{n(n+1)}\right|$

$= \left|\frac{S_n - nX_{n+1}}{n(n+1)}\right| \geq \underbrace{\frac{S_{n+1}}{n+1}}_{①} - \underbrace{\frac{|S_n|}{n} \cdot \frac{1}{n+1}}_{②}$ $\quad (\because |a-b| \geq |a|-|b| \text{ or } |b|-|a|)$

Since $w \in \{\frac{|X_n|}{n} \geq 1 \text{ i.o}\}$, so there are infinitely many n s.t ① ≥ 1.

If $w \in \{\lim \frac{S_n}{n} \text{ exists in } \mathbb{R}\}$ at the same time

$\exists N$ s.t $\forall n > N$ ② $< \frac{1}{2}$. $\quad (\because \underbrace{\frac{|S_n|}{n}}_{\text{converge}}, \frac{1}{n+1} \to 0)$

Hence there exists infinitely many n ① $-$ ② $> \frac{1}{2}$.

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So $\limsup \left|\frac{S_n}{n} - \frac{S_{n+1}}{n+1}\right| \geq \frac{1}{2}$

Thus $\left\{\frac{S_n(\omega)}{n}\right\}_{n\geq 1}$ is not a cauchy sequence

$\Rightarrow \left\{\frac{S_n(\omega)}{n}\right\}_{n\geq 1}$ does not converge.

However it contradicts to the fact that $\omega \in$

$\{\lim \frac{S_n}{n}$ exists in $\mathbb{R}\}$.

So $\{|X_n| \geq n \ i.o\}$ and $\{\lim \frac{S_n}{n}$ exists in $\mathbb{R}\}$

should be disjoint.

[15] Let $X_n \overset{def}{=} I_{A_n}(\omega)$ $(n=1,2,3\cdots)$ $(\therefore E[X_m] = P(A_m))$

$n_k \overset{def}{=} \inf\{n \mid \sum_{m=1}^{n} E[X_m] \geq k^2\}$

$P\left(\left|\sum_{m=1}^{n_k} X_m - \sum_{m=1}^{n_k} E[X_m]\right| > \varepsilon \sum_{m=1}^{n_k} E[X_m]\right)$

$$\underset{(*_1)}{\leq} \frac{\sum_{m=1}^{n_k} V[X_m]}{\varepsilon^2 \left(\sum_{m=1}^{n_k} E[X_m]\right)^2} \underset{(*_2)}{\leq} \frac{\sum_{m=1}^{n_k} E[X_m^2]}{\varepsilon^2 \left(\sum_{m=1}^{n_k} E[X_m]\right)^2}$$

$$\underset{(*_3)}{=} \frac{\sum_{m=1}^{n_k} E[X_m]}{\varepsilon^2 \left(\sum_{m=1}^{n_k} E[X_m]\right)^2} = \frac{1}{\varepsilon^2 \sum_{m=1}^{n_k} E[X_m]} \underset{(*_4)}{\leq} \frac{1}{k^2 \cdot \varepsilon^2}$$

$(*_1)$ ... Markov's inequality (chebyshev's inequality)

$(*_2)$ $V[X_m] = E[X_m^2] - E[X_m]^2 \leq E[X_m^2]$

$(*_3)$ $X_m = I_{A_m}(\omega) = 0$ or 1 $\therefore X_m = X_m^2$

$(*_4)$ By the definition of n_k.

$$\therefore \sum_{k=1}^{\infty} P\left(\left|\sum_{m=1}^{n_k} X_m - \sum_{m=1}^{n_k} E[X_m]\right| > \varepsilon \sum_{m=1}^{n_k} E[X_m]\right) < \infty$$

$\therefore$ By Borel-Cantelli's lemma, $P\left(\left|\frac{\sum_{m=1}^{n_k} X_m}{\sum_{m=1}^{n_k} E[X_m]} - 1\right| > \varepsilon \text{ i.o}\right) = 0$.

$$\left(\begin{array}{l} n_k = n(k) \\ n_{k+1} = n(k+1) \end{array} \right)$$

And by [37], we have $\dfrac{\sum_{m=1}^{n_k} X_m}{\sum_{m=1}^{n_k} E[X_m]} - 1 \xrightarrow{a.s} 0$

$$\therefore \frac{\sum_{m=1}^{n_k} X_m}{\sum_{m=1}^{n_k} E[X_m]} \to 1 \ (a.s) \qquad (\text{as } k \to \infty)$$

Now consider a general case. Let n: $\underset{(n(k))}{n_k} \leq n < \underset{(n(k+1))}{n_{k+1}}$

$$\underbrace{\frac{\sum_{m=1}^{n(k)} X_m}{\sum_{m=1}^{n(k+1)} E[X_m]}}_{①} \leq \frac{\sum_{m=1}^{n} X_m}{\sum_{m=1}^{n} E[X_m]} \leq \underbrace{\frac{\sum_{m=1}^{n(k+1)} X_m}{\sum_{m=1}^{n(k)} E[X_m]}}_{②}$$

$$① = \underbrace{\frac{\sum_{m=1}^{n(k)} X_m}{\sum_{m=1}^{n(k)} E[X_m]}}_{\substack{\to 1 \\ (a.s)}} \cdot \underbrace{\frac{\sum_{m=1}^{n(k)} E[X_m]}{\sum_{m=1}^{n(k+1)} E[X_m]}}_{\geq \frac{k^2}{(k+1)^2+1} \to 1} \quad (\Rightarrow \to 1 \ (a.s))$$

$$\left(\because \ 0 \leq E[X_m] \leq 1 \Rightarrow k^2 \leq \sum_{m=1}^{n(k)} E[X_m] < k^2+1 \right)$$

$\therefore$ ① $\to 1$ (a.s)

② : similarly $\to 1$ (a.s)

So $\dfrac{\sum_{m=1}^{n} X_m}{\sum_{m=1}^{n} E[X_m]} \to 1$ (a.s) ∎

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$\boxed{46}$

① Consider $P(l_n \geq (1+\varepsilon)\log_2(n))$

Since $P(l_n = k) = P(X_{n-k}=0, X_{n-k+1}=1, \dots, X_n=1)$
$= \left(\frac{1}{2}\right)^{k+1}$

$$P(l_n \geq (1+\varepsilon)\log_2(n)) \overset{(\leq)}{=} \left(\frac{1}{2}\right)^{(1+\varepsilon)\log_2(n)+1} \cdot \frac{1}{1-\frac{1}{2}}$$
$$= \left(\frac{1}{2}\right)^{(1+\varepsilon)\log_2(n)} = \frac{1}{n^{1+\varepsilon}}$$

Hence $\sum_{n=1}^{\infty} P(l_n \geq (1+\varepsilon)\log_2(n)) = \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} < \infty$

$(\because 1+\varepsilon > 1)$

Thus by Borel-Cantelli's lemma (I)

$P(\{l_n \geq (1+\varepsilon)\log_2(n) \text{ i.o}\}) = 0$

$\Rightarrow \#\{n \mid \frac{l_n}{\log_2(n)} \geq (1+\varepsilon)\} < \infty$ (a.s)

$\Rightarrow$ For suffiently lage n, $\frac{l_n}{\log_2(n)} < 1+\varepsilon$ (a.s)

$\Rightarrow \limsup \frac{l_n}{\log_2(n)} < 1+\varepsilon$ (a.s)

Since $l_n \geq 0$ $\limsup \frac{l_n^+}{\log_2(n)} < 1+\varepsilon$ (a.s)

$l_n = l_n^+$

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By $\boxed{60}$ $\overline{\lim}\sup \frac{\ell_n^{+}}{\log_2 n} = \overline{\lim}\sup \frac{L_n}{\log_2 n} < 1+\varepsilon$ (a.s)

$\ulcorner\ {}^{\forall}\varepsilon>0\ {}^{\exists}\Omega^{(\varepsilon)}\ P(\Omega^{(\varepsilon)})=1\ {}^{\forall}\omega\in\Omega^{(\varepsilon)}$

$\overline{\lim}\sup \frac{L_n}{\log_2 n} < 1+\varepsilon\ \lrcorner$

Let $\varepsilon = \frac{1}{k}$ $\Omega^{*} \overset{\text{def}}{=} \bigcap_{k=1}^{\infty} \Omega^{(\frac{1}{k})}$

$P(\Omega^{*})=1$ ${}^{\forall}\omega\in\Omega^{*}$ $\overline{\lim}\sup \frac{L_n^{(\omega)}}{\log_2 n} < 1+\frac{1}{k}$ $({}^{\forall}k\geq 1)$

$\Rightarrow \overline{\lim}\sup \frac{L_n^{(\omega)}}{\log_2 n} \leq 1$

Therefore $\ulcorner \overline{\lim}\sup \frac{L_n}{\log_2 n} \leq 1$ (a.s) $\lrcorner$

② Next we prove $\underline{\lim}\inf \frac{L_n}{\log_2(n)} \geq 1$ (a.s)

We consider $P(L_n \leq (1-\varepsilon)\log_2(n))$

$\{L_n \leq (1-\varepsilon)\log_2(n)\} \Rightarrow \{\ell_1, \ell_2 \ldots \ell_n \leq (1-\varepsilon)\log_2(n)\}$

Now separate n into blocks with length of

$(1-\varepsilon)\log_2(n)+1$ (See figure)

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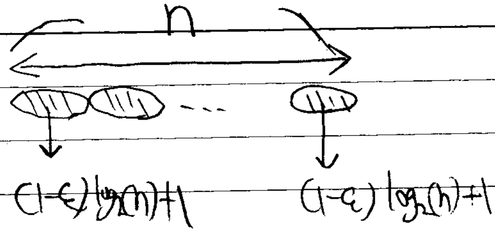

$\left(\because\ l_1 \sim l_n \leq (1-\epsilon)\log_2(n)\right)$

$\Downarrow$

In each block, they must not be all 1.

So $\left\{ \underbrace{\left(1-\left(\frac{1}{2}\right)^{(1-\epsilon)\log_2(n)+1}\right)}_{\text{(The probability of each blocks containing at least one '0')}} \right\}^{\overbrace{\frac{n}{(1-\epsilon)\log_2(n)+1}}^{}}$

$\frac{n}{(1-\epsilon)\log_2(n)+1} \rightarrow$ number of blocks

$\therefore\ P\left(\{l_1 \dots l_n \leq (1-\epsilon)\log_2(n)\}\right) \leq$

$$\left\{\left(1-\left(\frac{1}{2}\right)^{(1-\epsilon)\log_2(n)+1}\right)\right\}^{\frac{n}{(1-\epsilon)\log_2(n)+1}}$$

$$\| $$

$$\left(1-\frac{1}{2}\cdot\frac{1}{n^{1-\epsilon}}\right)^{\frac{n}{(1-\epsilon)\log_2(n)+1}} \leq \left(1-\frac{1}{2^{1-\epsilon}}\right)^{\frac{n}{\log_2 n}}$$

$\left(\because\right.$ For suffiently large n, $\left.\frac{n}{(1-\epsilon)\log_2(n)+1} \geq \frac{n}{\log_2 n}\right)$

$$\therefore\ \leq \left(1-\frac{1}{2}\frac{1}{n^{1-\epsilon}}\right)^{\frac{n}{\log_2(n)}}$$

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Moreover

$$\left(1-\frac{1}{2n^{1-\varepsilon}}\right)^{\frac{n}{\log_2 n}} = \left\{\left(1-\frac{1}{2n^{1-\varepsilon}}\right)^{(-2n^{1-\varepsilon})}\right\}^{\left(\frac{-n^{\varepsilon}}{2\log_2 n}\right)}$$

$\approx \exp\left(\frac{-n^{\varepsilon}}{2\log_2 n}\right)$ (for suffiently large $n \geq 1$)

And $\sum_{n\geq N}^{\infty} P(L_n \leq (1-\varepsilon)\log_2 n) \leq \sum_{n\geq N}^{\infty} \exp\left(\frac{-n^{\varepsilon}}{2\log_2 n}\right) < \infty$ ⊛

(N : suffiently large number)

Hence by Borel-Cantelli's lemma

$P(\{L_n \leq (1-\varepsilon)\log_2 n \ \text{i.o}\}) = 0$

$\therefore \#\{n \mid L_n \leq (1-\varepsilon)\log_2(n)\} < \infty$ (a.s)

For suffiently large n,

$L_n > (1-\varepsilon)\log_2 n$

$\Rightarrow \frac{L_n}{\log_2(n)} > (1-\varepsilon)$

$\inf_{m\geq n} \frac{L_m}{\log_2(m)} > (1-\varepsilon)$ (n : large)

$\Rightarrow \lim_{n\to\infty} \inf_{m\geq n} \frac{L_m}{\log_2 m} > 1-\varepsilon$ $\therefore$ $\liminf \frac{L_n}{\log_2 n} > 1-\varepsilon$ (a.s)

Using the similar argument with ① part $\left(\varepsilon = \frac{1}{k},\ k=1,2,\dots\right)$

We have $\liminf \frac{L_n}{\log_2 n} \geq 1$ (a.s) ■

$\boxed{\text{NOTE}}$ ⊛ $\sum_{n\geq N}^{\infty} \exp\left(\frac{-n^{\varepsilon}}{2\log_2 n}\right) < \infty$

For sufficiently large n, $2\log_2 n < n^{\frac{\varepsilon}{2}}$

$\therefore\ n^{-\frac{\varepsilon}{2}} < \frac{1}{2\log_2 n}$

$\Rightarrow\ n^{\frac{\varepsilon}{2}} < \frac{n^{\varepsilon}}{2\log_2 n}$

$\Rightarrow\ \frac{-n^{\varepsilon}}{2\log_2 n} < -n^{\frac{\varepsilon}{2}}$

$\Rightarrow\ \exp\left(\frac{-n^{\varepsilon}}{2\log_2 n}\right) < \exp\left(-n^{\frac{\varepsilon}{2}}\right)$

And $\delta > 0 \Rightarrow \sum_{n=1}^{\infty} \exp(-n^{\delta}) < \infty$

$$\infty > \int_0^{\infty} \exp(-x^{\delta}) = \sum_{n=0}^{\infty} \int_n^{n+1} \exp(-x^{\delta}) \geq \sum_{n=1}^{\infty} \exp(-n^{\delta})$$

$$\| \quad \int_0^{\infty} \frac{1}{\delta} t^{\left(\frac{1}{\delta}-1\right)} \exp(-t)\,dt \ (<\infty)$$

($\because$ Gamma Function)

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$\boxed{45}$ If $X_n \xrightarrow{P} X$ then $E[X] \leq \liminf E[X_n]$

where $X_n \geq 0$.

(proof) We may take a subsequence $\{n_k\}_{k\geq 1} \subseteq \mathbb{N}$

such that $\lim_{k\to\infty} E[X_{n_k}] = \liminf_n E[X_n]$ $\circledast$

And $X_{n_k} \xrightarrow{P} X$ ($\because X_n \xrightarrow{P} X$)

We may take subsub-sequence $\{n_{k_\ell}\}_{\ell\geq 1}$

$X_{n_{k_\ell}} \xrightarrow{a.s} X$. ($\because \boxed{39}$)

By Fatou's lemma. $E[\liminf_{\ell\to\infty} X_{n_{k_\ell}}] \leq \liminf_{\ell\to\infty} E[X_{n_{k_\ell}}]$

And $\liminf_{\ell\to\infty} E[X_{n_{k_\ell}}] = \liminf_{k\to\infty} E[X_{n_k}] = \lim_{k\to\infty} E[X_{n_k}]$

$= \liminf_{n\to\infty} E[X_n]$ ($\because \circledast$)

L.H.S $= E[\liminf_\ell X_{n_{k_\ell}}] = E[X]$

Hence we have $E[X] \leq \liminf E[X_n]$

$\boxed{48}$

(1) $X_n \xrightarrow{P} X$, $|X_n| \leq Y \in \mathcal{L}^1(P)$,

$\lim_{n\to\infty} E[X_n] = E[X]$

(proof) $X_n \xrightarrow{P} X \Rightarrow |X_n - X| \xrightarrow{P} 0$.

$|X_n - X| \leq |X_n| + |X| \underset{※_1}{\leq} 2 \sup_{n\geq 1} |X_n| \leq 2Y$

※1. Since there exists $\{n_k\}_{k\geq 1}$ s.t. $X_{n_k} \xrightarrow{(a.s)} X$

Thus $|X| \leq \sup_{k\geq 1} |X_{n_k}| \leq \sup_{n\geq 1} |X_n|$ (a.s)

Use Fatou's lemma (4.4.1) to $2Y - \underset{(\geq 0)}{|X_n - X|} \xrightarrow{P} 2Y$

$E[2Y] \leq \liminf_{n\to\infty} E[2Y - |X_n - X|]$

$= E[2Y] - \limsup_{n\to\infty} E[|X_n - X|]$

($\because E[2Y] < \infty$)

Therefore we have $\limsup_{n\to\infty} E|X_n - X| = 0 \Rightarrow E(X_n) \to E(X)$.

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In theorem 1.68 ($\xrightarrow{a.s}$).

(2) · $X_n \xrightarrow{P} X$.

· $g(x) \geq 0$, g, h are continuous

$\frac{|h(x)|}{g(x)} \to 0$ (as $x \to \infty$)

· $E[g_n(X)] \leq C < \infty$

Then $\lim E[h(X_n)] = E[h(X)]$

(proof)

· Without loss of generality $h(0) = 0$.

($\tilde{h}(x) \overset{def.}{=} h(x) - h(0)$)

· We may take infinitely large $M > 0$

$P(|X| = M) = 0$. ($\because$ Points of discontinuity are at most countable ch1. $\boxed{13}$)

· $\tilde{X}_n \overset{def}{=} X_n \cdot \mathbb{I}\{|X_n| > M\}$

$\tilde{X} \overset{def}{=} X \cdot \mathbb{I}\{|X| > M\}$

- $\left| E[h(X_n)] - E[h(X)] \right|$

$= \left| E[h(X_n)] - E[h(\tilde{X}_n)] + E[h(\tilde{X}_n)] - E[h(\tilde{X})] + E[h(\tilde{X})] - E[h(X)] \right|$

$\leq E\left| h(X_n) - h(\tilde{X}_n) \right| + E\left| h(\tilde{X}_n) - h(\tilde{X}) \right| + E\left| h(\tilde{X}) - h(X) \right|$

($\because$ Triangle inequality)

$= E\left| h(X_n) \cdot \mathbb{I}\{|X_n| > M\} \right| + E\left| h(\tilde{X}_n) - h(\tilde{X}) \right| + E\left| h(X) \cdot \mathbb{I}\{|X| > M\} \right|$

($\because h(u) = 0$)

$\left(\varepsilon_M \stackrel{\text{def}}{=} \sup\left\{ \frac{|h(x)|}{g(x)} \,\middle|\, |x| > M \right\} \right)$

$= E \frac{|h(X_n)|}{g(X_n)} \cdot \mathbb{I}\{|X_n| > M\} \cdot g(X_n) + E \frac{|h(X)|}{g(X)} \mathbb{I}\{|X| > M\} \, g(X) + E\left| h(\tilde{X}_n) - h(\tilde{X}) \right|$

$\leq \underbrace{\varepsilon_M \cdot E[g(X_n)]}_{①} + \underbrace{\varepsilon_M E[g(X)]}_{②} + \underbrace{E\left| h(\tilde{X}_n) - h(\tilde{X}) \right|}_{③}$

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① $\leq C \cdot \varepsilon_M$

③ $\leq C \cdot \varepsilon_M$

($\because$ By Fatou's lemma $\boxed{47}$. $g(X_n) \xrightarrow{P} g(X)$ $\because g$: continuous)

$$E\,g(X) \leq \liminf E[g(X_n)]$$
$$\leq \sup_{n\geq 1} E[g(X_n)] \leq C.$$

As $M \nearrow \infty$. ①+② $\leq 2C\varepsilon_M \downarrow 0$.

Finally $\hat{X}_n \xrightarrow{P} \hat{X}$ ($\because$ see $\boxed{49}$)

$\Rightarrow h(\hat{X}_n) \xrightarrow{P} h(\hat{X})$ ($\because$ h: continuous)

Since $|\hat{X}_n|, |\hat{X}| \leq M$.

And $h(x)$ is continuous on $[-M, M]$

hence bounded. $|h(\hat{X}_n) - h(x)| \leq 2K < \infty$

By bounded continuous theorem. ($\boxed{41}$,)

we have ③ $\to 0$ (as $n \to \infty$). ■

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$(A \triangle B = A^c \cap B \cup A \cup B^c)$

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$\boxed{49}$ In order to give a complete proof in $\boxed{48}$
it's enough to show if $p=1$ the statement is true

If $X_n \xrightarrow{P} X$ and $M \in C(|X|)$ (i.e $P(|X| = M) = 0$)

Then $|X_n| \cdot \mathbb{I}\{|X_n| \leq M\} \xrightarrow{P} |X| \cdot \mathbb{I}\{|X| \leq M\}$

(proof) Our first goal is to show that

$\lim_{n\to\infty} P(\{|X_n| \leq M \triangle |X| \leq M\}) = 0$

(i.e $P(|X_n| \leq M, |X| > M) + P(|X| \leq M, |X_n| > M) \to 0$)

① $P(|X_n| \leq M, |X| > M) - P(|X_n| \leq M, |X| > M+\delta)$

$= P(|X_n| \leq M, M < |X| \leq M+\delta)$

$\leq P(M < |X| \leq M+\delta)$ $\quad$ ($\because A \subseteq B \Rightarrow P(A) \leq P(B)$)

Since $M \in C(|X|)$, $\delta \downarrow 0$ $P(M < |X| \leq M+\delta) \downarrow 0$.

Hence $\forall \varepsilon > 0$ $\exists \delta_\varepsilon > 0$ st $P(M < |X| \leq M+\delta) < \varepsilon$

Therefore $P(|X_n| \leq M, |X| > M) < \varepsilon + P(|X_n| \leq M, |X| > M+\delta)$

And $|X_n| \leq M, |X| > M+\delta \Rightarrow |X_n - X| \geq |X| - |X_n| > \delta$

Thus $< \varepsilon + P(|X_n - X| > \delta)$

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$\limsup_n P(|X_n| \leq M, |X| > M) < \varepsilon + \lim P(|X_n - X| > \delta)$

$= \varepsilon$

$(\because X_n \xrightarrow{P} X)$

$\varepsilon \downarrow 0$, we have $\limsup P(|X_n| \leq M, |X| > M) = 0$

② Similary, $P(|X| \leq M, |X_n| > M) - P(|X| \leq M - \delta, |X_n| > M)$

$= P(M - \delta < |X| \leq M, |X_n| > M)$

$\leq P(M - \delta < |X| \leq M) < \varepsilon$ (When δ is small)

$\therefore P(|X| \leq M, |X_n| > M) < \varepsilon + P(|X| \leq M - \delta, |X_n| > M)$

$|X| \leq M - \delta, |X_n| > M \Rightarrow |X_n - X| \geq |X_n| - |X| > \delta$

Hence $\leq \varepsilon + P(|X_n - X| > \delta)$

By taking $n \to \infty$, we have

$\limsup P(|X| \leq M, |X_n| > M) = 0$

If you want to know the general proof ($p>0$)
please refer to the textbook of Beijing University
(測度論與概率論基礎 chapter 2
(程士宏) (chéng shì hóng)

① & ②, we have $\limsup P(\{|X_n| \le M\} \triangle \{|X| \le M\}) = 0$

Now we consider

$$P\left(\left| X_n \cdot I\{|X_n| \le M\} - X \cdot I\{|X| \le M\} \right| > \varepsilon\right)$$

$$= P\left(\left| X_n \cdot I\{|X_n| \le M\} - X \cdot I\{|X_n| \le M\} + X \cdot I\{|X_n| \le M\} - X \cdot I\{|X| \le M\} \right| > \varepsilon\right)$$

$\left(= I_{\{|X_n| \le M\} \triangle \{|X| \le M\}}\right)$

$$\le P\left(|X_n - X| \cdot I\{|X_n| \le M\} + |X| \cdot \left| I\{|X_n| \le M\} - I\{|X| \le M\} \right| > \varepsilon \right)$$

$$\le P\left(|X_n - X| + |X| \cdot I_{\{|X_n| \le M\} \triangle \{|X| \le M\}} > \varepsilon \right)$$

$$\le P\left(\left\{ |X_n - X| > \frac{\varepsilon}{2} \right\} \cup \left\{ |X| \cdot I_{\{|X_n| \le M\} \triangle \{|X| \le M\}} > \frac{\varepsilon}{2} \right\} \right)$$

$$\le P\left(|X_n - X| > \frac{\varepsilon}{2} \right) + P\left(|X| \cdot I_{\{|X_n| \le M\} \triangle \{|X| \le M\}} > \frac{\varepsilon}{2} \right)$$

$$\le (\cdot) + P\left(\{|X| > M\} \cup \left\{ I_{\{|X_n| \le M\} \triangle \{|X| \le M\}} > \frac{\varepsilon}{2M} \right\} \right)$$

$$\le (\cdot) + P(|X| > M) + P\left(I_{\{|X_n| \le M\} \triangle \{|X| \le M\}} > \frac{\varepsilon}{2M} \right)$$

$$= (\cdot) + P(|X| > M) + P\left(I\{\cdots\} = 1 \right) \quad (\because \text{ only takes } 0 \text{ or } 1)$$

$$= (\cdot) + P(|X| > M) + P\left(\{|X_n| \le M\} \triangle \{|X| \le M\} \right)$$

Take large $M \nearrow$, and $n \to \infty$. We will have the desired result.

$\boxed{50}$

(1) Let $h(x) = \frac{x}{1+x}$.

$h(0) = 0 \quad h'(x) = \frac{1}{(1+x)^2} > 0$ and $h'(x)$ is a monotone decreasing function

Let us recall $\boxed{18}$.

: If $\rho(x,y)$ is a metric, then $h \circ \rho(x,y)$ is also a metric on $\mathbb{R} \times \mathbb{R}$

$\rho(x,y) \overset{def}{=} |x-y|$. (is a metric on $\mathbb{R} \times \mathbb{R}$)

Then $d(X,Y) \overset{eq}{=} E[h \circ \rho(X,Y)]$

① $d(X,Y) = d(Y,X)$ ... trivial

② $d(X,Y) \geq 0$ ... trivial

$d(X,Y) = 0 \Rightarrow h \circ \rho(X,Y) = 0$ (as) (non-negative)

$\Rightarrow X = Y$ (as) ($\because h \circ \rho(x,y)$ is a metric on $\mathbb{R} \times \mathbb{R}$)

: $X = Y \Rightarrow d(X,Y) = 0$

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③ $h\circ\rho(X,Y) \leq h\circ\rho(X,Z) + h\circ\rho(Y,Z)$ ($\because$ $h\circ\rho$: metric)

Hence $E\,h\circ\rho(X,Y) \leq E\,h\circ\rho(X,Z) + E\,h\circ\rho(Y,Z)$

$\therefore d(X,Y) \leq d(X,Z) + d(Y,Z)$

$\therefore$ $(\mathcal{X}, d)$ is also a metric.

(2) $d(X_n, X) \to 0 \iff X_n \xrightarrow{P} X$

① $(\Rightarrow)$

$P(|X_n - X| > \varepsilon) = P(h(|X_n - X|) > h(\varepsilon))$

($\because$ $h(\cdot)$ is strictly increasing in $(0,\infty)$)

$= P(h\circ\rho(X_n, X) > h(\varepsilon))$

$\leq \frac{1}{h(\varepsilon)} E[h\circ\rho(X_n,X)] = \frac{1}{h(\varepsilon)} d(X_n, X)$

$d(X_n, X) \to 0 \Rightarrow P(|X_n - X| > \varepsilon) \to 0$ ($\forall \varepsilon > 0$) (as $n \to \infty$)

② $(\Leftarrow)$

$X_n \xrightarrow{P} X \Rightarrow X_n - X \xrightarrow{P} 0 \Rightarrow |X_n - X| \xrightarrow{P} 0$

$\Rightarrow h(|X_n - X|) \xrightarrow{P} h(0) \Rightarrow h\circ\rho(X_n, X) \xrightarrow{P} 0.$

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Since $h \circ \rho(X_n, X)$ is a bounded function

$\left(\sup_{n \ge 1} |h \circ \rho(X_n, X)| \le 1 \cdots \text{integrable} \right)$

By Bounded Convergence Theorem (or Lebesgue's Dominated Convergence Theorem)

$$\lim_{n\to\infty} E[h \circ \rho(X_n, X)] = \lim_{n\to\infty} d(X_n, X)$$

$$\parallel$$

$$E[0] = 0$$

∎

∴ The proof is complete.

$\boxed{5}$ $(\mathcal{X}, d)$, $\mathcal{X}$ are random variables on $(\Omega, \mathcal{F}, P)$

We show that $(\mathcal{X}, d)$ is a complete metric space.

(proof) Let $\{X_n\}_{n\geq 1} \subseteq \mathcal{X}$ and $\lim_{n,m\to\infty} d(X_n, X_m) = 0$.

Recall that $P(|X_n - X_m| > \varepsilon) = P(h(|X_n - X_m|) > h(\varepsilon))$

$\leq \frac{1}{h(\varepsilon)} E[h|X_n - X_m|] = \frac{1}{h(\varepsilon)} d(X_n, X_m)$

$\therefore \lim_{\substack{n\to\infty \\ m\to\infty}} P(|X_n - X_m| > \varepsilon) = 0$

, This implies that $\forall k = 1, 2, 3 \ldots$ $\exists \{n_k\}_{k\geq 1} \subseteq \mathbb{N}$

such that $P\left(|X_{n(k)} - X_{n(k+1)}| > \frac{1}{2^k}\right) \leq \frac{1}{2^k}$

$\left(\because \lim_{n,m\to\infty} P(|X_n - X_m| > \varepsilon) = 0 \Rightarrow \forall \varepsilon > 0\ \forall \delta > 0\ \exists M_{\varepsilon\delta} \text{ s.t.}\right.$

$\left. \sup_{n,m\geq M} P(|X_n - X_m| > \varepsilon) < \delta \right)$

By Borel-Cantelli's lemma,

$\sum_{k=1}^{\infty} P\left(|X_{n(k)} - X_{n(k+1)}| > \frac{1}{2^k}\right) < \infty$

$\Rightarrow P\left(\left\{|X_{n(k)} - X_{n(k+1)}| > \frac{1}{2^k} \text{ i.o}\right\}\right) = 0$

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$\#\{k \mid |X_{n(k)} - X_{n(k+1)}| > \frac{1}{2^k}\} < \infty$ (as)

$\Rightarrow \sum_{k=1}^{\infty} |X_{n(k)} - X_{n(k+1)}| < \infty$ (as)

$\Rightarrow \sum_{k=1}^{\infty} (X_{n(k+1)} - X_{n(k)})$: converses (as)

Let $X(w) = X_{n(1)} + \sum_{k\geq 1}^{\infty} (X_{n(k+1)} - X_{n(k)})$

$= \lim_{k\to\infty} X_{n(k)}$ (as)

be a random variable defined almost surely.
(By measurable modification, it will become a random variable)

$d(X, X_m) \leq d(X_{n(k)}, X_m) + d(X_{n(k)}, X)$

① ②

① $\to 0$ as $m\to\infty$ and $k\to\infty$ ($\because$ $\{X_n\}_{n\geq 1}$: Cauchy sequence)

② $\to 0$ as $k\to\infty$. $\because$ $X_{n(k)} \to X$ (as)

$\Rightarrow X_{n(k)} \xrightarrow{P} X$

$\Rightarrow d(X_{n(k)}, X) \to 0$ ($\because$ $\boxed{50}$)

So the proof is complete ■

[52] $\{A_n\}_{n\geq 1}$ are independent with $P(A_n)<1$

$P(\bigcup_{n=1}^{\infty} A_n) = 1 \Rightarrow \sum_{n=1}^{\infty} P(A_n) = \infty$

(proof) $S_n = \bigcup_{m=1}^{n} A_m \quad T_n = \bigcup_{m=n+1}^{\infty} A_m$

S_n and T_n are independent

$1 = P(\bigcup_{n=1}^{\infty} A_n) = P(S_n \cup T_n) = P(S_n) + P(T_n) - P(S_n \cap T_n)$

$= P(S_n) + P(T_n) - P(S_n)P(T_n) = -(1-P(S_n))(1-P(T_n)) + 1$

$\Leftrightarrow (1-P(S_n))(1-P(T_n)) = 0$

Hence $P(S_n)=1$ or $P(T_n)=1$

But $P(S_n)<1$. So $P(T_n)=1$ for all $n\geq 1$.

$(\because P(S_n^c) = P(\bigcap_{m=1}^{n} A_m^c) = \prod_{m=1}^{n} \underbrace{(1-P(A_m))}_{>0} > 0 \quad (P(A_m)<1)$

$\Rightarrow P(S_n) = 1 - P(S_n^c) < 1)$

Since $P(T_n)=1$ for all n. $P(\bigcap_{n=1}^{\infty} T_n) = \lim_{n\to\infty} P(T_n) = 1$

$\therefore P(A_n \text{ i.o}) = 1$

Recall the Borel-Cantelli's lemma

「 $\sum_{n=1}^{\infty} P(A_n) < \infty \Rightarrow P(A_n \text{ i.o}) = 0$ 」

$\Leftrightarrow$ 「 $P(A_n \text{ i.o}) > 0 \Rightarrow \sum_{n=1}^{\infty} P(A_n) = \infty$ 」

Hence $P(A_n \text{ i.o}) = 1 \Rightarrow P(A_n \text{ i.o}) > 1$

$\Rightarrow \sum_{n=1}^{\infty} P(A_n) = \infty$

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(1) $P(A_n \text{ i.o}) = P\left(\bigcap_{n=1}^{\infty} \bigcup_{m\ge n}^{\infty} A_m\right)$

$= \lim_{n\to\infty} P\left(\bigcup_{m\ge n}^{\infty} A_m\right) = \lim_{n\to\infty} P\left(A_n \cup \left(\bigcup_{m\ge n}^{\infty} A_{m+1} \setminus A_m\right)\right)$

$\le \lim_{n\to\infty} \left\{ P(A_n) + \sum_{m\ge n}^{\infty} P(A_{m+1} \setminus A_m) \right\}$

$= 0$

$\left(\begin{array}{l} \because \ P(A_n) \to 0 \quad (\text{as } n \to \infty) \\ \because \ \sum_{n=1}^{\infty} P(A_{n+1} \setminus A_n) < \infty \Rightarrow \lim_{n\to\infty} \sum_{m\ge n}^{\infty} P(A_{m+1} \setminus A_m) = 0 \end{array} \right.$

(2) $A_n = (0, \frac{1}{n})$ $\quad P$ = Lebesgue Measure

$A_{n+1} \setminus A_n = \phi$

$\sum_{n=1}^{\infty} P(A_{n+1} \setminus A_n) = 0 < \infty$

$P(A_n) = \frac{1}{n} \to 0 \qquad (\Rightarrow P(A_n \text{ i.o}) = 0)$

However $\sum_{n=1}^{\infty} P(A_n) = \infty$ so the Borel-Cantelli's Lemma is not applicable.

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$\boxed{54}$

(1) $X_n \xrightarrow{P} 0 \Leftrightarrow \lim_{n\to\infty} P(|X_n| > \varepsilon) = 0$

$= \lim_{n\to\infty} P(X_n = 1) = \lim_{n\to\infty} p_n = 0$ ⊗

(⊗ $X_n = 0$ or 1)

(2) $X_n \to 0$ (a.s) $\Leftrightarrow$ $\forall \varepsilon > 0$ $P(|X_n| > \varepsilon \text{ i.o}) = 0$

$\Leftrightarrow P(X_n = 1 \text{ i.o}) = 0$

① $\sum_{n=1}^{\infty} p_n < \infty \Rightarrow \sum_{n=1}^{\infty} P(X_n = 1) < \infty$

$\Rightarrow P(X_n = 1 \text{ i.o}) = 0$ ($\because$ Borel-Cantelli's lemma 1)

② $P(X_n = 1 \text{ i.o}) = 0 \Rightarrow$

Since $\{X_n = 1\}_{n\geq 1}$ are independent events

「 $\sum_{n=1}^{\infty} P(X_n = 1) = \infty \Rightarrow P(X_n = 1 \text{ i.o}) = 1$ 」

($\because$ Borel-Cantelli's lemma 2)

$\Leftrightarrow$ 「 $P(X_n = 1 \text{ i.o}) < 1 \Rightarrow \sum_{n=1}^{\infty} P(X_n = 1) < \infty$ 」

So $\sum_{n=1}^{\infty} P(X_n = 1) < \infty$. $\therefore \sum_{n=1}^{\infty} p_n < \infty$

($\because P(X_n = 1 \text{ i.o}) = 0$)

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$\boxed{55}$ $\{X_n\}_{n\geq1}$: random variables

We basically assume $X_n : \Omega \to \mathbb{R}$ ($(-\infty, \infty)$)

$^\forall \varepsilon>0$ $^\exists M>0$ s.t $P(|X_n|>M) < \varepsilon$

$(\because M \uparrow \infty$ $\{|X_n|>M\} \downarrow \phi$ $)$

So for any $\varepsilon>0$ and for each $n=1,2,\cdots$

We may find $\{C_n\}_{n\geq1}$ such that

$P\left(\frac{|X_n|}{\varepsilon} > C_n\right) < \frac{1}{2^n} \Rightarrow$

$P\left(\left|\frac{X_n}{C_n}\right| > \varepsilon\right) < \frac{1}{2^n}$

$\therefore \sum_{n=1}^{\infty} P\left(\frac{|X_n|}{C_n} > \varepsilon\right) < \infty$

By Borel-Cantelli's lemma

$P\left(\frac{|X_n|}{C_n} > \varepsilon \text{ i.o}\right) = 0$ (for all $\varepsilon>0$)

$\Rightarrow \frac{|X_n|}{C_n} \xrightarrow{a.s} 0$. $(\because \boxed{37})$

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$\boxed{56}$

① $\Rightarrow$

$X \overset{\text{def}}{=} \sup_{n \geq 1} X_n$

Since $X < \infty$ (a.s), $P\left(\bigcup_{A=1}^{\infty} \{X \leq A\}\right) = 1$

For suffiently large $A > 0$. $P(X \leq A) > 0$

$\Rightarrow P(X > A) < 1$.

$\{X_n > A \text{ i.o}\} \subseteq \{X > A\}$

$\therefore P(\{X_n > A \text{ i.o}\}) < 1$

$\Rightarrow \sum_{n=1}^{\infty} P(X_n > A) < \infty$ ⊛

⊛ Borel-Cantelli's lemma (II)

$\{X_n > A\}$: independant

「$\sum_{n=1}^{\infty} P(X_n > A) = \infty \Rightarrow P(X_n > A \text{ i.o}) = 1$」

「$P(X_n > A \text{ i.o}) < 1 \Rightarrow \sum_{n=1}^{\infty} P(X_n > A) < \infty$」

② $\Leftarrow$

$\sum_{n=1}^{\infty} P(X_n > A) < \infty$

$\Rightarrow P(X_n > A \text{ i.o}) = 0$

$\Rightarrow \#\{n \mid X_n > A\} < \infty$ (a.s)

So for suffiently large $N(\omega)$, $\forall n > N(\omega)$

$X_n \leq A \Rightarrow \sup_{n \geq 1} X_n \leq \max_{1 \leq m \leq N(\omega)} \{X_m\} \cup \{A\} < \infty$ (a.s)

Hence, $\sup_{n \geq 1} X_n(\omega) < \infty$ (a.s)

(∵ Only finite number of $\{X_n\}$ are larger than A

So they will eventually $\leq$ A)

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$\boxed{57}$ (1)

① $P\left(\frac{X_n}{\log(n)} > (1+\varepsilon)\right)$

$= P\left(X_n > \log(n)^{1+\varepsilon}\right) = \exp\left(-\log n^{1+\varepsilon}\right) = \frac{1}{n^{1+\varepsilon}}$

$\sum_{n=1}^{\infty} P\left(\frac{X_n}{\log(n)} > 1+\varepsilon\right) \leq \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} < \infty$

$\therefore P\left(\frac{X_n}{\log(n)} > 1+\varepsilon \text{ i.o}\right) = 0.$

$\#\left\{n \mid \frac{X_n}{\log n} > 1+\varepsilon\right\} < \infty$ (a.s)

$\therefore$ For suffiently large n, $\frac{X_n}{\log(n)} \leq 1+\varepsilon$ (a.s)

$\therefore \limsup_{n\to\infty} \frac{X_n}{\log(n)} \leq 1+\varepsilon$ (a.s)

$\Rightarrow \limsup \frac{X_n}{\log(n)} \leq 1$ (a.s)

$(\because \exists\, \Omega^{(\varepsilon)} \in \mathcal{F}$ $P(\Omega^{(\varepsilon)}) = 1$ st $\forall \omega \in \Omega^{(\varepsilon)}$

$\limsup_{n\to\infty} \frac{X_n}{\log(n)} \leq 1+\varepsilon.$

Now,

$\varepsilon = \frac{1}{k}$ $\Omega^{*} = \bigcap_{k=1}^{\infty} \Omega^{(\frac{1}{k})}$ $P(\Omega^{*}) = 1$

$\forall \omega \in \Omega^{*}$ $\limsup_{n\to\infty} \frac{X_n}{\log n} \leq 1+\frac{1}{k}$ for all $k \geq 1$)

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② $P\left(\frac{X_n}{\log n} > 1\right) = P(X_n > \log n) = \frac{1}{n}$

$\therefore \sum_{n=1}^{\infty} P\left(\frac{X_n}{\log n} > 1\right) = \infty$

$P\left(\frac{X_n}{\log n} > 1 \text{ i.o.}\right) = 1$ ($\because$ Borel-Cantelli's lemma II)

$\therefore \limsup \frac{X_n}{\log n} \geq 1$ (a.s)

$\left(\#\{n \mid \frac{X_n}{\log n} > 1\} = \infty\right.$ (a.s)

$\Rightarrow \sup_{m \geq n} \frac{X_m}{\log m} > 1$ (a.s) (for all $m \geq 1$)

$\Rightarrow \limsup \frac{X_n}{\log n} \geq 1$ (a.s)$\Big)$

① & ② $\Rightarrow \limsup X_n = 1$ (a.s)

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(2)

We are going to prove that $\limsup \frac{X_n^+}{\log n} = \limsup \frac{M_n}{\log n}$

in $\boxed{60}$. By using this fact, we have

$\limsup \frac{M_n}{\log n} = 1$ (a.s).

Next, we show that $\liminf_{n\to\infty} \frac{M_n}{\log n} \geq 1$ (a.s)

$P\left(\frac{M_n}{\log(n)} > (1-\varepsilon)\right)$

$= 1 - P\left(\frac{M_n}{\log(n)} \leq (1-\varepsilon)\right)$

$= 1 - P\left(M_n \leq \log(n)^{1-\varepsilon}\right)$

$= 1 - \left(1 - \frac{1}{n^{1-\varepsilon}}\right)^n$

$= 1 - \underbrace{\left(1 - \frac{1}{n^{1-\varepsilon}}\right)^{(-n^{1-\varepsilon})\cdot(-n^{\varepsilon})}}$

$\approx e$ (for suffiently lage n)

$\approx 1 - \exp(-n^{\varepsilon})$

$\therefore P\left(\frac{M_n}{\log n} \leq 1-\varepsilon\right) \approx \exp(-n^{\varepsilon})$ (for suffiently large n)

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$$\sum_{n=1}^{\infty} \exp(-n^{\varepsilon}) < \infty \quad \cdots \quad \circledast$$

$$\therefore \sum_{n=1}^{\infty} P\left(\frac{M_n}{\log n} \le 1-\varepsilon\right) < \infty$$

$$\therefore P\left(\frac{M_n}{\log n} \le 1-\varepsilon \text{ i.o.}\right) = 0$$

$$\#\left\{n \,\middle|\, \frac{M_n}{\log n} \le 1-\varepsilon\right\} < \infty \quad (a.s)$$

For sufficiently large n, $\frac{M_n}{\log(n)} > 1-\varepsilon$ (a.s)

$$\therefore \liminf \frac{M_n}{\log(n)} > 1-\varepsilon \quad (a.s)$$

$$\Rightarrow \liminf \frac{M_n}{\log n} \ge 1 \quad (a.s)$$

Now the proof is complete.

$$\circledast \quad \underset{(<\infty)}{\int_0^{\infty} \exp(-x^{\varepsilon})\,dx} = \sum_{n=0}^{\infty} \int_n^{n+1} \exp(-x^{\varepsilon})\,dx \ge \sum_{n=1}^{\infty} \exp(-n^{\varepsilon})$$

(put $t = x^{\varepsilon}$ and use the definition of Gamma Function)

$\boxed{58}$ $X_1, X_2 \dots$ iid $\sim F$

We show that

① $\sum_{n=1}^{\infty} (1-F(\lambda_n)) < \infty \Rightarrow P(A_n \text{ i.o}) = 0$
② $\sum_{n=1}^{\infty} (1-F(\lambda_n)) = \infty \Rightarrow P(A_n \text{ i.o}) = 1$

Let $B_n = \{X_n > \lambda_n\}$.

First we show that $\{A_n \text{ i.o}\} = \{B_n \text{ i.o}\}$

(I) $B_n \subseteq A_n \Rightarrow \{B_n \text{ i.o}\} \subseteq \{A_n \text{ i.o}\}$

(II) $\omega \in \{A_n \text{ i.o}\}$

We may find $n_1 \in \mathbb{N}$ such that $\omega \in A_{n_1}$

$\therefore \max_{1 \le m \le n_1} X_m(\omega) > \lambda_{n_1}$

There exists $m_1 \in \{1, 2, \dots, n_1\}$ s.t

$X_{m_1}(\omega) > \lambda_{n_1}$

$\Rightarrow X_{m_1}(\omega) > \lambda_{m_1}$ (λ_n is increasing.
$\therefore \lambda_{m_1} \le \lambda_{n_1}$)

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$\therefore \omega \in B_{m_1}$

In the same way, we may take n_2 ($n_2 > n_1$) s.t

$\omega \in A_{n_2}$: $\max_{1 \le m \le n_2} X_m(\omega) > \lambda_{n_2}$ (and $\lambda_{n_2} > X_{m_1}(\omega)$)
($\because \lambda_n \nearrow \infty$)

We may find $m_2 > m_1$ s.t $X_{m_2} > \lambda_{n_2}$
$(1 \le m_2 \le n_2)$

$\Rightarrow X_{m_2} > \lambda_{m_2} \Rightarrow \omega \in B_{m_2}$.

In this way we may find infinitely many

$\{B_{m_i}\}_{i \ge 1} \subseteq \{B_n\}_{n \ge 1}$ containing ω.

So $\{A_n \text{ i.o}\} \subset \{B_n \text{ i.o}\}$

(I) & (II) $\Rightarrow \{A_n \text{ i.o}\} = \{B_n \text{ i.o}\}$

Now ① and ② are easy to verify.

By Borel-Cantelli's lemma 1 and 2,

We have ① $\Rightarrow P(B_n \text{ i.o}) = 0 \Rightarrow P(A_n \text{ i.o}) = 0$
② $\Rightarrow P(B_n \text{ i.o}) = 1 \Rightarrow P(A_n \text{ i.o}) = 1$

$\boxed{59}$ $Y_1, Y_2 \cdots$ are iid. And We suppose $Y_n : \Omega \to \mathbb{R}$

(1) $\frac{Y_n}{n} \xrightarrow{a.s} 0 \iff E|Y_n| < \infty$

(1) ($\Leftarrow$) Let $\varepsilon > 0$.

$$E|Y_1| = \int_0^\infty P(|Y_1| > y)\,dy$$
$$= \sum_{n=0}^{\infty} \int_{n\varepsilon}^{(n+1)\varepsilon} P(|Y_1| > y)\,dy$$
$$\geq \sum_{n=0}^{\infty} \int_{n\varepsilon}^{(n+1)\varepsilon} P(|Y_1| > (n+1)\varepsilon)\,dy$$
$$= \sum_{n=0}^{\infty} P(|Y_n| > (n+1)\varepsilon) \quad (\because \text{iid})$$
$$= \sum_{n=1}^{\infty} P(|Y_n| > n\varepsilon)$$
$$= \sum_{n=1}^{\infty} P\left(\left|\frac{Y_n}{n}\right| > \varepsilon\right)$$

By Borel-Cantelli's Lemma

$$P\left(\frac{|Y_n|}{n} > \varepsilon \text{ i.o}\right) = 0$$

$$\Rightarrow \left|\frac{Y_n}{n}\right| \xrightarrow{a.s} 0$$

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($\Rightarrow$) We show $E|Y_1| = \infty \Rightarrow \frac{Y_n}{n} \overset{a.s}{\nrightarrow} 0$

$E|Y_1| = \int_0^\infty P(|Y_1| \geq y)\,dy = \sum_{n=0}^{\infty} \int_{n\varepsilon}^{(n+1)\varepsilon} P(|Y_1| \geq y)\,dy$

$\leq \sum_{n=0}^{\infty} \int_{n\varepsilon}^{(n+1)\varepsilon} P(|Y_1| > n\varepsilon)\,dy = \sum_{n=0}^{\infty} P(|Y_n| > n\varepsilon) = \infty$

$\{|Y_n| > n\varepsilon\}$ are independent and by Borel-Cantelli's

lemma, $P(\{|Y_n| > n\varepsilon \text{ i.o}\}) = 1$

$\therefore P(\{\frac{|Y_n|}{n} > \varepsilon \text{ i.o}\}) = 1 > 0$

$\therefore \frac{Y_n}{n} \overset{a.s}{\nrightarrow} 0$

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DATE $M_n = \max\{Y_1 \cdots Y_n\}$

(2) We show that $M_n/n \xrightarrow{as} 0 \Leftrightarrow E|Y_n^+| < \infty$

$\limsup \frac{M_n}{n} = \limsup \frac{Y_n^+}{n}$ (by $\boxed{60}$)

If $E|Y_n^+| < \infty$, then by (1),

$\lim_{n\to\infty} \frac{Y_n^+}{n} = 0$ (a.s) $\left(\Leftrightarrow \limsup \frac{Y_n^+}{n} = 0 \text{ (a.s)}\right)$

$\therefore \limsup_{n\to\infty} \frac{M_n}{n} = 0$ (a.s)

We show that $\liminf_{n\to\infty} \frac{M_n}{n} \geq 0$ (a.s)

① $\omega \in \{\omega \mid \exists n \text{ s.t } M_n \geq 0\}$

Then since $M_n \leq M_{n+1}$, thus

$\liminf \frac{M_n(\omega)}{n} \geq 0$.

② $\omega \in \{\omega \mid \forall n \ M_n < 0\}$

$\inf_{m\geq n} \frac{M_m}{m} = \frac{M_n}{n} \geq \frac{Y_1(\omega)}{n}$

$Y_1 : \Omega \to \mathbb{R} \Rightarrow Y_1(\omega) \in \mathbb{R}$. $\therefore \frac{Y_1(\omega)}{n} \to 0$.

$\therefore \liminf \frac{M_n}{n} \geq 0.$

So by ①, ②, $\liminf \frac{M_n}{n} \geq 0$

$\boxed{\text{NOTE}}$

- Some people may feel ③ is strange.

But $\frac{Y_n}{n} \xrightarrow{(a.s)} 0$ and $\frac{Y_1}{n} \xrightarrow{(a.s)} 0$ is quite different

since $\{\frac{Y_n}{n}\}_{n\geq 1}$ are independent but $\{\frac{Y_1}{n}\}$ are not.

$F_Y(\cdot)$: cdf of Y

(3)

$$P\left(\frac{|M_n|}{n}>\varepsilon\right)=P\left(\frac{M_n}{n}>\varepsilon\right)+P\left(\frac{M_n}{n}<-\varepsilon\right)$$

$$=1-P(M_n\leq n\varepsilon)+P(M_n<-n\varepsilon)$$

$$=1-F_Y(n\varepsilon)^n+\underbrace{F_Y(-n\varepsilon-)^n}_{\to 0}$$

$$(\because F_Y(-n\varepsilon)\downarrow 0)$$

So $\lim_{n\to\infty} P\left(\frac{|M_n|}{n}>\varepsilon\right)=0$

$\Leftrightarrow \lim_{n\to\infty} F_Y(n\varepsilon)^n=1 \qquad (\forall\varepsilon>0)$

Moreover $(1-(1-F_Y(n\varepsilon)))^n$

$$=\underbrace{\left\{(1-(1-F_Y(n\varepsilon)))^{\frac{-1}{1-F_Y(n\varepsilon)}}\right\}}_{\to e}^{(-1)(1-F_Y(n\varepsilon))n}$$

So $\Leftrightarrow n(1-F_Y(n\varepsilon))\to 0$

$\Leftrightarrow n\varepsilon(1-F_Y(n\varepsilon))\to 0$

$\Leftrightarrow xP(Y_1>x)\to 0 \quad (\text{as } x\to\infty)$

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(4) $P\left(\left|\frac{Y_n}{n}\right| > \varepsilon\right) = P\left(\frac{|Y_1|}{n} > \varepsilon\right) = P(|Y_1| > n\varepsilon) \to 0$

(as $n \to \infty$)

($\because \{|Y_1| > n\varepsilon\} \downarrow \phi$

$\because Y_1 : \Omega \to \mathbb{R}$)

So if $Y_1 : \Omega \to \mathbb{R}$ then $\frac{Y_n}{n} \xrightarrow{P} 0$

But if you assume $Y_n : \Omega \to \overline{\mathbb{R}}$ ($[-\infty, \infty]$)

$|Y_n| < \infty$ (a.s) will be the answer

When we talk about "random variables",

we often assume $Y_n : \Omega \to \mathbb{R}$.

But $\lim Y_n$ can be : $\Omega \to \overline{\mathbb{R}}$.

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$\boxed{60}$ $b_n \nearrow \infty$. $M_n = \max\{a_1, a_2 \cdots a_n\}$

Then $\limsup \frac{M_n}{b_n} = \limsup \frac{a_n^+}{b_n}$

$$\underset{\alpha}{\parallel} \qquad \underset{\beta}{\parallel}$$

① $\alpha \leqq \beta$

Since $\beta = \limsup \frac{a_n^+}{b_n} = \lim_{n\to\infty} \sup_{m \geq n} \frac{a_m^+}{b_m}$,

$\forall \varepsilon > 0$, $\exists N$ st $\frac{a_n^+}{b_n} \leqq \beta + \varepsilon$ (for all $n > N$)

Now consider $\frac{M_n}{b_n} = \frac{1}{b_n} \max\{a_1, a_2 \dots a_n\}$

$\leqq \frac{1}{b_n} \max\{a_1, a_2, \dots, a_N, \underset{\sim}{a_{N+1}^+}, \dots \underset{\sim}{a_n^+}\}$

$\leqq (\beta+\varepsilon) b_{N+1} \quad \leqq (\beta+\varepsilon) b_n$

And $b_n \leqq b_{n+1}$, $b_n \nearrow \infty$

$\Rightarrow \frac{M_n}{b_n} \leqq \frac{1}{b_n} \max\{a_1 \dots a_N, (\beta+\varepsilon) b_n\}$

$\leqq \max\{\frac{a_1}{b_n}, \dots, \frac{a_N}{b_n}, (\beta+\varepsilon)\}$

By taking sufficiently large n.

$\frac{a_1}{b_n} \sim \frac{a_N}{b_n} < (\beta+\varepsilon)$

$\Rightarrow \frac{M_n}{b_n} \leqq \beta+\varepsilon$

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$\therefore \limsup \frac{M_n}{b_n} \leq (\beta+\varepsilon)$

$\therefore \varepsilon \downarrow 0$ we have $\alpha \leq \beta$.

② $\alpha \geq \beta$

(I) If there exists n s.t. $a_n > 0$

Then $M_n \geq a_n^+$ ($\because M_n \leq M_{n+1}$)

Hence $\frac{M_n}{b_n} \geq \frac{a_n^+}{b_n}$

$\therefore \limsup \frac{M_n}{b_n} \geq \limsup \frac{a_n^+}{b_n}$ $\therefore$ $\alpha \geq \beta$

(II) If $a_n < 0$ … for all $n \geq 1$.

$\limsup \frac{a_n^+}{b_n} = 0$

$\frac{M_n}{b_n} \geq \frac{a_1}{b_n}$ $\limsup \frac{a_1}{b_n} = 0$

$\therefore \limsup \frac{M_n}{b_n} \geq 0$ and $M_n \leq 0$

So $\limsup \frac{M_n}{b_n} \leq 0$ $\therefore \limsup \frac{M_n}{b_n} = 0$

$\therefore \alpha = \beta = 0$ $\therefore \alpha \geq \beta$

[61] The technique is similar to [45]

Without loss of generality we may assume that
$0 < \lambda_n \leq 1$

($\because$ Y_1, Y_2 : independent $Y_1 \sim Po(\lambda_1)$ $Y_2 \sim Po(\lambda_2)$

$Y_1 + Y_2 \sim Po(\lambda_1 + \lambda_2)$. hence if $X \sim Po(\lambda)$ $\lambda > 1$

we may rewrite $X = X^{(1)} + \cdots + X^{(k)}$

$X^{(i)} \sim Po(\lambda(i))$ $\lambda(i) \in (0, 1]$. $\lambda(1) + \cdots + \lambda(k) = \lambda$)

Let $n(k) \stackrel{\text{def}}{=} \inf\{n \mid \lambda_1 + \cdots + \lambda_n \geq k^2\}$

Consider $P(|S_{n(k)} - E[S_{n(k)}]| > \varepsilon E[S_{n(k)}])$

By Markov's inequality,

$$\leq \frac{V[S_{n(k)}]}{\varepsilon^2 E[S_{n(k)}]^2}$$

$S_{n(k)} \sim Po(\lambda_1 + \cdots + \lambda_{n(k)})$

$\therefore V[S_{n(k)}] = E[S_{n(k)}]$ ($\because$ Poisson distribution)

$$= \frac{E[S_{n(k)}]}{\varepsilon^2 E[S_{n(k)}]^2} = \frac{1}{\varepsilon^2 E[S_{n(k)}]} \le \frac{1}{\varepsilon^2 \cdot k^2}$$

therefore $\displaystyle\sum_{k=1}^{\infty} P\left(\left|S_{n(k)} - E[S_{n(k)}]\right| > \varepsilon \cdot E[S_{n(k)}]\right) < \infty$

By Borel–Cantelli's lemma

$$P\left(\left\{\left|S_{n(k)} - E[S_{n(k)}]\right| > \varepsilon E[S_{n(k)}] \ \text{i.o.}\right\}\right) = 0$$

$$= P\left(\left\{\left|\frac{S_{n(k)}}{E[S_{n(k)}]} - 1\right| > \varepsilon \ \text{i.o.}\right\}\right) = 0 \quad (\forall \varepsilon > 0)$$

$$\Rightarrow \frac{S_{n(k)}}{E[S_{n(k)}]} \to 1 \ \text{(a.s)} \quad (\because \boxed{3\text{개}})$$

If $n(k) \le m < n(k+1)$

$$\underset{①}{\frac{S_{n(k)}}{E[S_{n(k+1)}]}} \le \frac{S_m}{E[S_m]} \le \underset{②}{\frac{S_{n(k+1)}}{E[S_{n(k)}]}}$$

$$① = \underbrace{\frac{S_{n(k)}}{E[S_{n(k)}]}}_{\to 1 \ \text{(a.s)}} \ \underbrace{\frac{E[S_{n(k)}]}{E[S_{n(k+1)}]}}_{\in \left[\frac{k^2}{(k+1)^2+1},\ \frac{k^2+1}{(k+1)^2}\right] \to 1}$$

② similarly $\to 1$ So $\dfrac{S_m}{E[S_m]} \to 1$ (a.s)

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$\boxed{62}$

(1) $\boxed{\text{Step 1}}$ $\sum_{n=1}^{\infty} P(\{X_n \neq Y_n\}) = \sum_{n=1}^{\infty} P(|X_n| > n) < \infty$ ... ⊛

(see below)

$\Rightarrow P(\{X_n \neq Y_n \text{ i.o}\}) = 0$

$\Rightarrow \#\{n \mid X_n \neq Y_n\} < \infty$ (a.s)

$\Rightarrow \sum_{n=1}^{\infty} |X_n - Y_n| < \infty$ (a.s)

$\boxed{\text{Step 2}}$ $\left|\frac{S_n}{n} - \frac{T_n}{n}\right| = \frac{1}{n}|S_n - T_n| \leq \frac{1}{n}\sum_{m=1}^{n}|X_m - Y_m|$ (triangle inequality)

$\leq \frac{1}{n}\underbrace{\sum_{m=1}^{\infty}|X_m - Y_m|}_{<\infty \text{ (a.s)}} \to 0$ (a.s)

$\therefore \frac{S_n}{n} - \frac{T_n}{n} \to 0$ (a.s)

$\therefore \frac{S_n}{n} \to \mu$ (a.s) $\Leftrightarrow \frac{T_n}{n} \to \mu$ (a.s) ∎

We show the main part ⊛ ...

$E|X_n| < \infty \Rightarrow \infty > \underset{\substack{\| \\ E|X_1|}}{\int_0^{\infty} P(|X_1| > x)\,dx} = \sum_{n=0}^{\infty}\int_n^{n+1} P(|X_1| > n)\,dx$

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$$\leq \sum_{n=0}^{\infty} \int_{n}^{n+1} P(|X_1| > n+1) = \sum_{n=0}^{\infty} P(|X_1| > n+1) = \sum_{n=1}^{\infty} P(|X_n| > n) < \infty$$

$(\because \text{ iid})$

So now the proof is complete ∎

(2) $$\frac{V[Y_k]}{k^2} \leq \frac{E[Y_k^2]}{k^2} = \frac{1}{k^2}\int_0^{\infty} 2y\, P(|Y_k| > y)\, dy$$

$$= \frac{1}{k^2}\int_0^{k} 2y\, P(|Y_k| > y)\, dy$$

$(\because |Y_k| = |X_k| \cdot \mathbb{I}\{|X_k| \leq k\}$

$\Rightarrow |Y_k| \leq k)$

$$\leq \frac{1}{k^2}\int_0^{k} 2y\, P(|X_k| > y)\, dy \quad (|Y_k| \leq |X_k|)$$

$$= \frac{1}{k^2}\int_0^{k} 2y\, P(|X_1| > y)\, dy$$

$$\sum_{k=1}^{\infty} \frac{1}{k^2} V[Y_k] \leq \sum_{k=1}^{\infty} \frac{1}{k^2} E[Y_k^2]$$

$$\leq \sum_{k=1}^{\infty} \frac{1}{k^2} \int_0^{k} 2y\, P(|X_1| > y)\, dy$$

by Tonelli's theorem swap Σ, $\int$

$$\leq \int_0^{\infty} \underbrace{\sum_{k \geq [y]+1} \left(\frac{1}{k^2} 2y\, P(|X_1| > y)\right)}_{\circledast} dy$$

$y \in (0,1) \cdots$ $$\circledast \leq \sum_{k=1} \frac{2}{k^2} P(|X_1| > y) \leq \frac{\pi^2}{3} P(|X_1| > y)$$

$$\leq 4P(|X_1| > y)$$

(see next page)

$y \in [1,\infty) \cdots$ $$\circledast \leq \frac{2y}{[y]} P(|X_1| > y) \leq 4P(|X_1| > y)$$

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So $\leq \int_0^\infty 4P(|X_1|>y)\,dy = 4E|X_1|$ ∎

$\boxed{\text{NOTE}}$ 「 $\sum_{k\geq m+1} \frac{1}{k^2} \leq \frac{1}{m}$ 」

We used this inequality. We show this statement.

$$\int_m^\infty \frac{1}{y^2}\,dy \left(=\frac{1}{m}\right) = \sum_{k=m}^\infty \int_k^{k+1} \frac{1}{y^2}\,dy$$

$$\geq \sum_{k=m}^\infty \frac{1}{(k+1)^2}$$

$$= \sum_{k=m+1}^\infty \frac{1}{k^2} \quad ∎$$

(3) We have already proved this.

• $2y \sum_{k>y} k^{-2} = 2y \sum_{k\geq[y]+1} \frac{1}{k^2}$

(if $y>1$) $\leq \frac{2y}{[y]} \leq 4$ $\quad (\because y=1.999\ldots)$

So if $y>1$) … $2y \sum_{k>y} k^{-2} \leq 4$

$y\in(0,1)$ … $\leq 2\cdot\frac{\pi^2}{6} \leq 4$

$\left(\sum_{k=1}^\infty \frac{1}{k^2} = \frac{\pi^2}{6}\right)$

$\boxed{63}$ As we have already shown in $\boxed{62}$, it's enough

to prove that $\frac{T_n}{n} \to \mu$ (a.s)

where $T_n = Y_1 + Y_2 + \cdots + Y_n$ $\quad \left(Y_m = X_m \cdot \mathbb{I}\{|X_m| \leq m\} \right)$

We define $k(n) \overset{\text{def}}{=} [\alpha^n]$ $\quad (\alpha > 1,\ \alpha : \text{fixed})$

Consider $P\left(\left| T_{k(n)} - E[T_{k(n)}] \right| > \varepsilon \cdot k(n) \right)$

By Markov's inequality

$$\leq \frac{V[T_{k(n)}]}{\varepsilon \cdot k(n)^2} = \frac{1}{\varepsilon \cdot k(n)^2} \sum_{m=1}^{k(n)} V[Y_m]$$

$$\leq \frac{1}{\varepsilon \cdot k(n)^2} \sum_{m=1}^{k(n)} E[Y_m^2]$$

$$\therefore \sum_{n=1}^{\infty} P\left(\left| T_{k(n)} - E[T_{k(n)}] \right| > \varepsilon \cdot k(n) \right)$$

$$\leq \sum_{n=1}^{\infty} \sum_{m=1}^{k(n)} \frac{1}{\varepsilon \cdot k(n)^2} E[Y_m^2]$$

$$= \sum_{m=1}^{\infty} \sum_{n:\ k(n) \geq m} (\cdot) \leq \sum_{m=1} \sum_{n:\ \alpha^n \geq m} (\cdot)$$

$\Leftrightarrow \underset{k(n)}{[\alpha^n]} \geq m \Rightarrow \alpha^n \geq m$

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$$= \sum_{m=1}^{\infty} \sum_{n: \alpha^n \geq m} \frac{1}{\varepsilon \cdot k(n)^2} E[Y_m^2]$$

$$\left(\alpha^n \geq k(n) \geq \frac{1}{2}\alpha^n \Rightarrow \frac{1}{k(n)^2} \leq \frac{4}{\alpha^{2n}} \right)$$

$$\therefore \leq \sum_{m=1}^{\infty} \sum_{n: \alpha^n \geq m} \frac{4}{\varepsilon \cdot \alpha^{2n}} E[Y_m^2]$$

$$= \sum_{m=1}^{\infty} E[Y_m^2] \cdot \sum_{n: \alpha^n \geq m} \frac{4}{\varepsilon \cdot \alpha^{2n}}$$

$$= \sum_{m=1}^{\infty} E[Y_m^2] \cdot \frac{4}{\varepsilon^2 \cdot m^2} \cdot \left(\frac{1}{\alpha^2 - 1} \right)$$

$$= \frac{4}{\varepsilon^2} \cdot \frac{1}{\alpha^2 - 1} \cdot \underbrace{\sum_{m=1}^{\infty} \frac{E[Y_m^2]}{m^2}}_{< \infty} < \infty \quad \left(\because \boxed{62} \ (2) \right)$$

Therefore by Borel-Cantelli's lemma

$$P\left(\left\{ \left| T_{k(n)} - E[T_{k(n)}] \right| > \varepsilon \cdot k(n) \ \text{i.o} \right\} \right) = 0$$

$$= P\left(\left\{ \left| \frac{T_{k(n)}}{k(n)} - \frac{E[T_{k(n)}]}{k(n)} \right| > \varepsilon \ \text{i.o} \right\} \right) = 0$$

$$\therefore \frac{T_{k(n)}}{k(n)} - \frac{E[T_{k(n)}]}{k(n)} \to 0 \ (a.s) \quad \left(\because \boxed{37} \right)$$

$\left(\{a_n\}_{n\ge1} \subseteq \mathbb{R} \quad a_n \to \mu \Rightarrow \frac{a_1+\cdots+a_n}{n} \to \mu \right)$

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$\widetilde{\mu_i} = E[Y_i]$

$$\frac{\widetilde{\mu_1}+\widetilde{\mu_2}+\cdots\widetilde{\mu_{k(n)}}}{k(n)} = \frac{E[T_{k(n)}]}{k(n)}$$

(L.D.C.T)

By L.D.C.T. $\widetilde{\mu_{k(n)}} \to \mu = E[X_1]$ $\left(\because \lim_{n\to\infty} E[Y_n] \overset{\text{(iid)}}{=} \lim_{n\to\infty} E[X_1 \cdot \mathbb{I}\{|X_1|\le n\}] \right.$ and $|X_1 \cdot \mathbb{I}\{|X_1|\le n\}| \le |X_1|$ (integrable) $\left. X_1 \cdot \mathbb{I}\{|X_1|\le n\} \to X_1 \right)$

So as $n\to\infty$ $\frac{E[T_{k(n)}]}{k(n)} \to \mu$

$\therefore \frac{T_{k(n)}}{k(n)} \to \mu$ (a.s).

Finally, if $k(n) \le m < k(n+1)$

$$\frac{T_{k(n)}}{k(n+1)} \le \frac{T_m}{m} \le \frac{T_{k(n+1)}}{k(n)} \quad (\mu>0. \text{ if } \mu\le 0 \dots \text{similar})$$

① ②

$$① = \frac{T_{k(n)}}{k(n)} \cdot \frac{k(n)}{k(n+1)} \xrightarrow{as} \frac{\mu}{\alpha} \quad (\text{when } n\to\infty)$$

$$② = \frac{T_{k(n+1)}}{k(n+1)} \cdot \frac{k(n+1)}{k(n)} \xrightarrow{a.s} \alpha\mu \quad (\text{when } n\to\infty)$$

$$\therefore \frac{\mu}{\alpha} \le \liminf \frac{T_m}{m} \le \limsup \frac{T_m}{m} \le \alpha\mu \quad (a.s)$$

By $\alpha \downarrow 1$ we have $\lim \frac{T_m}{m} \to \mu$ (a.s)

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$\boxed{64}$ $X_i^{(M)} \overset{\text{def}}{=} X_i \cdot \mathbb{I}\{X_i \leq M\}$ (M : fixed integer)

Then $\dfrac{X_1^{(M)} + \cdots + X_n^{(M)}}{n} \leq \dfrac{X_1 + \cdots + X_n}{n}$

$E[X_1^{(M)}] \in (-\infty, \infty)$

By $\boxed{63}$ (SLLN), $\dfrac{X_1^{(M)} + \cdots + X_n^{(M)}}{n} \to E[X_1^{(M)}]$ (a.s)

$\therefore \liminf \dfrac{X_1 + \cdots + X_n}{n} \geq E[X_1^{(M)}]$ (a.s)

$M \nearrow \infty$ $(M = 1, 2, 3, \ldots)$ ⊛

$\liminf \dfrac{X_1 + \cdots + X_n}{n} \geq E[X_1] = \infty$ (a.s)

$\therefore \dfrac{S_n}{n} \to \infty$ (a.s)

⊛
$\Omega^{(M)} \in \mathcal{F}$ $P(\Omega^{(M)}) = 1$ $\Rightarrow$

$\bigcap_{M=1}^{\infty} \Omega^{(M)} \in \mathcal{F}$, $P(\bigcap_{M=1}^{\infty} \Omega^{(M)}) = 1$

$\forall \omega \in \bigcap_{M=1}^{\infty} \Omega^{(M)}$ $\liminf \dfrac{X_1 + \cdots + X_n}{n} \geq E[X_1^{(M)}]$ for all $M = 1, 2, 3, \ldots$

(RHS $\nearrow \infty$ as $M \nearrow \infty$)

$\boxed{65}$ $X_{N(t)} \leq t < X_{N(t)+1}$ (by definition of N_t)

$$\frac{X_{N(t)}}{N(t)} \leq \frac{t}{N(t)} < \frac{X_{N(t)+1}}{N(t)} = \frac{X_{N(t)+1}}{N(t)+1} \cdot \frac{N(t)+1}{N(t)}$$

① ②

$t \to \infty \Rightarrow N(t) \to \infty$ ($\because 0 < X_i < \infty \Rightarrow 0 < T_n < \infty$)

$\therefore$ ① $\to \mu$ (as) (as $t \to \infty$) ($\because$ $\boxed{63}$, $\boxed{64}$)

② $\to \mu$ (as) (as $t \to \infty$)

Therefore $\frac{t}{N(t)} \to \mu$ (as)

$\Leftrightarrow \frac{N(t)}{t} \to \frac{1}{\mu}$ (as)

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$\boxed{66}$ $\mathcal{T} \stackrel{def}{=} \bigcap_{n=1}^{\infty} \sigma[X_n, X_{n+1} \cdots]$ $\sigma[X_n] \stackrel{def}{=} \{X_n^{-1}(B) \mid B \in \mathcal{B}(\mathbb{R})\}$

(1) $\{\omega \mid \lim S_n \text{ exists}\}$

$= \{\omega \mid \lim_{\substack{n \to \infty \\ m \to \infty}} |S_n - S_m| = 0\}$ ↓ Cauchy sequence

$= \{\omega \mid \lim_{N \to \infty} \sup_{n,m \geq N} |S_n - S_m| = 0\}$

• $W_N = \sup_{n,m \geq N} |S_n - S_m|$ is $\sigma[X_N, X_{N+1}, \cdots]$ -measurable

$= \{\omega \mid \lim_{N \to \infty} W_N = 0\}$

$= \{\omega \mid \lim_{N \to \infty} W_N > 0\}^c$ and $W_N \geq W_{N+1}$

$\{\lim_{N \to \infty} W_N \geq \frac{1}{k}\} = \bigcap_{N=1}^{\infty} \underbrace{\{W_N \geq \frac{1}{k}\}}_{\in \sigma[X_N, X_{N+1} \cdots]} \in \bigcap_{n=1} \sigma[X_n, X_{n+1} \cdots]$

hence $\{\lim W_N \geq \frac{1}{k}\} \in \mathcal{T}$

So $\bigcup_{k=1}^{\infty} \{\lim W_N \geq \frac{1}{k}\} = \{\lim_{N \to \infty} W_N > 0\} \in \mathcal{T}$

$\boxed{\text{NOTE}}$ $f_n \uparrow f$ $\{f > a\} = \bigcup_{n=1}^{\infty} \{f_n > a\}$

$g_n \downarrow g$ $\{g \geq a\} = \bigcap_{n=1} \{g_n \geq a\}$

(2) $\{\limsup S_n > 0\}$

We cannot ignore the influence of X_1.

So cannot be in $\bigcap_{n=1}^{\infty} \sigma[X_n, X_{n+1}, \cdots]$ generally.

(3) Since $C_n \nearrow \infty$, so eventually finite terms $X_1 + \cdots + X_{k-1}$

$\forall k \in \mathbb{N}$ will be cancelled.

$$\limsup \frac{S_n}{C_n} = \limsup_{n\to\infty} \frac{X_k + \cdots X_n}{C_n} \quad (k<n)$$

$$\text{So } \left\{\limsup \frac{S_n}{C_n}\right\} = \left\{\limsup_{n\to\infty} \frac{X_k + \cdots + X_n}{C_n}\right\}$$

$$\in \sigma[X_k, X_{k+1}, \cdots] \quad (\text{for all } k \geq 1)$$

$$\text{Hence} \quad \in \bigcap_{k=1}^{\infty} \sigma[X_k, X_{k+1} \cdots]$$

$$\in \mathcal{T}$$

$\boxed{67}$ We show that if $A \in \mathcal{I}$ then A is independent with itself. (then $P(A \cap A) = P(A) P(A)$
$= P(A) \Rightarrow P(A) = 0$ or 1)

$\boxed{\text{Step 1}}$ $\sigma[X_1, X_2 \dots X_k]$ and $\sigma[X_{k+1}, \dots X_{k+\ell}]$ are independent

Recall the statement $\boxed{8}$ (ch2) (theorem 2.1.9)

$$\sigma[X_1, \cdots X_k] = \sigma\left[\bigcup_{j=1}^{k} \sigma(X_j)\right] \quad \cdots (1)$$

$$\sigma[X_{k+1}, \dots, X_{k+\ell}] = \sigma\left[\bigcup_{j=k+1}^{k+\ell} \sigma(X_j)\right] \quad (2)$$

Since $\sigma[X_1] \cdots \sigma[X_{k+\ell}]$ are mutually independent

and are π-systems. by $\boxed{8}$, we conclude that

(1) and (2) are independent.

$\boxed{\text{Step 2}}$ $\sigma[X_1 \cdots X_k]$ and $\sigma[X_{k+1}, \dots\dots]$ are independent

It's easy to verify that

$$\sigma[X_{k+1} \dots] \overset{\text{def}}{=} \sigma\left[\bigcup_{j \geq k+1} \sigma(X_j)\right] \overset{\text{equal}}{=} \sigma\left[\bigcup_{\ell \geq 1} \sigma[X_{k+1}, \dots X_{k+\ell}]\right]$$

This is definition. It's easy to verify

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$\sigma[X_1, \dots X_k]$ and $\bigcup_{l \geq 1}^{\infty} \sigma[X_{k+1}, \dots X_{k+l}]$ are independent

($\because$ pick $A \in \sigma[X_1, \dots X_k]$ and $B \in \bigcup_{l \geq 1}^{\infty} \sigma[X_{k+1}, \dots, X_{k+l}]$

then there exists l_0 s.t. $B \in \sigma[X_{k+1}, \dots, X_{k+l_0}]$

by $\boxed{\text{step 1}}$ A and B are independent)

And $\sigma[X_1 \dots X_k]$ and $\bigcup_{l \geq 1}^{\infty} \sigma[X_{k+1} \dots X_{k+l}]$ are π-systems

by $\boxed{6}$ (ch2) (theorem 2.1.7)

We have $\sigma[\sigma[X_1 \dots X_k]]$ and $\sigma[\bigcup_{l \geq 1} \sigma[X_{k+1}, \dots, X_{k+l}]]$

$\sigma[\sigma[X_1 \dots X_k]] = \sigma(X_1 \dots X_k)$

are independent. $\therefore$ $\boxed{\text{Step 2}}$ is finished.

$\boxed{\text{Step 3}}$ $\forall k \in \mathbb{N}$ $\sigma[X_1, \dots X_k]$ and $\mathcal{T}$ is independent

If $A \in \sigma[X_1 \dots X_k]$ and $B \in \mathcal{T}$.

Then $B \in \sigma[X_{k+1}, X_{k+2} \dots]$ $\left(\because \mathcal{T} \overset{\text{def}}{=} \bigcap_{n=1}^{\infty} \sigma[X_n, X_{n+1} \dots]\right)$

So A and B are independent.

$\therefore$ $\sigma[X_1 \dots X_k]$, $\mathcal{T}$ are independent.

$\boxed{\text{Step 4}}$ $\sigma[X_1, X_2 \dots]$ and $\mathcal{T}$ are independent

It's easy to verify that $\sigma[X_1, X_2 \dots] \overset{\text{equal}}{=} \sigma\left[\bigcup_{k=1}^{\infty} \sigma[X_1 \dots X_k]\right]$

And $\bigcup_{k=1}^{\infty} \sigma[X_1 \dots X_k]$ and $\mathcal{T}$ are independent

(Similar to $\boxed{\text{Step 2}}$)

(Use $\boxed{6}$ (ch2))

And $\bigcup_{k=1}^{\infty} \sigma[X_1 \dots X_k]$: π-system $\mathcal{T}$: π-system

$\therefore$ $\sigma\left[\bigcup_{k=1}^{\infty} \sigma[X_1 \dots X_k]\right]$ and $\sigma[\mathcal{T}] = \mathcal{T}$ are independent

So $\sigma[X_1 X_2 \dots]$ and $\mathcal{T}$ are independent.

$\boxed{\text{Step5}}$ Let $A \in \mathcal{T}$

$$A \in \underbrace{\bigcap_{n=1}^{\infty} \sigma[X_n, X_{n+1} \cdots]}_{= \mathcal{T}} \Rightarrow A \in \sigma[X_1, X_2 \cdots]$$

So by $\boxed{\text{Step4}}$ A is independent with itself.

Now the proof is complete ∎

$\boxed{68}$

$$\{\max_{1\le k\le n}|S_k|\ge\lambda\}$$

$$\|$$

$$\sum_{k=1}^{n}\{|S_k|\ge\lambda \text{ and } |S_1|\sim|S_{k-1}|<\lambda\} \quad \text{(disjoint union)}$$

Let $A_k=\{|S_k|\ge\lambda,\ |S_1|\sim|S_{k-1}|<\lambda\}$

$$V[S_n]=E[S_n^2]=\int_\Omega S_n^2\,dP \qquad \left(\Omega\supseteq\sum_{k=1}^{n}A_k\right)$$

$$\ge\int_{\sum_{k=1}^{n}A_k} S_n^2\,dP=\sum_{k=1}^{n}\int_{A_k}S_n^2\,dP$$

$$=\sum_{k=1}^{n}\int_{A_k}(S_n-S_k+S_k)^2\,dP$$

$$=\sum_{k=1}^{n}\int_{A_k}\left\{\underbrace{(S_n-S_k)^2}_{\ge 0}+2(S_n-S_k)S_k+S_k^2\right\}dP$$

$$\ge\sum_{k=1}^{n}\int_{A_k}2(S_n-S_k)S_k\,dP+\sum_{k=1}^{n}\int_{A_k}S_k^2\,dP$$

$$=\sum_{k=1}^{n}\int_\Omega \underbrace{2(S_n-S_k)}_{\sigma[X_{k+1}\cdots X_n]\text{-measurable}}\cdot\underbrace{S_k\mathbb{I}_{A_k}}_{\sigma[X_1\cdots X_k]\text{-measurable}}\,dP+\underbrace{\sum_{k=1}^{n}\int_{A_k}\lambda^2\,dP}$$

$(\because\ \omega\in A_k \to \lambda^2\le S_k^2)$

$$\|$$

$$=E[2(S_n-S_k)]\cdot E[S_k\mathbb{I}_{A_k}]$$

$$E[2(S_n-S_k)]=0$$

$= \sum_{k=1}^{n} \lambda^2 P(A_k) = \lambda^2 P(\sum_{k=1}^{n} A_k) = \lambda^2 P(\max_{1\le k\le n} |S_k| \ge \lambda)$ ∎

$\boxed{\text{NOTE}}$ $S_k = X_1 + \cdots + X_k$ is $\sigma[X_1 \cdots X_k]$-measurable

$f(x_1 \cdots x_k) = x_1 + x_2 + \cdots x_k$: $\mathbb{R}^k \to \mathbb{R}$ is a Borel measurable function. In the proof of $\boxed{9}$, we proved that

$f(X_1 \cdots X_k)$ is $\mathcal{G}$-measurable. When $\mathcal{G} = \sigma[\bigcup_{j=1}^{k} \sigma[X_j]]$

So $S_k = f(X_1 \cdots X_k)$ is $\sigma[X_1 \cdots X_k]$-measurable

$\boxed{69}$ $S_n \overset{\text{def}}{=} \sum_{m=1}^{n} X_m$

$W_N \overset{\text{def}}{=} \sup_{n,m \geq N} |S_n - S_m|$

First we prove that $W_N \xrightarrow{P} 0$.

$$P(W_N > 2\varepsilon) = P\left(\sup_{n,m \geq N} |S_n - S_m| > 2\varepsilon\right)$$

$$= P\left(\sup_{n,m \geq N} |S_n - S_N + S_N - S_m| > 2\varepsilon\right)$$

$$\leq P\left(\sup_{n \geq N} |S_n - S_N| + \sup_{m \geq N} |S_m - S_N| > 2\varepsilon\right)$$ (They are the same)

$$= P\left(\sup_{n \geq N} |S_n - S_N| > \varepsilon\right)$$

$$= P\left(\lim_{M \to \infty} \underbrace{\max_{N \leq n \leq M} |S_n - S_N|}_{= g_M^{(N)}} > \varepsilon\right)$$

(Since $g_M^{(N)} \nearrow \sup_{n \geq N} |S_n - S_N|$, $P\left(\sup_{n \geq N} |S_n - S_N| > \varepsilon\right) = P\left(\bigcup_{M=N+1}^{\infty} \{g_M > \varepsilon\}\right)$

$$= P\left(\bigcup_{M=N+1}^{\infty} \underbrace{\left\{\max_{N \leq n \leq M} |S_n - S_N| > \varepsilon\right\}}_{= A_M^{(N)}}\right)$$

$A_M^{(N)} \nearrow$ so $P\left(\bigcup_{M=N+1}^{\infty} A_M\right) = \lim_{M \to \infty} P(A_M)$

$$= \lim_{M \to \infty} P\left(\max_{N \leq n \leq M} |S_n - S_N| > \varepsilon\right)$$

(NOTE $f_n \nearrow f$ $\{f > a\} = \{\sup_{n \geq 1} f_n > a\} = \bigcup_{n=1}^{\infty} \{f_n > a\}$

$A_n \nearrow A$ $\lim_{n} P(A_n) = P(A)$

$\leq \lim_{M\to\infty} P\left(\max_{N\leq n\leq M} |S_n - S_N| \geq \varepsilon\right)$ $\downarrow$ by $\boxed{6\delta}$

$\leq \lim_{M\to\infty} V[S_M - S_N] / \varepsilon^2$

$= \lim_{M\to\infty} \frac{1}{\varepsilon^2} \sum_{m=N+1}^{M} V[X_m] = \frac{1}{\varepsilon^2} \sum_{m\geq N+1}^{\infty} V[X_m] < \infty$

Hence $\lim_{N\to\infty} P(W_N > 2\varepsilon) \leq \lim_{N\to\infty} \frac{1}{\varepsilon^2} \sum_{m\geq N+1}^{\infty} V[X_m] = 0$

$\therefore W_N \xrightarrow{P} 0.$ This actually implies $W_N \xrightarrow{a.s} 0$

We may take a sub-sequence $\{W_{N_k}\}_{k\geq 1}$ $W_{N_k} \to 0$ (a.s)

But W_N is a decreasing sequence. thus $W_N \to 0$ (a.s.

$(W_N \geq W_{N+1})$

$\therefore \{S_n\}_{n\geq 1}$ is a cauchy sequence (a.s)

$\therefore S_n$ converges (a.s)

$\boxed{70}$ Kolmogorov's three series theorem. ($Y_n = X_n \cdot \mathbb{I}\{|X_n| \le A\}$)

① $\sum_{n=1}^{\infty} X_n(w)$: converges (a.s) $\Rightarrow$

$\forall A>0$, $\sum_{n=1}^{\infty} P(|X_n|>A) < \infty$ ... (i)

$\sum_{n=1}^{\infty} E[Y_n]$: converges ... (ii)

$\sum_{n=1}^{\infty} V[Y_n] < \infty$ ... (iii)

② $\exists A>0$, s.t $\sum_{n=1}^{\infty} P(|X_n|>A) < \infty$ ... (i)

$\sum_{n=1}^{\infty} E[Y_n]$: converges ... (ii)

$\sum_{n=1}^{\infty} V[Y_n] < \infty$ ... (iii)

$\Rightarrow \sum_{n=1}^{\infty} X_n(w)$: converges (a.s)

(proof)

First we prove ②...

By (i), $\sum_{n=1}^{\infty} X_n(w)$: converges $\overset{(a.s)}{\Leftrightarrow} \sum_{n=1}^{\infty} Y_n(w)$ converges $^{(a.s)}$

($\because \sum_{n=1}^{\infty} P(X_n \neq Y_n) = \sum_{n=1}^{\infty} P(|X_n|>A) < \infty$

$\Rightarrow P(\{X_n \neq Y_n \text{ i.o}\}) = 0 \Rightarrow \sum_{n=1}^{\infty} |X_n - Y_n| < \infty$ (a.s)

$\Rightarrow \sum_{n=1}^{\infty} X_n$ converges (a.s) $\Leftrightarrow \sum_{n=1}^{\infty} Y_n(w)$ converges (a.s)

By $\boxed{69}$ $\sum_{n=1}^{\infty}(Y_n - E[Y_n])$: converges (a.s)

($\because$ $Z_n = Y_n - E[Y_n]$ $\quad E[Z_n] = 0$ $\quad \sum_{n=1}^{\infty} V[Z_n] = \sum_{n=1}^{\infty} V[Y_n] < \infty$)

By (ii) $\Rightarrow$ $\sum_{n=1}^{\infty} Y_n(\omega)$: converges (a.s)

$\Rightarrow$ $\sum_{n=1}^{\infty} X_n(\omega)$ converges (a.s) ∎

Next we prove ①. (We will have an alternative proof in ch3 using Lindeberg-Feller's theorem.)

First (i) must hold.

($\because$ Suppose that $\exists A > 0$ s.t. $\sum_{n=1}^{\infty} P(|X_n| > A) = \infty$

By Borel-Cantelli's lemma (II). $P(\{|X_n| > A \text{ i.o}\}) = 1$

$\Rightarrow$ $\#\{n \mid |X_n| > A\} = \infty$ (a.s)

$\Rightarrow$ $\sum_{n=1}^{\infty} X_n(\omega)$: does not converge (a.s) $\Rightarrow$ contradiction

So (i) must hold.)

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Since (i') holds $\sum_{n=1}^{\infty} X_n(w)$ converses (as)

$\Rightarrow \sum_{n=1}^{\infty} Y_n(w)$ converses (as). (see ①)

($\Leftarrow$)

Now we consider $\{\tilde{Y}_n(w)\}_{n\geq1}$ which has the same distribution with $\{Y_n(w)\}_{n\geq1}$ but independent with $\{Y_n(w)\}_{n\geq1}$.

$\sum_{n=1}^{\infty} \tilde{Y}_n(w)$ converses (as) ($\because$ The distribution is same as $\sum_{n=1}^{\infty} Y_n(w)$ and by $\boxed{37}$)

$Z_n \stackrel{def}{=} Y_n(w) - \tilde{Y}_n(w)$.

Then $\sum_{n=1}^{\infty} Z_n(w)$ converses (as) ($\because \sum_{n=1}^{\infty} Y_n(w)$, $\sum_{n=1}^{\infty} \tilde{Y}_n(w)$ converses)

And $E[Z_n] = 0$ ($\because E[Y_n] = E[\tilde{Y}_n]$)

$V[Z_n] = V[Y_n] + V[\tilde{Y}_n] = 2V[Y_n]$

By Kolmogorov's two series theorem

if we prove that $\sum_{n=1}^{\infty} V[Z_n] < \infty$

Then $\sum_{n=1}^{\infty} V[Y_n] < \infty$ hence $\sum_{n=1}^{\infty} (Y_n - E[Y_n(w)])$ converses (a.s) ($\because \boxed{69}$)

And we have $\sum_{n=1}^{\infty} E[Y_n]$ converses. $\Rightarrow$ (i'')

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($\because \sum_{n=1}^{\infty} Y_n(w)$ converses (as) and $\sum_{n=1}^{\infty} (Y_n - E[Y_n])$ converses (as))

So the proof will be complete.

Now what we have to do is to prove that :

$$\left(\begin{array}{l} \cdot \ \sum_{n=1}^{\infty} Z_n(\omega) : \text{converges (a.s)} \\ \cdot \ E[Z_n]=0 \\ \cdot \ Z_n : \text{bounded} \end{array}\right. \qquad \Big(\because |Z_n| \le \underbrace{|Y_n| + |\tilde{Y}_n|}_{\Downarrow} \le \underset{\underset{M}{\shortparallel}}{2A} \Big)$$

Then $\sum_{n=1}^{\infty} V(Z_n) < \infty$ $\qquad$ So $|Z_n|$ is bounded

Let $S_n \overset{\text{def}}{=} Z_1 + Z_2 + \dots + Z_n$

$\tau(L) \overset{\text{def}}{=} \inf\{n \mid |S_n| > L\}$ $\quad (\tau_L = \tau(L))$

Now consider $S_{n \wedge \tau_L}$.

$|S_{n\wedge\tau_L}| \le M + L$ $\qquad (\cdot \wedge \cdot = \min\{\cdot, \cdot\})$

$\therefore |S_{n\wedge\tau_L}|^2 \le (M+L)^2$

$\therefore LHS = \int_\Omega (S_{n\wedge\tau_L})^2 dP$

$= \int_\Omega \left(\sum_{k=1}^{n} Z_k \, \mathbb{I}\{k \le \tau_L\}\right)\left(\sum_{l=1}^{n} Z_l \cdot \mathbb{I}\{l \le \tau_L\}\right) dP$

$=$

$$= \int_\Omega \left\{ \sum_{k=1}^{n} Z_k^2 \cdot \mathbb{I}\{k \le \tau_L\} + 2 \sum_{1 \le k < l \le n} Z_k Z_l \cdot \underbrace{\mathbb{I}\{k \le \tau_L\}\mathbb{I}\{l \le \tau_L\}}_{= \mathbb{I}\{l \le \tau_L\}} \right\} dp$$

$$= \sum_{k=1}^{n} \int_\Omega Z_k^2 \cdot \mathbb{I}\{k \le \tau_L\} \, dp + 2 \sum_{1 \le k < l \le n} Z_k \cdot \mathbb{I}\{l \le \tau_L\} \cdot Z_l$$

$$\{k \le \tau_L\} = \{k > \tau_L\}^c \in \sigma[X_1, X_2 \cdots X_{k-1}]$$

- So Z_k^2 and $\mathbb{I}\{k \le \tau_L\}$ are independent
- And $Z_k \mathbb{I}\{l \le \tau_L\}$ and Z_l are independent

$\sigma[X_1 \cdots X_{l-1}]$-measurable $\qquad \sigma[X_l]$-measurable

Hence $= \sum_{k=1}^{n} E[Z_k^2] \, P(k \le \tau_L) \qquad (E[Z_l] = 0)$

Therefore $(M+L)^2 \ge E|S_{n \wedge \tau_L}|^2 = \sum_{k=1}^{n} E[Z_k^2] \, P(k \le \tau_L)$

$\ge \sum_{k=1}^{n} E[Z_k^2] \cdot P(\tau_L = \infty) \quad \cdots (*)$

(Now we want to show that there exists L such that $P(\tau_L = \infty) > 0$)

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Since $\sum_{n=1}^{\infty} Z_n(\omega)$ converses (a.s)

We may take suffiently large $L>0$ such that

$P(|\sum_{n=1}^{\infty} Z_n(\omega)| \leq L) > 0$.

If $\omega \in \{ |\sum_{n=1}^{\infty} Z_n(\omega)| \leq L \} \Rightarrow \omega \in \{ T_L = \infty \}$

(See the definition of T_L)

$\therefore \exists L>0$ s.t $P(T_L=\infty) > 0$.

So $\sum_{k=1}^{n} E[Z_k^2] \leq \dfrac{(M+L)^2}{P(T_L=\infty)} < \infty$ $(\because ⊛)$

$\left(\underset{\sum_{k=1}^{n} V[Z_n]}{\parallel} \right)$

By taking $n \nearrow \infty$, we have $\sum_{n=1}^{\infty} V[Z_n] < \infty$

Now the proof is complete.

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[7] Kronecker's lemma

$a_n \nearrow \infty \quad \sum_{n=1}^{\infty} \frac{x_n}{a_n}$: converges $\Rightarrow \frac{1}{a_n}\sum_{m=1}^{n} x_m \to 0$ (as $n \to \infty$)

(proof) $b_n \stackrel{def}{=} \sum_{m=1}^{n} \frac{x_m}{a_m}$ $\quad (b_0 \stackrel{def}{=} 0 \quad a_0 \stackrel{def}{=} 0)$

$\Rightarrow b_n - b_{n-1} = \frac{x_n}{a_n}$

$\Rightarrow a_n(b_n - b_{n-1}) = x_n$

So $\frac{1}{a_n}\sum_{m=1}^{n} x_m = \frac{1}{a_n}\sum_{m=1}^{n} a_m(b_m - b_{m-1})$

$= \frac{1}{a_n}\left(\sum_{m=1}^{n} a_m b_m - \sum_{m=1}^{n} a_m b_{m-1}\right)$

$= \frac{1}{a_n}\left(a_n b_n + \sum_{m=1}^{n-1} a_m b_m - \sum_{m=1}^{n} a_m b_{m-1}\right)$

$= b_n + \frac{1}{a_n}\left(\sum_{m=1}^{n} a_{m-1} b_{m-1} - \sum_{m=1}^{n} a_m b_{m-1}\right) \quad (a_0 = b_0 = 0)$

$= b_n - \frac{1}{a_n}\sum_{m=1}^{n}(a_m - a_{m-1}) b_{m-1}$

And $b_n \nearrow b$ converges

We show that $\frac{1}{a_n}\sum_{m=1}^{n}(a_m - a_{m-1}) b_{m-1} - b \to 0$

$$\frac{1}{a_n}\sum_{m=1}^{n}(a_m-a_{m-1})b_{m-1}-b=\frac{1}{a_n}\sum_{m=1}^{n}(a_m-a_{m-1})(b_{m-1}-b)$$

$$\left|\frac{1}{a_n}\sum_{m=1}^{n}(a_m-a_{m-1})(b_{m-1}-b)\right|$$ ↓ triangle inequality $(a_n \leqq a_{n+1})$

$$\leqq \frac{1}{a_n}\sum_{m=1}^{M}(a_m-a_{m-1})\underbrace{|b_{m-1}-b|}_{\leqq 2B}+\underbrace{\frac{1}{a_n}\sum_{m=M+1}^{n}(a_m-a_{m-1})}_{\left(\frac{a_n-a_M}{a_n}<1\right)}\underbrace{|b_{m-1}-b|}_{<\frac{\varepsilon}{2}}$$
⊛

$$\leqq 2B\cdot\frac{a_M}{a_n}+\frac{\varepsilon}{2}$$

⊛ $\forall\varepsilon>0\ \exists M_\varepsilon$ st $\forall m>M_\varepsilon$ $|b_{m-1}-b|<\frac{\varepsilon}{2}$

$B=\sup_{n\geq 1}|b_n|$ so $|b_{m-1}-b|\leqq 2B$

Since $a_n\uparrow\infty$ we may take sufficiently large n
st. $\frac{a_M}{a_n}<\frac{\varepsilon}{4B}$.

Then $<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$

Now the proof is complete. ∎

$\boxed{172}$ We prove SLLN again.

$X_1, X_2 \cdots$ pairwise independent and identically distributed.

$(E|X_1| < \infty)$

$Y_m \overset{def}{=} X_m \mathbb{I}\{|X_m| \leq m\}$

$T_n = Y_1 + Y_2 + \cdots + Y_n$

We have already shown that $\frac{T_n}{n} \overset{as}{\to} \mu \Leftrightarrow \frac{S_n}{n} \overset{as}{\to} \mu$.

So we prove that $\frac{T_n}{n} \to \mu$ (as).

$a_n \overset{def}{=} n \qquad (a_n \uparrow \infty)$

$X_n \overset{def}{=} Y_n - E[Y_n]$

$\sum_{m=1}^{n} \frac{X_m}{a_m}$ converges $\Rightarrow \frac{1}{a_n} \sum_{m=1}^{n} X_m \to 0$ ⊛

(by Kronecker's theorem)

Consider $\sum_{m=1}^{n} \frac{X_m}{a_m} = \sum_{m=1}^{n} \frac{Y_m - E[Y_m]}{m}$

Let $Z_m = \frac{Y_m - E[Y_m]}{m} \Rightarrow E[Z_m] = 0 \quad \sum_{m=1}^{\infty} V[Z_m] = \sum_{m=1}^{\infty} \frac{V[Y_m]}{m^2} < \infty$

$(\because \boxed{62})$

By Kolmogorov's two series theorem

$\sum_{n=1}^{\infty} Z_n = \sum_{n=1}^{\infty} \frac{Y_n - E[Y_n]}{n}$ converges (a.s)

$\Rightarrow \frac{1}{n} \sum_{m=1}^{n} (Y_m - E[Y_m])$ converges (a.s)

($\because$ by ⊛)

And $\frac{1}{n} \sum_{m=1}^{n} E[Y_m] \to \mu$ (by L.D.C.T $\because$ see $\boxed{63}$)

So $\frac{T_n}{n} \to \mu$ (a.s)

Now the proof is complete

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$\boxed{13}$ We use Kronecker's lemma and Kolmogorov's two series theorem

Let $a_n = n^{\frac{1}{2}}(\log n)^{\frac{1}{2}+\varepsilon}$

If $\sum_{m=1}^{n} \frac{X_m}{a_m}$ converges (a.s) $\Rightarrow \frac{1}{a_n}\sum_{m=1}^{n} X_m \to 0$ (a.s)

$E\left[\frac{X_m}{a_m}\right] = 0$

$V\left[\frac{X_m}{a_m}\right] = \frac{\sigma^2}{a_m^2} = \frac{1}{n(\log n)^{1+2\varepsilon}} \cdot \sigma^2$

$\because \sum_{n=1}^{\infty} \frac{\sigma^2}{n(\log n)^{1+2\varepsilon}} < \infty$ ... ⊛

By Kronecker's two series theorem

$\sum_{n=1}^{\infty} \frac{X_n}{a_n}$ converges (a.s). Now the proof is complete ∎

⊛ $\because \int_2^{\infty} \frac{1}{x(\log x)^{1+2\varepsilon}} < \infty$

(hint)

$= \sum_{n=2}^{\infty} \int_n^{n+1} \frac{1}{x(\log x)^{1+2\varepsilon}}$

$\geq \sum_{n=2}^{\infty} \frac{1}{(n+1)(\log(n+1))^{1+2\varepsilon}}$ ...

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$\boxed{17}$. First we show that $\frac{X_n}{n^p} \to 0$ (a.s).

$$\frac{X_n}{n^p} = \frac{S_n - S_{n-1}}{n^p} = \underbrace{\frac{S_n}{n^p}}_{\substack{\to 0 \\ (a.s)}} - \underbrace{\frac{S_{n-1}}{(n-1)^p}}_{\substack{\to 0 \\ (a.s)}} \cdot \underbrace{\frac{(n-1)^p}{n^p}}_{\to 1} \to 0 \ (a.s)$$

So $\frac{X_n}{n^p} \to 0$ (a.s) -- ⊛

But if $E|X_1|^p = \infty$ then $E|X_1|^p = \int_0^\infty P(|X_1|^p > x)\,dx$

$$\leq \sum_{n=0}^{\infty} \int_n^{n+1} P(|X_1|^p > x)\,dx$$

$$= \sum_{n=0}^{\infty} P(|X_1|^p > n) = 1 + \sum_{n=1}^{\infty} P(|X_1|^p > n)$$

$$= 1 + \underbrace{\sum_{n=1}^{\infty} P\left(\frac{|X_1|}{n^p} > 1\right)}_{=\infty} = 1 + \sum_{n=1}^{\infty} P\left(\frac{|X_n|}{n^p} > 1\right)$$

(iid)

So, By Borel-Cantelli's lemma (II)

$$P\left(\left\{\frac{|X_n|}{n^p} > 1 \ \text{i.o}\right\}\right) = 1$$

So $\limsup \frac{|X_n|}{n^p} \geq 1$ (a.s)

This contradicts with the fact ⊛ ∎

So $E|X_1|^p < \infty$

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$\boxed{75}$ $X_1, X_2, \dots \overset{iid}{\sim} N(0,1)$

then $X_n \cdot \frac{1}{n} \sin(n\pi t) \sim N(0, \frac{1}{n^2} \sin^2(n\pi t))$

$Z_n \overset{def}{=} X_n \cdot \frac{1}{n} \sin(n\pi t)$

$E(Z_n) = 0$

$\sum_{n=1}^{\infty} V(Z_n) \le \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$

So by Kolmogorov's two series theorem,

$\sum_{n=1}^{\infty} Z_n(\omega)$ converges (a.s) ∎

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$\boxed{76}$ $X_n \geq 0$. We show ①⇔②⇔③

① ⇒ ②

By Kolmogorov's three series theorem

$\sum_{n=1}^{\infty} X_n(\omega) < \infty$ (as) $\Rightarrow$ $^{\forall}A(=1) > 0$

$\cdot\ \sum_{n=1}^{\infty} P(|X_n| > A) < \infty$

$\cdot\ \sum_{n=1}^{\infty} E[Y_n]$: converges

$\left(\cdot\ \sum_{n=1}^{\infty} V[Y_n] < \infty\right)$

$\sum_{n=1}^{\infty} P(|X_n| > 1) < \infty$ and $\sum_{n=1}^{\infty} E[X_n \cdot I\{X_n \leq 1\}] < \infty$

So the proof is complete.

② ⇒ ①

By Kolmogorov's three series theorem

$^{\exists}A > 0\ (A=1)$ s.t $\begin{cases} \cdot\ \sum_{n=1}^{\infty} P(|X_n| > A) < \infty \\ \cdot\ \sum_{n=1}^{\infty} E[X_n \cdot I\{|X_n| \leq A\}] : \text{converges} \end{cases}$

Moreover, $\sum_{n=1}^{\infty} V[X_n \cdot I\{|X_n| \leq A\}] \leq \sum_{n=1}^{\infty} E[X_n^2 \cdot I\{|X_n| \leq A\}]$

$\leq \sum_{n=1}^{\infty} E[A \cdot X_n\, I\{|X_n| \leq A\}]$

$(A=1) \quad = \sum_{n=1}^{\infty} E[X_n\, I\{|X_n| \leq A\}] < \infty$

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So $\{X_n\}_{n\geq1}$ satisfies three conditions for three series theorem.

Hence ① holds.

② ⟺ ③ :

$g(x) = \min\{1, x\} \quad (x \geq 0)$

$h(x) = \frac{x}{1+x} \quad (x \geq 0)$

② ... $\sum_{n=1}^{\infty} E\, g(X_n)$

③ ... $\sum_{n=1}^{\infty} E\, h(X_n)$

Since $0 \leq h(x) \leq g(x) \leq 2h(x) \quad (x \geq 0)$

So $\sum_{n=1}^{\infty} E\, g(X_n) < \infty \iff \sum_{n=1}^{\infty} E\, h(X_n) < \infty$

$\therefore$ ② ⟺ ③ ∎

$$\phi(x) = \begin{cases} |x| & (|x|>1) \\ x^2 & (|x|\le 1) \end{cases}$$

$\boxed{\text{VI}}$ We use Kolmogorov's three series theorem

$$\infty > \sum_{n=1}^{\infty} E[\phi(X_n)] = \sum_{n=1}^{\infty} E\left[X_n^2 \cdot \mathbb{I}\{|X_n| \le 1\} + X_n^2 \cdot \mathbb{I}\{|X_n|>1\}\right]$$

$$\ge \underbrace{\sum_{n=1}^{\infty} E\left[X_n^2 \cdot \mathbb{I}\{|X_n| \le 1\}\right]}_{<\infty \ (i)} + \underbrace{\sum_{n=1}^{\infty} E\left[X_n^2 \cdot \mathbb{I}\{|X_n|>1\}\right]}_{<\infty \ (ii)}$$

First $\displaystyle\sum_{n=1}^{\infty} P(|X_n|>1) = \sum_{n=1}^{\infty} E\left[\mathbb{I}\{|X_n|>1\}\right] \le \sum_{n=1}^{\infty} E\left[|X_n|^2 \cdot \mathbb{I}\{|X_n|>1\}\right] < \infty$ (by (ii)) ①

Next $\displaystyle\sum_{n=1}^{\infty} E\left[X_n \cdot \mathbb{I}\{|X_n| \le 1\}\right]$

$$= \sum_{n=1}^{\infty} E\left[-X_n \cdot \mathbb{I}\{|X_n|>1\}\right] \quad (\because E[X_n]=0)$$

So $\displaystyle\left|\sum_{n=1}^{\infty} E\left[X_n \cdot \mathbb{I}\{|X_n| \le 1\}\right]\right| \le \sum_{n=1}^{\infty} E\left[|X_n| \cdot \mathbb{I}\{|X_n|>1\}\right]$

$$\le \sum_{n=1}^{\infty} E\left[|X_n|^2 \cdot \mathbb{I}\{|X_n|>1\}\right] < \infty \quad \text{(by (ii))}$$

$\cdots$ ②

Hence $\displaystyle\sum_{n=1}^{\infty} E\left[X_n \cdot \mathbb{I}\{|X_n| \le 1\}\right]$ converges

Finally $\displaystyle\sum_{n=1}^{\infty} V\left[X_n \cdot \mathbb{I}\{|X_n| \le 1\}\right] \le \sum_{n=1}^{\infty} E\left[X_n^2 \cdot \mathbb{I}\{|X_n| \le 1\}\right] < \infty$ (by (i))

$\cdots$ ③

①~③ $\Rightarrow$ Kolmogorov's three series theorem

$\Rightarrow$ The proof is complete.

$\boxed{78}$ We use Kolmogorov's three series theorem.

$(A=1)$

① $\sum_{n=1}^{\infty} P(|X_n| > 1)$

$= \sum_{n=1}^{\infty} E[\mathbb{I}\{|X_n|>1\}] \le \sum_{n=1}^{\infty} E[\underbrace{|X_n|^{p(n)}}_{>1} \cdot \mathbb{I}\{|X_n|>1\}]$

$\le \sum_{n=1}^{\infty} E[|X_n|^{p(n)}] < \infty$

② $\sum_{n=1}^{\infty} E[X_n \cdot \mathbb{I}\{|X_n| \le 1\}]$

$= \sum_{n=1}^{\infty} E[X_n \cdot \mathbb{I}\{|X_n| \le 1\}] \cdot \mathbb{I}\{p(n)>1\} +$

$E[X_n \cdot \mathbb{I}\{|X_n| \le 1\}] \cdot \mathbb{I}\{p(n) \le 1\}$

$= \sum_{n=1}^{\infty} E[-X_n \cdot \mathbb{I}\{|X_n| > 1\}] \cdot \mathbb{I}\{p(n)>1\}$ $\quad \left(\begin{array}{c} E[X_n]=0 \\ \text{if } p(n)>1 \end{array} \right)$

$+ E[X_n \cdot \mathbb{I}\{|X_n| \le 1\}] \cdot \mathbb{I}\{p(n) \le 1\}$

$\therefore \left| \sum_{n=1}^{\infty} E[X_n \mathbb{I}\{|X_n| \le 1\}] \right|$

$\le \sum_{n=1}^{\infty} E[|X_n| \cdot \mathbb{I}\{|X_n| > 1\}] \cdot \mathbb{I}\{p(n)>1\}$

$+ E[|X_n| \cdot \mathbb{I}\{|X_n| \le 1\}] \cdot \mathbb{I}\{p(n) \le 1\}$

$$\leq \sum_{n=1}^{\infty} E\left[|X_n|^{p(n)} \cdot \mathbb{I}\{|X_n|>1\}\right] \cdot \mathbb{I}\{p(n)>1\}$$

$$+ E\left[|X_n|^{p(n)} \mathbb{I}\{|X_n|\leq 1\}\right] \cdot \mathbb{I}\{p(n)\leq 1\}$$

$$\leq \sum_{n=1}^{\infty} E\left[|X_n|^{p(n)}\right] < \infty$$

③ $\sum_{n=1}^{\infty} V[X_n \cdot \mathbb{I}\{|X_n|\leq 1\}) \leq \sum_{n=1}^{\infty} E(X_n^2 \cdot \mathbb{I}\{|X_n|\leq 1\})$

Since $p(n)\leq 2$, $\leq \sum_{n=1}^{\infty} E\left[|X_n|^{p(n)} \mathbb{I}\{|X_n|\leq 1\}\right]$

$$\leq \sum_{n=1}^{\infty} E(|X_n|^{p(n)}) < \infty$$

Now the proof is complete ■

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$\boxed{79}$ $R(\omega) = \dfrac{1}{\limsup_{n\to\infty} |X_n|^{\frac{1}{n}}} = \exp\left(-\limsup_{n\to\infty} \frac{1}{n}\log|X_n|\right)$

① $E[\log^+|X_1|] < \infty$ .. we show $R(\omega) = 1$ (a.s)

(We want to show $\limsup_{n\to\infty} \frac{1}{n}\log|X_n| = 0$ (a.s))

Let $\varepsilon > 0$. By assumption, $E\left[\frac{\log^+|X_1|}{\varepsilon}\right] < \infty$.

$= \int_0^\infty P\left(\frac{\log^+|X_1|}{\varepsilon} > \lambda\right) d\lambda$ ($\because$ by formula)

$= \sum_{n=0}^{\infty} \int_{(n,n+1]} P\left(\frac{\log^+|X_1|}{\varepsilon} > x\right) dx$

$\geq \sum_{n=0}^{\infty} \int_{(n,n+1]} P\left(\frac{\log^+|X_1|}{\varepsilon} > n+1\right) dx$

$= \sum_{n=1}^{\infty} P\left(\frac{\log^+|X_1|}{\varepsilon} > n\right) = \sum_{n=1}^{\infty} P\left(\frac{\log^+|X_n|}{\varepsilon} > n\right)$ $(<\infty)$

$\rightarrow$ (iid)

By Borel-Cantelli's lemma (I)

$P\left(\frac{\log^+|X_n|}{\varepsilon} > n.\ i.o\right) = 0$

$\Leftrightarrow P\left(\frac{\log^+|X_n|}{n} > \varepsilon\ i.o\right) = 0$

$\Rightarrow \#\left\{n \,\middle|\, \frac{\log^+|X_n|}{n} > \varepsilon\right\} < \infty$ (a.s)

$\Rightarrow \limsup \frac{\log^+|X_n|}{n} \leq \epsilon$ (a.s) $(\forall \varepsilon > 0)$

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$\Rightarrow \limsup \frac{\log^+|X_n|}{n} = 0$ (a.s)

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by lemma (see below) $\overline{\lim}\sup \frac{1}{n}\log^+|X_n| = \overline{\lim}\sup \frac{1}{n}\log|X_n|$ (as)

Hence $\overline{\lim}\sup \frac{1}{n}\log|X_n| = 0$ (as) ∎

② $E[\log^+|X_1|] = \infty$, Similarly we derive $\overline{\lim}\sup \frac{1}{n}\log|X_n| = \infty$
(Then $R(w) = 0$ (as))

$\infty = E[\log^+|X_1|] \Rightarrow E[\varepsilon\log^+|X_1|] = \infty$ $(\varepsilon > 0)$

$\therefore \infty = E[\varepsilon\log^+|X_1|] = \int_0^\infty P(\varepsilon\log^+|X_1| > \lambda)\,dx$

$= \sum_{n=0}^\infty \int_{(n,n+1]} P(\varepsilon\log^+|X_1| > x)\,dx$

$\leq \sum_{n=0}^\infty \int_{(n,n+1]} P(\varepsilon\log^+|X_1| > n)\,dx$

$= \sum_{n=0}^\infty P(\varepsilon\log^+|X_1| > n) = \sum_{n=0}^\infty P(\varepsilon\log^+|X_n| > n)$ (iid)

$= \sum_{n=0}^\infty P\left(\frac{\log^+|X_n|}{n} > \frac{1}{\varepsilon}\right)$

By Borel-Cantelli's lemma (II) $P\left(\frac{\log^+|X_n|}{n} > \frac{1}{\varepsilon}\ \text{i.o}\right) = 1$

$\therefore \overline{\lim}\sup \frac{\log^+|X_n|}{n} \geq \frac{1}{\varepsilon}$ (as) $(\forall \varepsilon > 0)$ $\left(\varepsilon = \frac{1}{k}\ (k = 1, 2, 3, \dots)\right.$
$\left. P\left(\bigcap_{k=1}^\infty \Omega^{(k)}\right) = 1\right)$

$\Rightarrow \overline{\lim}\sup \frac{\log^+|X_n|}{n} = \infty$ (a.s) (lemma)

$\Rightarrow \overline{\lim}\sup \frac{\log|X_n|}{n} = \infty$ (AS) ∎

$\boxed{\text{Lemma}}$ Finally we prove the lemma which we used in ①, ②

「 $X_1, X_2, \dots$ iid and $P(|X_1|=0)<1$ then ...
(not $\equiv 0$)

$\Downarrow$

- $\limsup \frac{\log|X_n|}{n} \geq 0$ (a.s)
- $\limsup \frac{\log^+|X_n|}{n} = \limsup \frac{\log|X_n|}{n}$ (a.s) 」

(proof)

Let $\varepsilon>0$. Since $P(|X_1|=0)<1$, we may find sufficiently large $n\in\mathbb{N}$ s.t. $P(|X_n|\geq e^{-\varepsilon n})>0$.

Hence $\sum_{n=1}^{\infty} P(|X_n|\geq e^{-\varepsilon n})=\infty$

By Borel-Cantelli's lemma (II),

We have $P(|X_n|\geq e^{-\varepsilon n} \text{ i.o.})=1$

$$\parallel$$

$$P\left(\frac{\log|X_n|}{n}\geq -\varepsilon \text{ i.o.}\right)=1 \quad ({}^\forall\varepsilon>0)$$

So $\limsup \frac{\log|X_n|}{n}\geq -\varepsilon$ (a.s) $\quad$ ($\varepsilon=\frac{1}{k}$. $P(\Omega^{(k)})=1$

$\Rightarrow \limsup \frac{\log|X_n|}{n}\geq 0$ (a.s) $\quad$ $\Rightarrow P(\bigcap_{k=1}^{\infty}\Omega^{(k)})=1$).

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Now we prove $\limsup \frac{\log^+|X_n|}{n} = \limsup \frac{\log|X_n|}{n}$.

$\log^+|X_n| = \max\{0, \log|X_n|\}$

$\frac{1}{n}\log^+|X_n| = \frac{1}{n}\max\{0, \log|X_n|\} = \max\{0, \frac{1}{n}\log|X_n|\}$

$\limsup \frac{1}{n}\log^+|X_n| = \limsup \max\{0, \frac{1}{n}\log|X_n|\}$

$= \max\{0, \limsup \frac{1}{n}\log|X_n|\}$

$= \limsup \frac{1}{n}\log|X_n|$ (as)

$(\because \limsup \frac{1}{n}\log|X_n| \geq 0$ (as))

Now the proof of lemma is complete ∎

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$\boxed{80}$ Let $X_1, X_2 \cdots$ be independent and let $S_{m,n} =$

$X_{m+1} + \cdots + X_n$. We show $P\left(\max_{m<j\le n} |S_{m,j}| > 2a\right)$

$\cdot \min_{m<k\le n} P\left(|S_{k,n}| \le a\right) \le P\left(|S_{m,n}| > a\right)$.

(proof) We define $A_{m,j} = \{\omega \in \Omega \mid |S_{m,j}| > 2a;$

$|S_{m,k}| \le 2a$ for all $k = m+1 \sim j-1\}$

Then $\{\max_{m<j\le n} |S_{m,j}| > 2a\} = \sum_{j=m+1}^{n} A_{m,j}$ ---(*)

Next if $\omega \in A_{m,j} \cap \{|S_{j,n}| \le a\}$

then $|S_{m,n}| = |S_n - S_m| = |\underbrace{S_n - S_j} + \underbrace{S_j - S_m}|$

$\ge |S_j - S_m| - |S_n - S_j|$

$= \underbrace{|S_{m,j}|}_{>2a} - \underbrace{|S_{j,n}|}_{\le a} > a$

Hence $A_{m,j} \cap \{|S_{j,n}| \le a\} \subseteq \{|S_{m,n}| \ge a\}$ $\underbrace{\forall j = m+1 \sim n}$

$\Rightarrow \sum_{j=m+1}^{n} A_{m,j} \cap \{|S_{j,n}| \le a\} \subseteq \{|S_{m,n}| \ge a\}$

(RHS is not related to "j".)

$$\sum_{j=m+1}^{n} P(A_{m,j} \cap \{|S_{j,n}| \leqq a\}) \leqq P(|S_{m,n}| \geqq a) \quad (\forall j$$

$A_{m,j} \in \sigma[X_{m+1}, \cdots X_j]$

$\{S_{j,n} \leqq a\} \in \sigma[X_{j+1} \cdots X_n]$

Here these two sets (events) are independent)

$$= \sum_{j=m+1}^{n} P(A_{m,j}) P(\{|S_{j,n}| \leqq a\}) \leqq P(|S_{m,n}| \geqq a)$$

$\uparrow$

$$\text{L.H.S} \geqq \sum_{j=m+1}^{n} P(A_{m,j}) \cdot \min_{k=m+1 \sim n} P(|S_{k,n}| \leqq a)$$

$$= \min_{k=m+1 \sim n} P(|S_{k,n}| \leqq a) \cdot \sum_{j=m+1}^{n} P(A_{m,j})$$

$\{A_{m,j}\}_{j=m+1}^{n}$ are disjoint

$$\sum_{j=m+1}^{n} P(A_{m,j}) = P\left(\sum_{j=m+1}^{n} A_{m,j}\right)$$

(*)

$$P\left(\sum_{j=m+1}^{n} A_{m,j}\right) = P\left(\max_{m<j\leqq n} |S_{m,j}| > 2a\right)$$

So $P(|S_{m,n}| \geqq a) \geqq \min_{k=m+1 \sim n} P(|S_{k,n}| \leqq a) \cdot P\left(\max_{m<j\leqq n} |S_{m,j}| > 2a\right)$

$\boxed{81}$ $S_n \xrightarrow{P} S \Rightarrow S_n \xrightarrow{(a.s)} S$

(proof) $W_N(\omega) \overset{def}{=} \sup_{m,n>N} |S_n - S_m|$

First we show $W_N \xrightarrow{P} 0$.

$P(W_N > 8\varepsilon) = P(\sup_{m,n>N} |S_n - S_N + S_N - S_m| > 8\varepsilon)$

$\leq P(\sup_{m>N} |S_m - S_N| + \sup_{n>N} |S_n - S_N| > 8\varepsilon)$

$= P(\sup_{m>N} |S_m - S_N| > 4\varepsilon)$

$= P(\lim_{n\to\infty} \max_{N<m\leq n} |S_m - S_N| > 4\varepsilon)$

$(f_n \nearrow f = \sup_{n\geq 1} f_n \Rightarrow \{f > \varepsilon\} = \bigcup_{n=1} \{f_n > \varepsilon\})$

$= P(\bigcup_{n\geq N+1}^{\infty} \{\max_{N<m\leq n} |S_m - S_N| > 4\varepsilon\})$

$(A_n \nearrow A \Rightarrow \lim_{n\to\infty} P(A_n) = P(A))$

$= \lim_{n\to\infty} P(\max_{N<m\leq n} |S_m - S_N| > 4\varepsilon)$ ... (*1)

Now we apply $\boxed{80}$ to (*1)

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$(*_1)$ .. $P\left(\max_{N<m\leq n}|S_m-S_N|>4\varepsilon\right)$ $\downarrow$ $\boxed{80}$

$$\leq \frac{P(|S_n-S_N|>2\varepsilon)}{\min_{N<k\leq n} P(|S_n-S_k|\leq 2\varepsilon)}$$

$$= \frac{P(|S_n-S_N|>2\varepsilon)}{1-\max_{N<k\leq n} P(|S_n-S_k|>2\varepsilon)}$$ $\downarrow$ triangle inequality

$$\leq \frac{P(|S_n-S|>\varepsilon)+P(|S_N-S|>\varepsilon)}{1-\max_{N<k\leq n} P(|S_n-S_k|>2\varepsilon)} \quad \cdots (*_2)$$

See $(*_2)$ $\max_{N<k\leq n} P(|S_n-S_k|>2\varepsilon)$

$$\leq P(|S_n-S|>\varepsilon)+\max_{N<k\leq n} P(|S_k-S|>2\varepsilon)$$

$$\leq 2\max_{N<k\leq n} P(|S_k-S|>2\varepsilon)$$

So $(*_2)$ $\leq$ $$\frac{P(|S_n-S|>\varepsilon)+P(|S_N-S|>\varepsilon)}{1-\max_{N<k\leq n} P(|S_k-S|>2\varepsilon)} \quad \cdots (*_3)$$

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$$(*3) = \frac{P(|S_n - S| > \varepsilon) + P(|S_N - S| > \varepsilon)}{1 - \max_{N<k\le n} P(|S_k - S| > 2\varepsilon)} \quad (\ge P(W_N > \delta_\varepsilon))$$

(for all $n \ge 1$)

By taking $n \nearrow \infty$, we have

$$P(W_N > \delta\varepsilon) \le \frac{P(|S_N - S| > \varepsilon)}{1 - \sup_{k>N} P(|S_k - S| > 2\varepsilon)} \quad \cdots (*4)$$

$(\because S_n \xrightarrow{P} S \Rightarrow P(|S_n - S| > \varepsilon) \to 0)$

In $(*4)$ $N \nearrow \infty$. $\cdot \sup_{k>N} P(|S_k - S| > 2\varepsilon) \to 0$

$\cdot P(|S_N - S| > \varepsilon) \to 0$

So $\limsup_{N \nearrow \infty} P(W_N > \delta\varepsilon) = 0$

$\left(\because S_n \xrightarrow{P} S \Leftrightarrow \overset{\forall}{\varepsilon > 0}\ \overset{\forall}{\delta > 0}\ \overset{\exists}{N_{\varepsilon\delta}} \text{ s.t. } \sup_{k > N_{\varepsilon\delta}} P(|S_k - S| > \varepsilon) < \delta\right)$

Therefore $W_N \xrightarrow{P} 0$.

We may take a subsequence $\{W_{N_k}\}_{k\ge1}$

$W_{N_k} \to 0$ (a.s)

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But W_N is a decreasing sequence $W_N \geq W_{N+1}$

This means $W_N \to 0$ (a.s)

So $\{S_n\}_{n\geq 1}$ is a Cauchy Sequence (a.s)

$\therefore$ $S_n \to S$ (a.s)

$\boxed{82}$ $\frac{S_n}{n} \xrightarrow{P} 0 \Rightarrow \frac{1}{n} \max_{1\le m\le n} S_m \xrightarrow{P} 0$

(proof)

$$P\left(\left|\frac{1}{n}\max_{1\le m\le n} S_m\right| > 8\varepsilon\right) = P\left(\left|\max_{1\le m\le n} S_m\right| > 8n\varepsilon\right)$$

$$\le P\left(\left\{\left|\max_{1\le m\le N} S_m\right| > 8n\varepsilon\right\} \cup \left\{\left|\max_{N+1\le m\le n} S_m\right| > 8n\varepsilon\right\}\right)$$

$$\left(\because \max_{1\le m\le n} S_m = \max_{1\le m\le N} S_m \text{ or } \max_{N+1\le m\le n} S_m\right)$$

$$\le P\left(\left|\max_{1\le m\le N} S_m\right| > 8n\varepsilon\right) + P\left(\left|\max_{N+1\le m\le n} S_m\right| > 8n\varepsilon\right)$$

$$\le \underbrace{P\left(\max_{1\le m\le N}|S_m| > 8n\varepsilon\right)}_{①} + \underbrace{P\left(\max_{N+1\le m\le n}|S_m| > 8n\varepsilon\right)}_{②}$$

First we consider ②. By using $\boxed{80}$, we have:

$$② = P\left(\max_{N+1\le m\le n}|S_m - S_N + S_N| > 8n\varepsilon\right)$$

$$= P\left(\max_{N+1\le m\le n}|S_m - S_N| + |S_N| > 8n\varepsilon\right)$$

$$\le P\left(\max_{N+1\le m\le n}|S_m - S_N| > 4n\varepsilon\right) + P\left(|S_N| > 4n\varepsilon\right)$$

$$\le \frac{P\left(|S_n - S_N| > 2n\varepsilon\right)}{1 - \underbrace{\max_{N+1\le k\le n} P\left(|S_n - S_k| > 2n\varepsilon\right)}_{③}} + \underbrace{P\left(|S_N| > 4n\varepsilon\right)}_{④}$$

③ ... $\max\limits_{N+1 \leq k \leq n} P(|S_n - S_k| > 2n\varepsilon) \leq \max\limits_{N+1 \leq k \leq n} \{P(|S_n| > n\varepsilon) + P(|S_k| > n\varepsilon)\}$

$\leq 2\max\limits_{N+1 \leq k \leq n} P(|S_k| > n\varepsilon) \leq 2\max\limits_{N+1 \leq k \leq n} P(|S_k| > k\varepsilon)$

$(\because k \leq n)$

$= 2\max\limits_{N+1 \leq k \leq n} P\left(\frac{|S_k|}{k} > \varepsilon\right)$ ... ③'

And $P(|S_n - S_N| > 2n\varepsilon) \leq P(|S_N| > n\varepsilon) + P(|S_n| > n\varepsilon)$ .. ③''

So ③ $\leq \dfrac{P(|S_N| > n\varepsilon) + P(|S_n| > n\varepsilon)}{1 - 2\max\limits_{N+1 \leq k \leq n} P\left(\frac{|S_k|}{k} > \varepsilon\right)}$

Since $\frac{|S_n|}{n} \xrightarrow{P} 0$. By taking large $N^{(\varepsilon)}$, $\max\limits_{N+1 \leq k \leq n} P\left(\frac{|S_k|}{k} > \varepsilon\right) < \frac{1}{4}$

So ③ $\leq \cdots \leq 2\{P(|S_N| > n\varepsilon) + P(|S_n| > n\varepsilon)\}$

Therefore $P\left(\left|\frac{1}{n}\max\limits_{1 \leq m \leq n} S_m\right| > 8\varepsilon\right) \leq$

$\ulcorner P\left(\max\limits_{1 \leq m \leq N} |S_m| > 8n\varepsilon\right) + 2\{P(|S_N| > n\varepsilon) + P(|S_n| > n\varepsilon)\}$

$+ P(|S_N| > 4n\varepsilon) \lrcorner$

By $n \to \infty$, all of them $\downarrow 0$. ■

$(\because \max\limits_{1 \leq m \leq N} |S_m|, |S_N|$ is not related with $n)$

$\boxed{83}$. $a(n) \nearrow \infty$, $\dfrac{a(2^n)}{a(2^{n-1})}$ is bounded. $\left(\dfrac{a(2^n)}{a(2^{n-1})} \leq M\right)$

$\dfrac{S_{2^n}}{a(2^n)} \to 0$ (a.s), $\dfrac{S_n}{a(n)} \xrightarrow{P} 0$.

We show that $\dfrac{S_n}{a(n)} \xrightarrow{a.s} 0$.

(proof) Our first goal is to show that

$$\ulcorner \sum_{n=1}^{\infty} P\left(\max_{m \in (2^{n-1}, 2^n]} \frac{|S_m - S_{2^{n-1}}|}{a(2^{n-1})} > 4\varepsilon \right) < \infty \lrcorner$$

$$① = P\left(\max_{m \in (2^{n-1}, 2^n]} \frac{|S_m - S_{2^{n-1}}|}{a(2^{n-1})} > 4\varepsilon \right) \quad \downarrow \text{ by } \boxed{80}$$

$$\leq \frac{P\left(\frac{|S_{2^n} - S_{2^{n-1}}|}{a(2^{n-1})} > 2\varepsilon \right)}{1 - \max_{k \in (2^{n-1}, 2^n]} P\left(\frac{|S_{2^n} - S_k|}{a(2^{n-1})} > 2\varepsilon \right)} = \frac{②-(1)}{②-(1')}$$

Next we see ②-(1').

$$P\left(\frac{|S_{2^n} - S_k|}{a(2^{n-1})} > 2\varepsilon \right) = P\left(\frac{|S_{2^n} - S_k|}{a(2^n)} > 2 \cdot \frac{a(2^n)}{a(2^{n-1})} \varepsilon \right)$$

$$\leq P\left(\frac{|S_{2^n} - S_k|}{a(2^n)} > 2M\varepsilon \right) \quad \left(\because \frac{a(2^n)}{a(2^{n-1})} \text{ bounded}\right)$$

$$\leq P\left(\frac{|S_{2^n}|}{a(2^n)} + \frac{|S_k|}{a(k)} > 2M\varepsilon \right) \quad \left(\because k \in (2^{n-1}, 2^n],\ a(2^n) \geq a(k)\right)$$

$$\leq P\left(\frac{|S_{2^n}|}{a(2^n)} + \frac{|S_k|}{a(k)} > 2M\varepsilon \right) \leq P\left(\frac{|S_{2^n}|}{a(2^n)} > M\varepsilon \right) + P\left(\frac{|S_k|}{a(k)} > M\varepsilon \right)$$

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$$\max_{k\in(2^{n-1},2^n]} P\left(\frac{|S_{2^n}-S_k|}{a(2^{n-1})}>2\varepsilon\right) \le \max_{k\in(2^{n-1},2^n]}\left\{P\left(\frac{|S_{2^n}|}{a(2^n)}>M\varepsilon\right)+P\left(\frac{|S_k|}{a(k)}>M\varepsilon\right)\right\}$$

$$\le 2\max_{k\in(2^{n-1},2^n]} P\left(\frac{|S_k|}{a(k)}>M\varepsilon\right)$$

$$\frac{1}{\text{②}-\text{(1')}} \le \frac{1}{1-2\max_{k\in(2^{n-1},2^n]} P\left(\frac{|S_k|}{a(k)}>M\varepsilon\right)} \le 2$$

(if we take suffiently large n, $\max_{k\in(2^{n-1},2^n]} P\left(\frac{|S_k|}{a(k)}>M\varepsilon\right) \le \frac{1}{4}$

$\because \frac{S_n}{a(n)} \xrightarrow{P} 0$)

Hence $\frac{\text{③}-\text{(1')}}{\text{②}-\text{(1')}} \le 2P\left(\frac{|S_{2^n}-S_{2^{n-1}}|}{a(2^{n-1})}>2\varepsilon\right)$.

$$\therefore \sum_{n=1}^{\infty} P\left(\max_{m\in(2^{n-1},2^n]}\frac{|S_m-S_{2^{n-1}}|}{a(2^{n-1})}>4\varepsilon\right) \le 2\sum_{n=1}^{\infty} \underbrace{P\left(\frac{|S_{2^n}-S_{2^{n-1}}|}{a(2^{n-1})}>2\varepsilon\right)}_{=A_n}$$

$A_n \overset{def}{=} \left\{\frac{|S_{2^n}-S_{2^{n-1}}|}{a(2^{n-1})}>2\varepsilon\right\}$ then $\{A_n\}_{n\ge1}$ are independent

$$\frac{|S_{2^n}-S_{2^{n-1}}|}{a(2^{n-1})} \le \frac{a(2^n)}{a(2^{n-1})}\cdot\frac{|S_{2^n}|}{a(2^n)}+\frac{|S_{2^{n-1}}|}{a(2^{n-1})} \le M\cdot\frac{|S_{2^n}|}{a(2^n)}+\frac{|S_{2^{n-1}}|}{a(2^{n-1})} \to 0 \ \text{(a.s)}$$

($\because$ by assumption $\frac{S_{2^n}}{a(2^n)} \to 0$ (a.s))

$\Leftrightarrow P(A_n \text{ i.o}) = 0 \quad (\forall \varepsilon>0)$

By Borel-Cantelli's lemma (II) $\sum_{n=1}^{\infty} P(A_n)=\infty \Rightarrow P(A_n \text{ i.o}) = 1$

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$\therefore P(A_n \text{ i.o}) = 0 < 1 \Rightarrow \sum_{n=1}^{\infty} P(A_n) < \infty$

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So $\sum_{n=1}^{\infty} P\left(\max_{m\in(2^{n-1},2^n]} \frac{|S_m - S_{2^{n-1}}|}{a(2^{n-1})} > 4\varepsilon\right) < \infty$

By Borel-Cantelli's Lemma (I). and $\boxed{3'}$

$$\max_{m\in(2^{n-1},2^n]} \frac{|S_m - S_{2^{n-1}}|}{a(2^{n-1})} \to 0 \quad (a.s)$$

Now pick $m \in (2^{n-1}, 2^n]$

$$\frac{|S_m|}{a(m)} \leq \frac{|S_m - S_{2^{n-1}} + S_{2^{n-1}}|}{a(2^{n-1})} \leq \frac{|S_m - S_{2^{n-1}}|}{a(2^{n-1})} + \frac{|S_{2^{n-1}}|}{a(2^{n-1})}$$

$$\therefore \max_{m\in(2^{n-1},2^n]} \frac{|S_m|}{a(m)} \leq \underbrace{\max_{m\in(2^{n-1},2^n]} \frac{|S_m - S_{2^{n-1}}|}{a(2^{n-1})}}_{\to 0 \ (a.s)} + \underbrace{\frac{|S_{2^{n-1}}|}{a(2^{n-1})}}_{\to 0 \ (a.s)}$$

This implies $\frac{S_n}{a(n)} \to 0$ (a.s)

Finally $a(n) \overset{def}{=} n^{\frac{1}{2}} (\log_2(n))^{\frac{1}{2}+\varepsilon}$ $(\nearrow \infty)$

It's enough to confirm $\frac{a(2^n)}{a(2^{n-1})}$ is bounded

$\frac{S_{2^n}}{a(2^n)} \to 0$ (a.s) and $\frac{S_n}{a(n)} \xrightarrow{P} 0$.

- $P\left(\frac{|S_{2^n}|}{a(2^n)} > \delta\right) = \frac{\sigma^2 \cdot 2^n}{a(2^n)^2 \cdot \delta^2} = \frac{\sigma^2}{\delta^2 \cdot n^{1+2\varepsilon}} \Rightarrow \sum_{n=1}^{\infty}(\cdot) < \infty$ (BC & $\boxed{3'}$)
- $P\left(\frac{|S_n|}{a(n)} > \delta\right) \leq \frac{\sigma^2 \cdot n}{a(n)^2 \cdot \delta^2} = \frac{1}{\delta^2 (\log_2(n))^{1+2\varepsilon}} \to 0$