r05246013 @ ntu.edu.tw

Chapter 1 Solutions by Toshinari Morimoto

$\boxed{1}$.

(1) monotonicity : If $A, B \in \mathcal{F}$ and $A \subseteq B$

then $P(A) \leq P(B)$

(proof) $B = (B \setminus A) \cup (A)$

Since μ is measure and $(B \setminus A)$, A are disjoint

then $\mu(B) = \mu(B \setminus A) + \mu(A)$ and $P(B \setminus A) \geq 0$

So $\mu(B) \geq \mu(A)$

(2) sub-additivity : $\{A_n\}_{n \geq 1} \subseteq \mathcal{F}$

$P\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} P(A_n)$

(proof) $B_1 \overset{def}{=} A_1$ and $B_n \overset{def}{=} A_n \setminus \bigcup_{j=1}^{n-1} A_j$ $(n \geq 2)$

Then $\bigcup_{n=1}^{\infty} A_n = \sum_{n=1}^{\infty} B_n$ (= disjoint countable union)

$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu\left(\sum_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \mu(B_n) \leq \sum_{n=1}^{\infty} \mu(A_n)$

($\because$ measure) ($\because$ $B_n \subset A_n$)

CHA SHIN

(3) (4)

- If $A_n \nearrow A$ $\left(\begin{array}{c} \{A_n\} \subseteq \mathcal{F}. \\ A_n \subseteq A_{n+1}. \end{array} \right)$

$\lim_{n\to\infty} \mu(A_n) = \mu(A)$

(proof) $B_n = A_n \setminus A_{n-1}$ $(B_1 = A_1)$

Then $A = \bigcup_{n=1}^{\infty} A_n = \sum_{n=1}^{\infty} B_n$

$\mu(A) = \mu(\sum_{n=1}^{\infty} B_n) = \sum_{n=1}^{\infty} \mu(B_n) = \lim_{n\to\infty} \sum_{m=1}^{n} \mu(B_m)$

$= \lim_{n\to\infty} \sum_{m=1}^{n} \{\mu(A_m \setminus A_{m-1})\} = \lim_{n\to\infty} \sum_{m=1}^{n} (\mu(A_m) - \mu(A_{m-1}))$

$= \lim_{n\to\infty} \mu(A_n)$

- If $A_n \searrow A$ and $\mu(A_1) < \infty$ $\lim_{n\to\infty} \mu(A_n) = \mu(A)$

(proof) $B_n = A_1 \setminus A_n$ then $B_n \nearrow B$.
$B = A_1 \setminus A$

By the previous result. $\lim_{n\to\infty} \mu(B_n) = \mu(B)$

(left) $\lim_{n\to\infty} \mu(A_1 \setminus A_n) = \lim_{n\to\infty} (\mu(A_1) - \mu(A_n))$
(right) $= \mu(A_1) - \mu(A)$
Since $\mu(A_1) < \infty$, we have $\lim_{n} \mu(A_n) = \mu(A)$

PAGE

DATE

$\boxed{2}$

(1) $F(x) \overset{\text{def}}{=} P(\{w \mid X(w) \le x\})$

If $x_1 < x_2$ then $\{w \mid X(w) \le x_1\} \subseteq \{w \mid X(w) \le x_2\}$

By monotonicity of measure (probability)

$P(\{w \mid X(w) \le x_1\}) \le P(\{w \mid X(w) \le x_2\})$

Thus $F(x_1) \le F(x_2)$

(2) $x \uparrow \infty$ ... $\{w \mid X(w) \le x\} \nearrow \{w \mid X(w) < \infty\} = \Omega$

$\therefore \underbrace{\lim_{x \to \infty} P(\{w \mid X(w) \le x\})}_{= \lim_{x \to \infty} F(x)} = P(\{w \mid X(w) < \infty\}) = P(\Omega) = 1$

($\because$ continuity of measure (I))

$x \downarrow -\infty$ $\{w \mid X(w) \le x\} \downarrow \phi$

And $P(\Omega) = 1 < \infty$ thus $\lim_{x \to -\infty} P(\{w \mid X(w) \le x\}) = P(\phi) = 0$

($\because$ continuity of measure) (II)

CHA SHIN

(3) Suppose $x_n \downarrow x_0$ (as $n \to \infty$)

$$\lim_{n\to\infty} P(\{\omega \mid X(\omega) \leq x_n\}) = P\left(\bigcap_{n=1}^{\infty} \{\omega \mid X(\omega) \leq x_n\}\right)$$

$$= P(\{\omega \mid X(\omega) \leq x_0\}) = F(x_0)$$

(left) $\lim_{n\to\infty} F(x_n) = F(x_0)$ (right)

(4) $x_n \uparrow x$

$$\lim_{n\to\infty} F(x_n) = \lim_{n\to\infty} P(\{\omega \mid X(\omega) \leq x_n\}) = P\left(\bigcup_{n=1}^{\infty} \{\omega \mid X(\omega) \leq x_n\}\right)$$

$$\overset{\parallel}{F(x-)} \qquad = P(\{\omega \mid X(\omega) < x\})$$

$$= P(X < x)$$

(5) $P(X = x) = P(\{\omega \mid X(\omega) \in (-\infty, x]\} \setminus \{\omega \mid X(\omega) \in (-\infty, x)\})$

$$= P(\{\omega \mid X(\omega) \in (-\infty, x]\}) - P(\{\omega \mid X(\omega) \in (-\infty, x)\})$$

$$P(X \leq x) \quad - \quad P(X < x)$$

$$= F(x) - F(x-)$$

Borel measurable set.

$\boxed{3}$ $\Omega = (0,1)$ $\mathcal{F} = \mathbb{B}((0,1))$ P: Lebesgue Measure.

$X(w) \overset{\text{def}}{=} \inf\{x \in \mathbb{R} \mid F(x) \geq w\}$

Then $P(X(w) \leq x) = F(x)$ $(= P(\{w \mid w \leq F(x)\}))$

(proof) We show that $\{w \mid X(w) \leq x\} = \{w \mid w \leq F(x)\}$

① $\{w \mid X(w) \leq x\} \subseteq \{w \mid w \leq F(x)\}$

By definition of $X(w)$, $\exists \{x_n\}_{n \geq 1}$ $x_n \downarrow X(w)$: $F(x_n) \geq w$.

Right continuity : " $w \leq \lim_n F(x_n) = F(X(w))$ " ... (a)

And $X(w) \leq x \Rightarrow$ " $F(X(w)) \leq F(x)$ " ... (b)

By (a), (b), we have $w \leq F(x)$.

② $\{w \mid X(w) \leq x\} \supseteq \{w \mid w \leq F(x)\}$

$w \leq F(x) \Rightarrow x \in \{y \in \mathbb{R} \mid F(y) \geq w\}$

$\Rightarrow x \geq \inf\{y \in \mathbb{R} \mid F(y) \geq w\} = X(w)$.

PAGE
DATE

$\boxed{4}$

① $\int_x^\infty \exp\left(\frac{-y^2}{2}\right) dy$

$z = y - x \qquad \frac{dz}{dy} = 1$

$\int_0^\infty \exp\left(\frac{-1}{2}(z+x)^2\right) dz$

$= \int_0^\infty \underbrace{\exp\left(\frac{-z^2}{2}\right)}_{\leq 1} \cdot \exp(-zx) \exp\left(\frac{-x^2}{2}\right) dz$

$\int_0^\infty \exp(-zx)\, dz \cdot \exp\left(\frac{-x^2}{2}\right) = x^{-1} \exp\left(\frac{-x^2}{2}\right)$

② $\frac{d}{dx}\left((x^{-1} - x^{-3}) \exp\left(\frac{-x^2}{2}\right)\right)$

$= (-x^{-2} + 3x^{-4}) \exp\left(\frac{-x^2}{2}\right) - (1 - x^{-2}) \exp\left(\frac{-x^2}{2}\right)$

$= (-1 + 3x^{-4}) \exp\left(\frac{-x^2}{2}\right)$

Thus $\int_x^\infty (1 - 3y^{-4}) \exp\left(\frac{-y^2}{2}\right) dy = (x^{-1} - x^{-3}) \exp\left(\frac{-x^2}{2}\right)$

$\leq \int_x^\infty 1 \cdot \exp\left(\frac{-y^2}{2}\right) dy$

Now the proof is completed

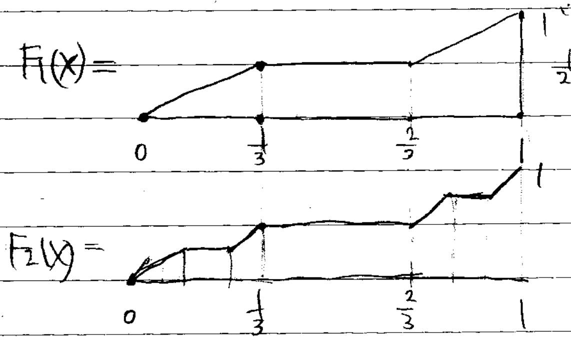
$\boxed{5}$

(1) cdf : $F(x) = P(\{w \mid X(w) \leq x\})$

If there exists a Borel measurable function such that $F(x) = \int_{-\infty}^{x} f(y)\,dy$.

Then $f(\cdot)$ is called a probability density function of X.

(2) No. It does not always exists



($F_n \to F$)

(F can be a cdf)

We consider the Cantor - Lebesgue function : $F(x)$

As we see $F(x)$ is flat almost everywhere.

Thus $F'(x) = 0$ (a.e). Suppose $P(X \leq x) = F(x)$

So $\int_{-\infty}^{x} F'(y)\,dy = 0$ (for all x)

Thus its density function does not exist

$\boxed{6}$ $\mathcal{g} \overset{def}{=} \{ (a,b] \mid a,b \in \mathbb{R} \}$

$\mu((a,b]) \overset{def}{=} F(b) - F(a)$.

where $F(\cdot)$ is a right-continuous real-valued function. Then $(\mathcal{g}, \mu)$ can be extended to $(\sigma[\mathcal{g}], \mu)$.

(Proof) If $\mathcal{g}$: semi-algebra, $(\mathcal{g}, \mu)$: finite additive and $\forall \{G_n\}_{n\geq 1} \cup \{G\} \subseteq \mathcal{g}$: $G = \sum_{n=1}^{\infty} G_n$ (disjoint union)

$\mu(G) \leq \sum_{n=1}^{\infty} \mu(G_n)$, then $(\mathcal{g}, \mu)$ can be extended to $(\sigma[\mathcal{g}], \mu)$.

Here we prove that $G = \sum_{n=1}^{\infty} G_n$ ($\{G\} \cup \{G_n\}_{n\geq 1} \subset \mathcal{g}$)

then $\mu(G) = \sum_{n=1}^{\infty} \mu(G_n)$. (thus $\mu(G) \leq \sum_{n=1}^{\infty} \mu(G_n)$)

(You may refer to the textbook 測度與概率論基礎 程士宏 (北京大學) chapter 2. prop 2.1.2

$\cup$ … union. $\sum$ … disjoint union

We separate the proof into $\boxed{\text{Step 1}} \sim \boxed{\text{Step 4}}$

$\boxed{\text{Step 1}}$ $(a, b] = \sum_{j=1}^{n} (a_j, b_j]$ $(n < \infty)$

Then $\mu((a, b]) = \sum_{j=1}^{n} \mu((a_j, b_j])$ holds

(proof) We rearrange the order : $\{(a_j, b_j]\}_{j=1}^{n}$

$\rightarrow \{(a_{(j)}, b_{(j)}]\}_{j=1}^{n}$ so that

$a = a_{(1)} \leq b_{(1)} = a_{(2)} \leq b_{(2)} \cdots \leq b_{(n)} = b$.

$$\sum_{j=1}^{n} \mu((a_j, b_j]) = \sum_{j=1}^{n} \mu((a_{(j)}, b_{(j)}])$$

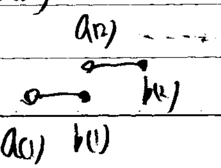
$$= \sum_{j=1}^{n} F(b_{(j)}) - F(a_{(j)})$$

$$= F(b_{(n)}) - F(a_{(n)}) + F(b_{(n-1)}) - F(a_{(n-1)}) + \cdots$$

$$\cdots F(b_{(1)}) - F(a_{(1)})$$

$$= F(b_{(n)}) - F(a_{(1)}) = F(b) - F(a) = \mu((a, b])$$

$\left(\begin{array}{ll} ※\ F(a_{(n)}) = F(b_{(n-1)}) & (\because a_{(n)} = b_{(n-1)}) \\ F(a_{(n-1)}) = F(b_{(n-2)}) & \vdots \\ \vdots & \\ F(a_{(2)}) = F(b_{(1)}) & \end{array} \right.$



disjoint union

$\boxed{\text{Step2}}$ $(a,b] \supseteq \sum_{n=1}^{\infty}(a_n,b_n]$ Then $\mu((a,b]) \geq \sum_{n=1}^{\infty}\mu((a_n,b_n])$

In the same way, we rearrange the order.

But there are infinite intervals. thus.

$b(1)=a(2)$, $b(2)=a(3)$ ... does not alway hold.

We can only say that "$a(1) \leq b(1) \leq a(2) \leq b(2)$ ..."

(proof) pick $1 \leq N < \infty$

$$\mu((a,b]) = F(b)-F(a) = \underbrace{F(b)-F(b(N))}_{\geq 0} + \sum_{n=1}^{N} F(b(n))-F(a(n)) + \underbrace{F(a(1))-F(a)}_{\geq 0}$$

$\geq \sum_{n=1}^{N} F(b(n))-F(a(n))$ (for all N)

$N \uparrow \infty$ we have $\mu((a,b]) \geq \sum_{n=1}^{N} \mu((a_n,b_n])$

It is not necessary disjoint union

$\boxed{\text{step3}}$ If $(a,b] \subseteq \bigcup_{n=1}^{N} (a_n, b_n]$ then

$$\mu((a,b]) \leq \sum_{n=1}^{N} \mu((a_n, b_n]) \qquad (N<\infty)$$

(proof) We prove by induction

N=1 … It's trivial.

N=k We assume the statement above holds.

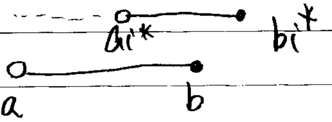
N=k+1 … $(a,b] \subseteq \bigcup_{n=1}^{k+1} \mu((a_n, b_n])$

We pick $b_{i^*} \overset{def}{=} \max_{1\leq n \leq k+1} \{b_n\}$.

(case I) $a_{i^*} \leq a$. then $(a,b] \subseteq (a_{i^*}, b_{i^*}]$

$$\sum_{n=1}^{k+1} \mu((a_n, b_n]) \geq \mu((a_{i^*}, b_{i^*}]) \geq \mu((a,b])$$

(case II) $a_{i^*} > a$ then $(a, a_{i^*}] \subseteq \bigcup_{n \neq i^*} (a_n, b_n]$.

⊛ 

→ $(a, a_{i^*}]$ must be covered by intervals

$\{(a_n, b_n]\}_{n \neq i^*}$.

By assumption $(N=k)$, we have

$$\mu((a, a_{i^*}]) \leq \sum_{n \neq i^*} \mu((a_n, b_n]) . \quad (\because \text{ there are } k\text{-intervals})$$

$$\Rightarrow \mu((a, a_{i^*}]) + \mu((a_{i^*}, b_{i^*}])$$
$$\leq \sum_{n \neq i^*} \mu((a_n, b_n]) + \mu((a_{i^*}, b_{i^*}])$$
$$= \sum_{n=1}^{k+1} \mu((a_n, b_n])$$

The left hand side $= F(b_{i^*}) - F(a)$

$$\geq F(b) - F(a) = \mu((a,b])$$

So we have $\mu((a,b]) \leq \sum_{n=1}^{k+1} \mu((a_n, b_n])$

Thus the statement is true for $N=k+1$.

disjoint union.

$\boxed{\text{Step4}}$ $(a,b] = \sum_{n=1}^{\infty} (a_n, b_n] \Rightarrow$

$$\mu((a,b]) = \sum_{n=1}^{\infty} \mu((a_n,b_n])$$

(proof) $(a,b] \supseteq \sum_{n=1}^{\infty} (a_n,b_n]$. So by $\boxed{\text{Step2}}$

we have $\mu((a,b]) \geq \sum_{n=1}^{\infty} \mu((a_n,b_n])$.

Next we prove $\mu((a,b]) \leq \sum_{n=1}^{\infty} \mu((a_n,b_n])$.

Since $F(\cdot)$ is right continuous. for all $\varepsilon>0$ and for each $n \geq 1$ there exists $\delta_n > 0$. such that

$F(b_n+\delta_n) - F(b_n) \leq \frac{\varepsilon}{2^n}$;

Let $\eta > 0$ be an arbitrary positive number.

It's obviously true that $(a+\eta, b] \subseteq [a+\eta, b]$

$\subseteq (a,b] = \sum_{n=1}^{+\infty} (a_n, b_n] \subseteq \bigcup_{n=1}^{\infty} (a_n, b_n+\delta_n)$

So $[a+\eta, b]$: bounded closed set is covered by

$\{(a_n, b_n+\delta_n)\}_{n\geq1}$: open-intervals.

PAGE
DATE

By Heine-Borel Cover theorem,

We may cover $[a+\eta, b]$ by finite number of

$\{(a_n, b_n+\delta_n)\}_{n=1}^{N}$.

So $(a+\eta, b]$ is covered by $\{(a_n, b_n+\delta_n]\}_{n=1}^{N}$

By $\boxed{\text{STEP 3}}$ $(a+\eta, b] \subseteq \bigcup_{n=1}^{N} (a_n, b_n+\delta_n] \Rightarrow$

$$\mu((a+\eta, b]) \leq \sum_{n=1}^{N} \mu((a_n, b_n+\delta_n])$$

$$\overset{\shortparallel}{F(b)-F(a+\eta)} \leq \sum_{n=1}^{N} F(b_n+\delta_n) - F(a_n)$$

$$\leq \sum_{n=1}^{N} \{F(b_n+\delta_n) - F(b_n) + F(b_n) - F(a_n)\}$$

$$\leq \sum_{n=1}^{N} \left(\frac{\varepsilon}{2^n} + F(b_n) - F(a_n)\right)$$

$$\leq \varepsilon + \sum_{n=1}^{\infty} (F(b_n) - F(a_n))$$

$$= \sum_{n=1}^{\infty} \mu((a_n, b_n]) + \varepsilon$$

$\eta \downarrow 0$ (Left) $\rightarrow F(b) - F(a)$ ($\because$ right continuous)

$\varepsilon \downarrow 0$ (Right) $\rightarrow \sum_{n=1}^{\infty} \mu((a_n, b_n])$ $\blacksquare$

[17] Dynkin's π-λ theorem.

- $\mathcal{G}$ ... π-system. (i.e $\forall A, B$, if $A, B \in \mathcal{G} \Rightarrow A \cap B \in \mathcal{G}$)
- $\lambda[\cdot]$ – the smallest λ-system (Dynkin system) containing $\cdot$.
- $\sigma[\cdot]$ the smallest σ-algebra containing $\cdot$

Then $\lambda[\mathcal{G}] = \sigma[\mathcal{G}]$.

① σ-algebra is also a λ-system (easy to verify)

thus $\lambda[\mathcal{G}] \subseteq \sigma[\mathcal{G}]$.

② If a λ-system is also a π-system then it's a σ-algebra. (easy to verify)

When $\mathcal{G}$ is a π-system, then $\lambda[\mathcal{G}]$ is also a π-system (We prove it in ③) ...... ⊛

So $\lambda[\mathcal{G}]$ is a λ-system and a π-system

$\Rightarrow \lambda[\mathcal{G}]$ is a σ-algebra $\therefore \lambda[\mathcal{G}] \supseteq \sigma[\mathcal{G}]$ ∎

The proof is done but we should show ⊛.

PAGE

DATE

③ Finally we prove $\mathcal{G} = \pi\text{-system} \Rightarrow \lambda[\mathcal{G}] = \pi\text{-system}$

$\mathcal{D}_1 \overset{\text{def}}{=} \{A \mid \forall B \in \mathcal{G} \quad A \cap B \in \lambda[\mathcal{G}]\}$

$\mathcal{D}_2 \overset{\text{def}}{=} \{B \mid \forall A \in \lambda[\mathcal{G}] \quad A \cap B \in \lambda[\mathcal{G}]\}$

Since $\mathcal{G}$ is a π-system, thus $\mathcal{G} \subseteq \mathcal{D}_1$. $(*_1)$

And $\mathcal{D}_1$ is a λ-system.$^{(*_2)}$ So $\lambda[\mathcal{G}] \subseteq \mathcal{D}_1$

Since $\lambda[\mathcal{G}] \subseteq \mathcal{D}_1$, we find that $\mathcal{G} \subseteq \mathcal{D}_2$. $^{(*_3)}$

$\mathcal{D}_2$ is also a λ-system.$^{(*_4)}$ thus $\lambda[\mathcal{G}] \subseteq \mathcal{D}_2$

This implies $\lambda[\mathcal{G}]$ is a π-system.$^{(*_5)}$ ■

$(*_1)$ $\mathcal{G}$ is a π-system $\Rightarrow$ Let $A \in \mathcal{G}$. and $\forall B \in \mathcal{G}$

$A \cap B \in \mathcal{G} \subseteq \lambda[\mathcal{G}] \Rightarrow A \in \mathcal{D}_1$. Thus $\mathcal{G} \subseteq \mathcal{D}_1$.

$(*_2)$ $\mathcal{D}_1$ is a λ-system because

- $\Omega \in \mathcal{D}_1$ ... $\forall B \in \mathcal{G}$ $\quad \Omega \cap B = B \in \mathcal{G} \subseteq \lambda[\mathcal{G}]$
- $A, \hat{A} \in \mathcal{D}_1$ $(A \subseteq \hat{A})$ $\quad A \cap B \in \lambda[\mathcal{G}]$

$\hat{A} \cap B \in \lambda[\mathcal{G}]$

$\Rightarrow (\hat{A} \cap B) \setminus (A \cap B) = (\hat{A} \setminus A) \cap B \in \lambda[\mathcal{G}]$

PAGE
DATE

If $\{A_n\}_{n\geq 1} \subseteq \mathcal{D}_1$ $A_n \nearrow A$. Then,

Since $A_n \cap B \in \lambda(\mathcal{G})$ and $A_n \cap B \nearrow A \cap B$

So $A \cap B \in \lambda(\mathcal{G})$ (for all $B \in \mathcal{G}$)

$\Rightarrow$ $A \in \mathcal{D}_1$

($*_3$) $\lambda[\mathcal{G}] \subseteq \mathcal{D}_1$ implies $\mathcal{G} \subset \mathcal{D}_2$

because $\lambda[\mathcal{G}] \subseteq \mathcal{D}_1 \Rightarrow$ $\forall A \in \lambda[\mathcal{G}]$ $\forall B \in \mathcal{G}$ $A \cap B \in \lambda(\mathcal{G}]$.

This means $\forall B \in \mathcal{G}$ $B \in \mathcal{D}_2$. $\Rightarrow$ $\mathcal{G} \subset \mathcal{D}_2$.

($*_4$) $\mathcal{D}_2$ is a λ-system.. The proof is similar to ($*_2$)

($*_5$) $\lambda[\mathcal{G}] \subset \mathcal{D}_2$... implies $\forall A, B \in \lambda[\mathcal{G}]$ $A \cap B \in \lambda[\mathcal{G}]$

So $\lambda(\mathcal{G})$ is a π-system

PAGE
DATE

[8] Since $(\mathcal{P}, \mu)$ is σ-finite, there exists

$\{\Omega_n\}_{n\geq 1} \subseteq \mathcal{P}$ $\Omega_n \uparrow \Omega$, $\mu_1(\Omega_n) < \infty$

Define $\mathcal{D}^{(n)} = \{A \in \mathcal{F} \mid \mu_1(A \cap \Omega_n) = \mu_2(A \cap \Omega_n)\}$

- $\mathcal{P} \subseteq \mathcal{D}^{(n)}$; $\forall A \in \mathcal{P}$. $A \cap \Omega_n \in \mathcal{P}$

  ($\because$ $\Omega_n, A \in \mathcal{P}$ and $\mathcal{P}$ is a π-system)

  So $\mu_1(A \cap \Omega_n) = \mu_2(A \cap \Omega_n)$

  hence $A \in \mathcal{D}^{(n)}$ for all $A \in \mathcal{P}$.

- $\mathcal{D}^{(n)}$ is a λ-system because

  - $\Omega \in \mathcal{D}^{(n)}$ (easy)
  - $A, \hat{A} \in \mathcal{D}^{(n)} \Rightarrow \hat{A} \setminus A \in \mathcal{D}^{(n)}$

  $\mu_1(A \cap \Omega_n) = \mu_2(A \cap \Omega_n) < \infty$
  $\mu_1(\hat{A} \cap \Omega_n) = \mu_2(\hat{A} \cap \Omega_n) < \infty$

  Since finite, $\mu_1(\hat{A} \cap \Omega_n) - \mu_2(A \cap \Omega_n)$
  $= \mu_2(\hat{A} \cap \Omega_n) - \mu_1(A \cap \Omega_n)$

PAGE

DATE

$\therefore \mu_1((\hat{A}\setminus A)\cap\Omega_n) = \mu_2((\hat{A}\setminus A)\cap\Omega_n)$

$\therefore \hat{A}\setminus A \in \mathcal{D}^{(n)}$

- $\{A_m\}_{m\geq 1} \subset \mathcal{D}^{(n)}$ $\quad A_m \uparrow A$

$\mu_1(A_m\cap\Omega_n) = \mu_2(A_m\cap\Omega_n)$

By letting $m\uparrow\infty$, $\rightarrow$ $\mu_1(A\cap\Omega_n) = \mu_2(A\cap\Omega_n)$

$\therefore A \in \mathcal{D}^{(n)}$

So $\mathcal{P} \subseteq \mathcal{D}^{(n)}$ and $\mathcal{D}^{(n)}$: λ-system

$\Rightarrow \lambda[\mathcal{P}] \subseteq \mathcal{D}^{(n)}$

$\quad \parallel$

$\sigma[\mathcal{P}]$ $\quad$ ($\because$ π-λ-theorem)

$\therefore$ $^{\forall}A \in \sigma[\mathcal{P}]$ $\quad \mu_1(A\cap\Omega_n) = \mu_2(A\cap\Omega_n)$

Finally $n\uparrow\infty$ we have $\mu_1(A\cap\Omega) = \mu_2(A\cap\Omega)$

$= \mu_1(A) = \mu_2(A)$ $\quad$ ($^{\forall}A\in\sigma[\mathcal{P}]$)

PAGE

DATE

$\boxed{9}$ When $\mathcal{g}$ is a semi-algebra,

We define $\mathcal{A} \overset{def}{=} \{A \mid A = \sum_{k=1}^{n} G_k, \{G_k\}_{k=1}^{n} \subset \mathcal{g}\}$

(ie $\mathcal{A} = \{A \mid A$ is expressed as a finite disjoint union of sets from $\mathcal{g}\}$)

Then $\sigma_0[\mathcal{g}] \overset{equal}{=} \mathcal{A}$.

(I) $\sigma_0[\mathcal{g}] \supseteq \mathcal{A}$

This is obviously true by the definition of $\mathcal{A}$.

because $\sigma_0(\mathcal{g})$ is an algebra containing $\mathcal{g}$.

(II) $\sigma_0[\mathcal{g}] \subseteq \mathcal{A}$

$\mathcal{g} \subset \mathcal{A}$ is obviously true.

It's enough to show that $\mathcal{A}$ is an algebra.

<claim> $\mathcal{A}$ is an algebra.

- $\phi \in \mathcal{A}$ since $\phi \in \mathcal{g}$

• $A, B \in \mathcal{A} \Rightarrow A \cap B \in \mathcal{A}$

$A = \sum_{j=1}^{J} G_{1,j}$ $\quad B = \sum_{\ell=1}^{L} G_{2,\ell}$ $\quad$ (Σ: disjoint union)

$A \cap B = \left(\sum_{j=1}^{J} G_{1,j}\right) \cap \left(\sum_{\ell=1}^{L} G_{2,\ell}\right) = \sum_{j=1}^{J} \sum_{\ell=1}^{L} \left(G_{1,j} \cap G_{2,\ell}\right)$

$\{G_{1,j} \cap G_{2,\ell}\}_{(j,\ell) \in \{1..J\} \times \{1..L\}}$

$\subseteq \mathcal{G}$ ($\because \mathcal{G}$: semi-algebra $\Rightarrow$ π-system)

And $J, L < \infty$: finite disjoint union

$\therefore A \cap B \in \mathcal{A}$

• $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$

$A = \sum_{j=1}^{J} G_j$ $\quad A^c = \bigcap_{j=1}^{J} G_j^c$

Since $\{G_j^c\} \subseteq \mathcal{A}$ ($\because \mathcal{G}$ .. semi-algebra)

($\forall G \in \mathcal{G}$, G^c ... disjoint finite union of sets from $\mathcal{G}$ $\Rightarrow \in \mathcal{A}$)

So $\bigcap_{j=1}^{J} G_j^c \in \mathcal{A}$ ($\because$ we have already shown that $A, B \in \mathcal{A} \Rightarrow A \cap B \in \mathcal{A}$) ■

The proof is complete. ($\because A, B \in \mathcal{A} \Rightarrow A \cup B = (A^c \cap B^c)^c \in \mathcal{A}$)

$\boxed{10}$ X is $\mathcal{F}/\mathcal{A}$-measurable

$\overset{def}{\Leftrightarrow}$ $\forall A \in \mathcal{A}$ $\quad X^{-1}(A) \in \mathcal{F}$

① $\forall A \in \mathcal{A}$ $X^{-1}(A) \in \mathcal{F} \Rightarrow \forall G \in \mathcal{G}$ $X^{-1}(G) \in \mathcal{F}$

Since $\mathcal{G} \subseteq \sigma[\mathcal{G}] = \mathcal{A}$. so it's obviously true

② $\forall G \in \mathcal{G}$ $X^{-1}(G) \in \mathcal{F} \Rightarrow \forall A \in \mathcal{A}$ $\quad X^{-1}(A) \in \mathcal{F}$

Consider $B = \{A \mid X^{-1}(A) \in \mathcal{F}\}$.

Since $\mathcal{F}$ is a σ-algebra, thus B is also a σ-algebra. (easy to verify)

By assumption, $\mathcal{G} \subseteq B$. And B is a σ-algebra

Thus $\sigma[\mathcal{G}] \subseteq B \Rightarrow \mathcal{A} \subset B$.

$\therefore \forall A \in \mathcal{A}$ $\quad X^{-1}(A) \in \mathcal{F}$.

[1] $\forall B \in \mathcal{B}$ $(f \circ X)^{-1}(B) = X^{-1} \circ f^{-1}(B)$

Since f is a measurable map from $(S, \mathcal{A}) \to (T, \mathcal{B})$

thus $f^{-1}(B) \in \mathcal{A}$. And since $f^{-1}(B) \in \mathcal{A}$

$X^{-1}(f^{-1}(B)) \in \mathcal{F}$. ($\because$ X is a measurable map)

[2]

① $\sup_{n \geq 1} X_n(w)$

$\{ w \in \Omega \mid \sup_{n \geq 1} X_n(w) > a \}$

$= \bigcup_{n \geq 1}^{\infty} \{ w \in \Omega \mid X_n(w) > a \} \in \mathcal{F}$

$\because \{ w \in \Omega \mid X_n(w) > a \} \in \mathcal{F}$

② $\inf X_n(w) = -\sup_{n \geq 1} (-X_n(w))$

$X_n(w)$: measurable $\Rightarrow -X_n(w)$: measurable

$\Rightarrow \sup_{n \geq 1} (-X_n(w)$: measurable$) \Rightarrow -\sup_{n \geq 1} (-X_n(w))$: measurable

③ $\lim\inf X_n = \lim_{n\to\infty} \inf_{m\geq n} X_m(w) = \sup_{n\geq 1} \inf_{m\geq n} X_m(w)$

Since ②, $\inf_{m\geq n} X_n(w)$: is measurable

Since ① $\sup_{n\geq 1} \left(\inf_{m\geq n} X_n(w)\right)$ is measurable

④ $\lim\sup X_n(w)$: (Similar to ③)

$\boxed{13}$ $D_n = \{x \in R \mid F(x) - F(x-) > \frac{1}{n}\}$

Since $0 \leq F(x) \leq 1$, thus $\# D_n < \infty$.

(Otherwise, $F(x)$ will be out of $[0,1]$)

$\bigcup_{n=1}^{\infty} \{x \in \mathbb{R} \mid F(x) - F(x-) > \frac{1}{n}\}$

$= \{x \in \mathbb{R} \mid F(x) - F(x-) > 0\}$

will be countable

$\boxed{\text{NOTE}}$ # : number of elements in a set (card)

$\boxed{4}$ We show that $P(\{w \mid F(X(w)) \leq y\}) = y$

when $y \in (0,1)$.

Consider $\{x \mid F(x) = y\}$ $(y \in (0,1))$

It will be a closed set.

$\because \{x \in \mathbb{R} \mid F(x) \neq y\} = \{x \in \mathbb{R} \mid F(x) \in (y, \infty)\}$ ①
$\cup \{x \in \mathbb{R} \mid F(x) \in (-\infty, y)\}$ ②

Since $F(x)$ is continuous, ①, ② are open.

$\therefore \{x \in \mathbb{R} \mid F(x) \neq y\}^c = \{x \in \mathbb{R} \mid F(x) = y\}$ is closed

And $F(x)$ is monotone increasing, thus

$\{x \in \mathbb{R} \mid F(x) \neq y\}$ will be a closed interval.

$x_0 \overset{\text{def}}{=} \max\{x \mid F(x) = y\}$

$\therefore P(F(X(w)) \leq y) = P(X \leq x_0) = F(x_0) = y$

by definition of x_0.

Now the proof is complete ■

PAGE
DATE

15

① f is lower-semi continuous : $\forall x_0 \in \mathbb{R}$

「$\liminf_{x \to x_0} f(x) \geq f(x_0)$」 $\Leftrightarrow$ 「$\lim_{\delta \to 0} \inf_{y \in B(x_0,\delta)} f(y) \geq f(x_0)$」

$\Rightarrow$ We show that $\{x \mid f(x) \leq a\}$ is closed for $\forall a \in \mathbb{R}$.

Take $\{x_n\}_{n\geq 1} \subseteq \{x \mid f(x) \leq a\}$ where $x_n \to x_0$

We prove that $f(x_0) \leq a$.

Let $\varepsilon > 0$. $\lim_{\delta \to 0} \inf_{y \in B(x_0,\delta)} f(y) \geq f(x_0)$

$\Rightarrow \exists \delta_\varepsilon > 0$ s.t. $\inf_{y \in B(x_0,\delta_\varepsilon)} f(y) > f(x_0) - \varepsilon$.

Since $x_n \to x_0$, thus for sufficiently large n,

$x_n \in B(x_0, \delta_\varepsilon)$. Therefore $f(x_n) \geq \inf_{y \in B(x_0,\delta_\varepsilon)} f(y)$

$> f(x_0) - \varepsilon$

$\therefore a \geq f(x_n) > f(x_0) - \varepsilon$. $\Rightarrow f(x_0) \leq a$ (by letting $\varepsilon \downarrow 0$)

CHA SHIN

② $\{x\in\mathbb{R} \mid f(x)\le a\}$ is a closed set for $\forall a\in\mathbb{R}$

$\Rightarrow f(\cdot)$ is lower-semi continuous

We consider its contra position.

「f is NOT lower-semi continuous」 $\Rightarrow$ 「$\exists a\in\mathbb{R}$ s.t. $\{x \mid f(x)\le a\}$ is not closed」

(「f is lower-semi continuous」 $\overset{\text{def}}{=}$ 「$\forall x_0,\ \forall \varepsilon>0\ \exists \delta_\varepsilon>0$ s.t. $\forall\, 0<\delta^*\le\delta_\varepsilon\ \inf_{y\in B(x_0,\delta^*)} f(y) > f(x_0)-\varepsilon$」)

「f is NOT lower-semi continuous」 $\overset{\text{def}}{=}$ 「$\exists x_0\ \exists\varepsilon>0\ \forall\delta_n>0$ s.t. $\exists\, 0<\delta_n^*<\delta_n$ s.t. $\inf_{y\in B(x_0,\delta_n^*)} f(y)\le f(x_0)-\varepsilon$」

We consider $\{\delta_n\}_{n\ge1}$ $\delta_n\downarrow 0$.

We may find $x_n\in B(x_0,\delta_n^*)$ s.t. $f(x_n)-\frac{\varepsilon}{2}\le \inf_{y\in B(x_0,\delta_n^*)} f(y)$.

Therefore $f(x_n)\le f(x_0)-\frac{\varepsilon}{2}$. Let $a=f(x_0)-\frac{\varepsilon}{2}$

$\therefore \{x_n\}_{n\ge1}\subseteq\{f(x)\le a\}$ and $x_n\to x_0$. ($\because |x_n-x_0|\le\delta_n^*\le\delta_n\to0$)

But $f(x_0)\le a$ does not hold. ($\because a=f(x_0)-\frac{\varepsilon}{2}$)

So $\{f(x)\le a\}$ is not closed. ∎

PAGE

DATE

(1)

$\boxed{16}$ We show that $f^{\delta}(x)$ is lower semi continuous

Take $\{x_n\}_{n\geq 1} \subset \{f^{\delta}(x) \leq a\}$ and $x_n \to x_0$.
We show that $x_0 \in \{f^{\delta}(x) \leq a\}$.

$\boxed{\text{claim}}$ $\bigcup_{n\geq 1}^{\infty} B(x_n, \delta) \supseteq B(x_0, \delta)$

pick $x_0^* \in B(x_0, \delta)$.

$|x_0^* - x_n| \leq \underbrace{|x_0 - x_0^*|}_{= \delta_0 < \delta} + |x_n - x_0|$

Since $x_n \to x_0$ We may find sufficiently large

n such that $|x_n - x_0| < \delta - \delta_0$.

Thus $|x_0^* - x_n| < \delta$.

$\therefore x_0^* \in B(x_n, \delta) \subset \bigcup_{n\geq 1}^{\infty} B(x_n, \delta)$

$\therefore B(x_0, \delta) \subset \bigcup_{n\geq 1}^{\infty} B(x_n, \delta)$. ■

Next, we consider $f^{\delta}(x_0) \overset{(\text{def})}{=} \sup_{z \in B(x_0, \delta)} \{f(z)\}$

We may find $z_0 \in B(x_0, \delta)$ $f^{\delta}(x_0) \leq f(z_0) + \varepsilon$.

where $\varepsilon > 0$ is an arbitrary positive number

CHA SHIN

PAGE
DATE

Since $y_0 \in B(x_0,\delta) \subseteq \bigcup_{n=1}^{\infty} B(x_n,\delta)$.

We may find $n \geq 1$ such that $y_0 \in B(x_n,\delta)$ (by $\boxed{\text{claim}}$)

$\therefore f(y_0) \leq \sup_{y \in B(x_n,\delta)} f(y) = f^{\delta}(x_n)$.

So we have $f^{\delta}(x_0) \leq f(y_0) + \varepsilon \leq f^{\delta}(x_n) + \varepsilon \leq a + \varepsilon$

Since $\varepsilon > 0$ is arbitrary, $\varepsilon \downarrow 0$ we have $f^{\delta}(x_0) \leq a$. ∎

$f_{\delta}(x)$ is upper-semi continuous...

$$f_{\delta}(x) = \inf_{y \in B(x,\delta)} \{f(y)\} = -\underbrace{\sup_{y \in B(x,\delta)} \{-f(y)\}}_{\text{lower semi continuous}}$$

$\therefore -(l.s.c) = u.s.c$

(2)

Next we prove $\{f^0 \neq f_0\}$ is measurable

$\{f^0 \neq f_0\} = \{f^0 - f_0 > 0\}$ ($\because f^\delta \geq f_\delta$ for all $\delta > 0$)

$= \bigcup_{m=1}^{\infty} \{f^0 - f_0 \geq \frac{1}{m}\}$.

We prove that $\{f^0 - f_0 \geq a\}$ is measurable for all $a > 0$.

$g_n \overset{\text{def}}{=} f^{\frac{1}{n}} - f_{\frac{1}{n}}$ .... is lower semi continuous.

g_n decreases as $n \nearrow \infty$.

$g_n \downarrow f^0 - f_0$. So $-g_n \nearrow f_0 - f^0$.

$\left(\bigcup_{n=1}^{\infty} \{-g_n > -a\}\right)^c = \bigcap_{n=1}^{\infty} \{a \leq g_n\}$

$\{f_0 - f^0 > -a\}^c$

$\{f^0 - f_0 < a\}^c$

$\{f^0 - f_0 \geq a\}$

(We prove $\{g_n \geq a\}$ is measurable)

PAGE

DATE

Since $\{g_n \leq a\}$ is closed ($\because g_n \dots$ l.s.c)

$\{g_n > a\}$ is open ($\rightarrow$ measurable)

$\{g_n > a - \frac{1}{k}\}$ is open ($\Rightarrow$ measurable)

$\bigcap_{k=1}^{\infty} \{g_n > a - \frac{1}{k}\} = \{g_n \geq a\}$ is measurable

Now the proof is complete.

[17] $\{M_t\}_{t\in T}$ ... family of functions

$^\forall t\in T$ ... M_t satisfies :

$\left(\begin{array}{l}\cdot\ M_t \text{ contains } \mathcal{F}\text{-measurable simple functions} \\ \cdot\ ^\forall f_n \in M_t \quad \text{if } f_n \to f \text{ then } f \in M_t\end{array}\right.$

Suppose f is a $\mathcal{F}$-measurable function

We prove that $f \in M_t$ for all $t\in T$.

There exists f_n^+, f_n^- : non-negative

$\mathcal{F}$-measurable functions s.t. $f_n^+ \nearrow f^+$, $f_n^- \nearrow f^-$

$f_n \overset{def}{=} f_n^+ - f_n^-$ f_n is also a $\mathcal{F}$-measurable

simple function And $f_n \to f^+ - f^- = f$

Since $\{f_n\} \subset M_t$ $(^\forall t\in T)$ Thus $f \in M_t$ $(^\forall t\in T)$

$(\because f_n \to f)$

So the proof is complete

· $\boxed{\text{Note}}$ Let f : be a non-negative $\mathcal{F}$-measurable function.

$$f_n \overset{def}{=} \min\{n, \frac{[2^n f(x)]}{2^n}\}$$

Then f_n : non-negative measurable simple function

$f_n \uparrow f$.

(proof) $g_n(x) \overset{def}{=} \frac{[2^n f(x)]}{2^n}$

$$0 \le f(x) - g_n(x) \le \frac{1}{2^n}$$

$$\therefore \lim_{n\to\infty} g_n(x) = f(x)$$

$$\therefore \lim_{n\to\infty} f_n(x) = \min\{\infty, \lim_{n\to\infty} \frac{[2^n f(x)]}{2^n}\} = f(x)$$

$$f_n \le f_{n+1} \quad (\because \ g_n(x) \le g_{n+1}(x))$$

$$(\because 2[a] \le [2a])$$

· So $f_n \uparrow f$.

$$\left(f_n : \text{measurable} \ \because \ f_n = \sum_{k=0}^{2^n-1} \frac{k}{2^n} \cdot \mathbb{I}\{\frac{k}{2^n} \le f(x) < \frac{k+1}{2^n}\} + n \cdot \mathbb{I}\{f(x) \ge n\} \right)$$