Selected Exercises and Solutions of Probability Theory

Edited by Toshinari Morimoto

This document was created by a student of National Taiwan University when he was taking a course in Probability Theory. Most of the questions are from R.Durrett's Textbook [1] and were selected by our teacher. Please make use of this note to prepare for examinations.

Chapter 1. Review of Measure Theory

1. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. State and prove the following properties of a measure.

(1) monotonicity

(2) subadditivity

(3) continuity from above

(4) continuity from below

2. (Theorem 1.2.1) Let $(\Omega, \mathcal{F}, P)$ be a proability space and let X be a random variable on it. Let $F(x)$ be a distribution function of X. Prove the following properties.

(1) F is non-decreasing

(2) $\lim_{x\to\infty} F(x) = 1$ and $\lim_{x\to-\infty} F(x) = 0$

(3) F is right-continuous

(4) If $F(x-) = \lim_{y\nearrow x} F(y)$, then $F(x-) = P(X < x)$

(5) $P(X = x) = F(x) - F(x-)$

3. (Theorem 1.2.2) Suppose F satisfies 1,2, and 3 in Theorem 1.2.1. Prove that there exists a probability space and a random variable X whose distribution function is F.

4. (Theorem 1.2.6) Show that $x > 0 \Rightarrow (x^{-1} - x^{-3}) \exp(-x^2/2) \le \int_x^\infty \exp(-y^2/2)dy \le x^{-1}\exp(-x^2/2)$.

5.

(1) (Formula 1.2.1) State the definition of density function of a random variable X on a probability space $(\Omega, \mathcal{F}, \mu)$.

(2) (Example 1.2.7) Can a density function always exist? If no, give a counter example.

6. (Theorem 1.1.4) State a measure extension theorem associated with a Stieltjes measure function F on the Borel measurable space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. You may want to prove the following fact. Let $\mathcal{G} = \{(a,b] | a,b \in \mathbb{R}\}$ and $\mu((a,b]) = F(b) - F(a)(b > a)$ or $0(a \geq b)$. Then $(\mathcal{G}, \mu)$ has countable additivity.

7. (Theorem 2.1.6) State and prove the Dynkin's $\pi - \lambda$ theorem.

8. (Uniqueness of measure) Let $(\Omega, \mathcal{F})$ be a measurable space and let μ_1, μ_2 be a measure on it. Let $\mathcal{P}$ be a π-system and suppose $\mathcal{P} \subset \mathcal{F}$. Suppose $(\mathcal{P}, \mu_1)$ is σ-finite and $\mu_1 = \mu_2$ on $\mathcal{P}$. Show that $\mu_1 = \mu_2$ on $\sigma[\mathcal{P}]$.

9. (Lemma 1.1.7) Let $\sigma_0[\bullet]$ be a smallest algebra which contains $\bullet$. (We often call it an algebra generated from $\bullet$.) Let $\mathcal{G}$ be a semi-algebra. Give a family of sets which is equal to $\sigma_0[\mathcal{G}]$. Also show that they are equivalent.

10. (Thorem 1.3.1) Let X be a measurable map from $(\Omega, \mathcal{F})$ to $(A, \mathcal{A})$. Suppose $\mathcal{A} = \sigma[\mathcal{G}]$. Show that X is a $\mathcal{F}/\mathcal{A}$-measurable map $\Leftrightarrow \forall A \in \mathcal{A}\ X^{-1}(A) \in \mathcal{G}$. Here $X^{-1}(A) = \{\omega \in \Omega | X(\omega) \in A\}$ where $A \in \mathcal{B}(\mathbb{R})$.

11. (Theorem 1.3.4) Let X is a measurable map from $(\Omega, \mathcal{F})$ to $(S, \mathcal{A})$ and let f be a measurable map from $(S, \mathcal{A})$ to $(T, \mathcal{B})$. Then show that $f(X(\omega))$ is a measurable map from $(\Omega, \mathcal{F})$ to $(T, \mathcal{B})$.

12. (Theorem 1.3.7) Let $\{X_n\}_{n\in\mathbb{N}}$ be a series of random variables. Show that $\inf_{n\geq 1} X_n, \sup_{n\geq 1} X_n, \liminf_{n\to\infty} X_n, \limsup_{n\to\infty} X_n$ are also random variables. Discuss a set $\{\omega \in \Omega | \lim_{n\to\infty} X(\omega) \text{ exists}\}$ is measurable or not.

13. (Exercise 1.2.3) Show that a distribution function F has at most countably many discontinuities.

14. (Exercise 1.2.4) Show that if $F(x) = P(\{\omega | X \leq x\})$ is continuous then $Y = F(X)$ has a uniform distribution on $(0, 1)$.

15. (Exercise 1.3.5) Show that f is lower semi continuous $\Leftrightarrow \{x | f(x) \leq a\}$ is closed for each $a \in \mathbb{R}$. Finally conclude that semicontinuous functions are measurable.

16. (Exercise 1.3.6) Let $f : \mathbb{R}^d \to \mathbb{R}$ be an arbitrary function and let $f^\delta(x) = \sup\{f(y) \mid |y - x| < \delta\}$ $f_\delta(x) = \inf\{f(y) \mid |y - x| < \delta\}$.

(1) Show that f^δ, f_δ are lower semi continuous and upper semi continuous respectively.

(2) Let $f^0 = \lim_{\delta\searrow 0} f^\delta, f_0 = \lim_{\delta\searrow 0} f_\delta$. Show that the set of points at which f is discontinuous $= \{f^0 \neq f_0\}$ is measurable.

17. (Exercise 1.3.7) A function $\phi : \Omega \to \mathbb{R}$ is said to be simple if $\phi(\omega) = \sum_{m=1}^n c_m \mathbb{I}_{A_m}(\omega)$ where $\{c_m\}_{m=1}^n \subset \mathbb{R}$ and $\{A_m\}_{m=1}^n \subset \mathcal{F}$. Show that $\mathcal{F}-$measurable functions is the smallest class containing the simple functions and closed under pointwise limits.

18. (Exercise 1.3.8) Use Exercise 1.3.7 and show that Y is measurable with respect to $\sigma(X)$ if and only if there exists a Borel measurable function $f : \mathbb{R} \to \mathbb{R}$ such that $Y = f(X)$.

19. (Exercise 1.3.9) To get a constructive proof of the last result note that $\{\omega : m2^{-2} \leq Y < (m+1)2^{-n}\} = \{X \in B_{m,n}\}$ for some $B_{m,n} \in \mathcal{B}(\mathbb{R})$ and set $f_n = m2^{-n}$ for $x \in B_{m,n}$ and show that as $n \to \infty$ $f_n(x) \to f(x)$ and $Y = f(X)$.

20. (Exercise 1.4.4) Prove the Riemann-Lebesgue lemma. If g is integrable then $\lim_{n\to\infty} \int g(x) \cos nx dx = 0$. In order to prove this statement, it might be helpful for you to prove the following fact. Suppose $\mathcal{G}$ be a semi-algebra and $(\mathcal{G}, \mu_0)$ is non-negative $\mu_0(\phi) = 0$ and has finite additivity. Let μ be an outer measure derived from $(\mathcal{G}, \mu_0)$. Let $\sigma_0[\mathcal{G}]$ be an smallest algebra containing $\mathcal{G}$. Show that $\forall \epsilon > 0$ and $A \in \sigma[\mathcal{G}]$ with $\mu(A) < \infty$, $\exists B \in \sigma_0[\mathcal{G}]$ such that $\mu(A \Delta B) < \epsilon$.

21. (Theorem 1.5.1 Jensen's Inequality.)

(1) State the definition of a function ϕ being convex.

(2) State and prove Jensen's Inequality.

22. (Theorem 1.5.2) State and prove Hölder's Inequality.

23. (Theorem 1.5.7) State and prove Monotone Convergence Theorem.

24. (Theorem 1.5.5) State and prove Fatou's lemma.

25. (Theorem 1.5.7) State and prove Lebesgue's dominated convergence theorem.

26. (Exercise 1.5.7) Let $f \geq 0$ be an integrable funtion. Show that for all $\epsilon > 0$, there exists a positive number δ such that $\int_A f d\mu < \epsilon (\forall A \in \mathcal{F} : \mu(A) < \delta)$.

27. (Theorem 1.6.4) State and prove Chebyshev's (or Markov's) inequality.

28. (Theorem 1.6.8) Suppose $X_n \to X$ a.s.. Let g, h be continuous functions with the following properties.

(1) $g \geq 0$ and $g(x) \to \infty$ as $|x| \to \infty$.

(2) $|h(x)|/g(x) \to 0$ as $|x| \to \infty$.

(3) $E[g(X_n)] \leq K < \infty$ for all $n \in \mathbb{N}$.

Then $E[h(X_n)] \to E[h(X)]$.

29. (Theorem 1.6.9 Change Variable Formula) Let X be a random element of $(S, \mathcal{S})$ with distribution μ. (i.e. $\mu(A) = P(X \in A)$.) If f is a measurable function from $(S, \mathcal{S})$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ so that $f \geq 0$ or $E[f(X)] < \infty$, then $E[f(X)] = \int_S f(y)\mu(dy)$.

30. (Exercise 1.6.6) Let $Y \geq 0$ with $E[Y^2] < \infty$. Show that $P(Y > 0) \geq (E[Y]^2)/E[Y^2]$. (hint: you may use Cauch Shwartz's inequality.)

31. (Exercise 1.6.9 Inclusion Exclusin Formula) Show that $P(\cup_{i=1}^n A_i) = \sum_{i=1}^n P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cup A_k) + ...$ (hint: you may first show $\mathbb{I}_A = 1 - \prod_{i=1}^n (1 - \mathbb{I}_{A_i})$.)

32. (Exercise 1.6.10 Bonferroni inequalities)

(1) $P(\cup_{i=1}^n A_i) \leq \sum_{i=1}^n P(A_i)$

(2) $P(\cup_{i=1}^n A_i) \geq \sum_{i=1}^n P(A_i) - \sum_{i<j} P(A_i \cap A_j)$

(3) $P(\cup_{i=1}^n A_i) \leq \sum_{i=1}^n P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k)$

33. (Exercise 1.6.14) Let $X \geq 0$ and $E[1/X] \leq \infty$. Show the following statements.

(1) $\lim_{y\to\infty} yE[1/X\mathbb{I}_{\{X>y\}}] = 0$

(2) $\lim_{y\searrow+0} yE[1/X\mathbb{I}_{\{X>y\}}] = 0$

34. (Theorem 1.7.1) Consider two σ-finite measure spaces $(X, \mathcal{A}, \mu_1)$ and $(Y, \mathcal{B}, \mu_2)$. Let $\mathcal{F} = \mathcal{A} \otimes \mathcal{B}$ be a product measurable space of $\mathcal{A}, \mathcal{B}$. Show that there exists a unique measure μ on $(X \times Y, \mathcal{A} \otimes \mathcal{B})$ satisfying $\mu = \mu_1 \times \mu_2$ on $\mathcal{S} = \mathcal{A} \times \mathcal{B}$ (rectangles).

35. (Theorem 1.7.2) State (and prove) Fubini's theorem.

36. (Completion of measure space) Let $(\Omega, \mathcal{F}, \mu)$ be a measure space.

(1) Define its completion $(\Omega, \tilde{\mathcal{F}}, \tilde{\mu})$.

(2) Show that $\tilde{\mathcal{F}}$ is a σ-algebra.

(3) Show that $\tilde{\mu}$ is well-defined.

(4) Show that $(\Omega, \tilde{\mathcal{F}}, \tilde{\mu})$ is a measure space

(5) Show that $(\Omega, \tilde{\mathcal{F}}, \tilde{\mu})$ is complete.

Chapter 2. Law of Large Numbers

1. (Definition) State the definition of independence for the following items.

(1) Events $A, B \in \mathcal{F}$

(2) Random variables X, Y on $\mathcal{F}$

(3) σ-algebras $\mathcal{F}, \mathcal{G}$

(4) σ-algebras $\mathcal{F}_1, ..., \mathcal{F}_n$

(5) Random variables $X_1, ..., X_n$

(6) Events $A_1, ..., A_n \in \mathcal{F}$

2. (Theorem 2.1.1)

(1) Show that random variables X, Y are independent $\Rightarrow \sigma[X], \sigma[Y]$ are independent.

(2) Suppose that sub $\sigma-$algebras $\mathcal{F}_1, \mathcal{F}_2$ are independent and X, Y are $\mathcal{F}_1$-measurable and $\mathcal{F}_2$-measurable. Show that X, Y are independent.

3. (Theorem 2.1.2)

(1) Two events A, B are independent. Show that $A^c, B,\ A, B^c,\ A^c, B^c$ are also independent.

(2) Show that A, B are independent $\Leftrightarrow \mathbb{I}_A, \mathbb{I}_B$ are independent.

4. (Theorem 2.1.3) Show that independence of $A_1, ..., A_n$ implies independence of $\mathbb{I}_{A_1}, ...\mathbb{I}_{A_2}$. (hint: You may first consider the independence of $A_1^c, A_2, ..., A_n$.)

5. (Example 2.1.4) Show that pairwise independence does not imply independence by giving an example.

6. (Theorem 2.1.7) Suppose that $\mathcal{A}_1, ...\mathcal{A}_n$ are independent families of events and are π-systems. Show that $\sigma[\mathcal{A}_1], ..., \sigma[\mathcal{A}_n]$ are also independent.

7. (Theorem 2.1.8) Show that $P(X_1 \leq x_1, \cdots, X_n \leq x_n) = \prod_{i=1}^n P(X_i \leq x_i)$ implies independence of $X_1, ..., X_n$.

8. (Theorem 2.1.9) Suppose that $\{\mathcal{F}_{i,j}\}_{i=1,2...n;j=1,2...m_i}$ are independent of each other and are sub σ-algebras of $\mathcal{F}$. Let $\mathcal{G}_i = \sigma(\cup_{j=1}^{m_i} \mathcal{F}_{i,j})$. Show that $\mathcal{G}_1, ...\mathcal{G}_n$ are independent.

9. (Theorem 2.1.10) Suppose that random variables $\{X_{i,j}\}_{i=1,2...n;j=1,2...m_i}$ on a probability space $(\Omega, \mathcal{F}, P)$ are independent. Let f_i be a Borel measurable function $\mathbb{R}^{m_i} \to \mathbb{R}$. Show that random variables $\{f_i(X_{i,1}(\omega), ..., X_{i,m_i}(\omega))\}(i = 1, 2, ...n)$ are independent.

10. (Theorem 2.1.11) Suppose $X_1, X_2, ...X_n$ are mutually independent. Each random variable has distribution $\mu_i((-\infty, x]) = P(X_i \leq x)$. Show that a random vector $(X_1, ..., X_n)$ has a distribution of $\mu_1 \times ... \times \mu_n$, where $\mu_1 \times ... \times \mu_n$ is a measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ which satiefies $\mu_1 \times ... \times \mu_n((-\infty, x_1] \times ... \times (-\infty, x_n]) = \prod_{i=1}^n \mu_i((-\infty, x_i])$ for all $x_1, ..., x_n \in \mathbb{R}$.

11. (Theorem 2.1.12) Suppoose X, Y are independent and have distributions μ, ν. Let h be a borel measurable function from $\mathbb{R}^2$ to $\mathbb{R}^1$ which is non-negative or satisfies $E[|h|(X, Y)] < \infty$. Show that $E[h(X, Y)] = \int_{\mathbb{R}} \int_{\mathbb{R}} h(x, y)\mu(dx)\nu(dy)$.

12. (Example 2.1.4) Show that $E[XY] = E[X]E[Y]$ (uncorrelated) does not imply independence by giving a counter example.

13. (Theorem 2.1.15) If X, Y are independent and have distribution function F, G respectively. Show that $P(X + Y \leq z) = \int F(z - y)dG(y)$.

14. (Theorem 2.1.16) Suppose that X with density f and Y with distribution function G are independent. Then $X + Y$ has density $h(x) = \int f(x - y)dG(y)$. Prove this statement.

15. (Theorem 2.1.21) Prove the Kolmogorov's extension theorem. Suppose we are given probability measure μ_n on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. Each of them is consistent, that is, $\mu_{n+1}((a_1, b_1] \times ... \times (a_n, b_n] \times \mathbb{R}) = \mu_n((a_1, b_1] \times ... \times (a_n, b_n])$. Show that there is a unique measure on $\mathbb{R}^{\mathbb{N}}, \mathcal{B}(\mathbb{R}^{\mathbb{N}})$ with $P(\{\omega | \omega_i \in (a_i, b_i], 1 \leq i \leq n\}) = \mu_n((a_1, b_1] \times ...(a_n, b_n])$.

16. (Additional Theorem) In the proof of Theorem 2.1.21, it might be helpful to show the following facts. Let $\mathcal{R}$ be a ring (an algebra is also a ring), and let μ be a set function with finite additivity on $\mathcal{R}$. Then, (a) μ has countable additivity $\Leftrightarrow$ (b) μ has sub-countable additivity $\Leftrightarrow$ (c) μ is continuous from below $\Rightarrow$ (d) μ is continuous from abobe $\Leftarrow$ (e) $\{A_n\} : \mu(A_1) < \infty, A_n \searrow \emptyset$, then $\lim_{n\to} \mu(A_n) = 0$. Especially when μ is a finite, then (e) $\Rightarrow$ (a).

17. (Theorem 2.1.22)

(1) State the definition for a measurable space $(S, \mathcal{S})$ to be nice.

(2) If S is a Borel subset of a complete separable metric space M, and $\mathcal{S}$ is the collection of Borel subsets of S. Show that $(S, \mathcal{S})$ is nice.

18. (Exercise 2.1.3) Let $\rho(x, y)$ be a metric. Suppose h is differentiable with $h(0) = 0, h'(x) > 0 (x > 0)$ and $h'(x)$ is decreasing on $[0, \infty)$. Show that $h(\rho(x, y))$ is a metric. Show that $h(x) = x/(x + 1)$ satisfies the conditions.

19. (Exercise 2.1.4) Let $\Omega = (0, 1), \mathcal{F} = \mathcal{B}((0, 1)), P = \mu, X_n(\omega) = sin(2\pi n\omega)$. Show that $X_n(\omega)$ are uncorrelated but not independent.

20. (Exercise 2.1.5) Show that if X, Y are independent with distributions μ, ν then $P(X+Y = 0) = \sum_y \mu(\{-y\})\nu(\{y\})$. Also conclude that if X has continuous distribution $P(X = Y) = 0$. (hint) You should be careful of the fact that the number of discontinuous points are at most countable.

21. (Exercise 2.1.15) If we want an infinite sequence of coin tossings, we do not have to use Kolmogorov's theorem. Let Ω be the unit interval $(0, 1)$ equipped with the Borel sets $\mathcal{F}$ and Lebesgue measure P. Let $Y_n(\omega) = 1$ if $[2^n\omega]$ is odd and $= 0$ if $[2^n\omega]$ is even. Show that $Y_1, Y_2....$ are independent with $P(Y_k = 0) = P(Y_k = 1) = 1/2$.

22. (Theorem 2.2.1) State and prove the finite additivity of variance, that is, $V[X_1 + ... + V_n] = \sum_{i=1}^{n} V[X_i]$ when $X_i's$ are uncorrelated.

23. (Lemma 2.2.1) Give an sufficient condition for Z_n to converge in probability.

(hint: You may think of the relationship between L^p convergence and converge in measure.)

24. (Example 2.2.4) Prove the Weierstrass approximation via Bernstein Polynomial. $\lim_{n\to\infty} \sup_{x\in[0,1]} |f(x) - f_n(x)| = 0$ where f_n is Bernstein polynomial of degree n.

25. (Theorem 2.2.6) Let $\mu = E[S_n], \sigma_n^2 = V[S_n]$. If $\sigma_n^2/b_n^2 \to 0$, then $(S_n - \mu_n)/b_n \xrightarrow{P} 0$.

26. (Example 2.2.7) State the Coupon Collector's problem. You must verify the number of total trials T_n has $T_n/n\log(n) \xrightarrow{P} 1$.

27. (Theorem 2.2.11) Prove week law for triangular arrays. Let $X_{n,k}(1 \le k \le n)$ be independent. Let $b_n \to \infty$ and let $\tilde{X_{n,k}} = X_{n,k}\mathbb{I}_{\{|X_{n,k}| \le b_n\}}$. Let $a_n = E[\sum_{k=1}^n \tilde{X}_{n,k}]$. First give two sufficient conditions for $(S_n - a_n)/b_n \xrightarrow{P} 0$. And also show that under the conditions it will converge to 0 in probability.

28. (Theorem 2.2.12 Week law of large numbers) Let $X_1, X_2 ... X_n$ be i.i.d random variables. Suppose $xP(|X_i| > x) \to 0$ as $x \to \infty$. Let $S_n = \sum_{i=1}^n X_i$ and $\mu_n = E[X_1\mathbb{I}_{\{|X_1| \le n\}}]$. Show that $S_n/n - \mu_n \xrightarrow{P} 0$.

29. (Theorem 2.2.14) Let $X_1, X_2 ... X_n$ be i.i.d random variables with $E|X_i| < \infty$.. Show that $S_n/n \xrightarrow{P} \mu$.

30. (Exercise 2.2.1) Let $X_1, X_2 ...$ be uncorrelated with $E[X_i] = \mu_i$ and $V[X_i]/i \to 0$ as $i \to \infty$. Let $S_n = X_1 + ... + X_n$ and $\nu_n = E[S_n]/n$ then as $n \to \infty$, $S_n/n - \nu_n \xrightarrow{L^2} 0$ thus also in probability.

31. (Exercise 2.2.2) The L^2 week law generalizes immediately to cetertain dependent sequences. Suppose $E[X_n] = 0$ and $E[X_nX_m] \le r(n-m)(n \ge m)$ with $r(k) \to 0$ as $k \to \infty$. ($r(k)$ is a function of $k \in \mathbb{N}$) Show that $(X_1 + ... X_n)/n \xrightarrow{P} 0$.

32. (Exercise 2.2.3 Monte Carlo integration) Let f be a measurable function on $[0, 1]$ with $\int_0^1 |f(x)|dx < \infty$. Let $U_1, ...$ be independent and uniformly distributed on $[0, 1]$ and let $I_n = n^{-1}(f(U_1) + f(U_2) + ... f(U_n))$. Show that $I_n \xrightarrow{P} I := \int_0^1 f dx$. Moreover, we suppose $\int_0^1 |f(x)|^2 dx < \infty$. Then use Chevyshev's inequality to estimate $P(|I_n - I| > a/n^{1/2})$.

33. (Exercise 2.2.4) Let $X_1, X_2 ...$be iid with $P(X_i = (-1)^k k) = C/k^2 \log(k)(k \ge 2)$ where C is a constatnt to make the sum of the probabilities equals to 1. Show that $E|X_i| = \infty$, but there is a finite constatnt μ so that $S_n/n \xrightarrow{P} \mu$.

34. (Exercise 2.2.5) Let $X_1, X_2 ...$ be iid with $P(X_i > x) = e/x\log(x)$ for $x \ge e$. Show that $E|X_i| = \infty$ but there is a sequence of constants $\mu_n \to \infty$ so that $S_n/n - \mu_n \xrightarrow{P}$

0.

35. (Exercise 2.2.6) Show that when X is a non-negative integer valued random variable. Show that $E[X] = \sum_{n\geq 1} P(X \geq n)$. Find a similar expression for $E[X^2]$.

36. (Exercise 2.2.7) Generalize Lemma 2.2.13 to conlude that if $H(x)0 \int_{(-\infty,x]} h(y)dy$ with $h(y) \geq 0$, then $E[H(X)] = \int_{\mathbb{R}} h(y)P(X \geq y)dy$. An important special case is $H(x) = \exp(\theta x)$ with $\theta > 0$.

37. (Ch2.3) Give an equivalent statement with $X_n \xrightarrow{a.s} 0$. Show that these two statements are equivalent.

38. (Theorem 2.3.1 Borel-Cantelli's Lemma I) State the First Borel-Cantelli's lemma and prove it.

39. (Theorem 2.3.2) Give an equivalent statement with $X_n \xrightarrow{P} X$ and prove that they are equivalent. If possible, give an equivalent statement with $f_n \xrightarrow{\mu} f$ and prove that they are equivalent. (μ is not necessarily a finite measure.)

40. (Theorem 2.3.3) Give an equivalent statement with $y_n \to y$ on a topological space and prove that they are equivalent.

41. (Theorem 2.3.4) First recall the bouded convergence theorem. When $X_n \xrightarrow{a.s} X$ and $\sup_{n\geq 1} |X_n| < \infty$, $\lim_{n\to\infty} E[X_n] = E[X]$. Show that $X_n \xrightarrow{a.s} X$ can be replaced with $X_n \xrightarrow{P} X$.

42. (Theorem 2.3.5 SLLN with finite fourth moment) Suppose $X_1, X_2, ...X_n...$ are identically independently distributed with $E[X_i] = \mu, E[X_i^4] < \infty$. Show that $S_n/n \xrightarrow{a.s} \mu$ where $S_n = \sum_{j=1}^n X_j$.

43. (Theorem 2.3.7 Borel-Cantelli's lemma II) State the Second Botel-Cantelli's lemma and prove it.

44. (Theorem 2.3.8) Suppose that $X_1, ...X_n$ are iid with $E[|X_1|] = \infty$. Show the following statements.

(1) $P(|X_n| \geq n \ i.o) = 1$

(2) $P(\lim_{n\to} S_n/n \text{ exists in } \mathbb{R}) = 0$

45. (Theorem 2.3.9) Suppose that $A_1, A_2, ...$ are pairwise independent and $\sum_{n=1}^\infty P(A_n) = \infty$. Show that $\sum_{k=1}^n \mathbb{I}_{A_k}(\omega)/\sum_{k=1}^n P(A_k) \xrightarrow{a.s} 1$.

46. (Example 2.3.12 Head runs) Let $X_n, n \in \mathbb{Z}$ be iid with $P(X_n = 1) = P(X_n = -1) = 1/2$. Let $l_n = \max\{m|X_{n-m+1} = X_{n-m+2} \cdots = X_n = 1\}$ be the longest run at time n and let $L_n = \max_{1\leq m\leq n}\{l_m\}$. Now show that $L_n/\log_2(n) \xrightarrow{a.s} 1$.

47. (Exercise 2.3.4) In Theorem 2.3.4, we have alreday shown that in bounded convergence theorem, the almost surely convergence can be replaced with convergence in probability. How about Fatou's lemma? Now suppose $X_n \geq 0$ and $X_n \xrightarrow{P} X$ and state and prove the alternative version of Fatou's lemma.

48. (Exercise 2.3.5 Lebesgue's Dominated Convergence Theorem)

(1) Similaly, state and prove the Lebesgue's Dominated Convergence Theorem with convergence probability.

(2) In Theorem 1.6.8, we supposed that $X_n \xrightarrow{a.s} X$. Show that this condition also can be replaced with $X_n \xrightarrow{P} X$.

49. (Additional Theorem) In Exercise 2.3.5, it might be helpful for you to show the following lemma. If $X_n \xrightarrow{P} X$, then for any positive number $p \in (0, \infty)$ and $M \in C(|X|)$ (set of continuous points of cdf of $|X|$), $|X_n|^p \mathbb{I}_{\{|X_n| \leq M\}} \xrightarrow{P} |X|^p \mathbb{I}_{\{|X| \leq M\}}$

50. (Exercise 2.3.6 Metric for convergence in probability) Consider $(\mathcal{X}, d), d : \mathcal{X}^2 \to [0, \infty]$ where $\mathcal{X}$ is set of random variables on $(\Omega, \mathcal{F}, P)$. We define $d(X, Y) = E[\frac{|X-Y|}{1+|X-Y|}]$.

(1) Show that $(\mathcal{X}, d)$ is a matric space. .

(2) Show that $d(X_n, X) \to 0 (n \to \infty) \Leftrightarrow X_n \xrightarrow{P} X$.

51. (Exercise 2.3.7 Completeness of a metric space) Show that the metric space defined in the previous question is complete. Completeness means: when $\{X_n\}_{n\geq 1}$ is a Cauchy sequence on $(\mathcal{X}, d)$, there always exists $X \in \mathcal{X}$ where $d(X_n, X) \to 0$ $(asn \to \infty)$.

52. (Exercise 2.3.8) Suppose that $\{A_n\}$ are independent with $P(A_n) < 1$. Show that $P(\cup A_n) = 1$ implies $\sum_{n\geq 1} P(A_n) = \infty$.

53. (Exercise 2.3.9) Show the following statements.

(1) $P(A_n) \to 0$ and $\sum_{n=1}^{\infty} P(A_{n+1} \setminus A_n) < \infty$, then $P(A_n \ i.o) = 0$

(2) Find an example of a sequence A_n to which the result of the previous question is applied but the Borel-Cantelli's lemma cannot.

54. (Exercise 2.3.11) Let $X_1, X_2 ...$ be independent with $P(X_n = 1) = p_n$ and $P(X_n = 0) = 1 - p_n$. Show the following statements.

(1) $X_n \xrightarrow{P} 0 \Leftrightarrow p_n \to 0$

(2) $X_n \xrightarrow{a.s} 0 \Leftrightarrow \sum_{n=1}^{\infty} p_n < \infty$

55. (Exercise 2.3.13) If X_n is any sequence of random variables, there are constants $c_n \to \infty$ so that $X_n/c_n \to 0 (a.s)$.

56. (Exercise 2.3.14) Let $X_1, X_2, ...$ be independent. Show that $\sup_{n\geq 1} X_n < \infty (a.s) \Leftrightarrow \sum_{n\geq 1} P(X_n > A) < \infty$ for some A.

57. (Exercise 2.3.15) Let $X_1, X_2 ...$ be iid with $P(X_i > x) = \exp(-x)$, let $M_n = \max_{1 \le m \le n} X_m$. Show the following statements.

(1) $\limsup_{n\to\infty} X_n/\log(n) = 1(a.s)$

(2) $M_n/\log(n) \xrightarrow{a.s} 1$

58. (Exercise 2.3.16) Let $X_1, X_2 ...$ be iid with distribution F, let $\lambda_n \nearrow \infty$ and let $A_n = \{\max_{1\le m\le n} X_m > \lambda_n\}$. Show that $P(A_n i.o) = 0, 1$ according as $\sum_{n\ge1}(1 - F(\lambda_n)) < \infty, = \infty$. (hint) You may first show that $\{A_n i.o\} = \{X_n > \lambda_n i.o\}$. Try to put $r_{n_1} = \min\{m | X_m > \lambda_{n_1}\}$.

59. (Exercise 2.3.17) Let $Y_1, Y_2 ...$ be iid. Find necessary and sufficient conditions for the following statements. (hint) $\limsup_{n\to\infty}(\max_{1\le m\le n} Y_m)/n = \limsup_{n\to\infty} Y_n^+/n$. Also, $\liminf_{n\to\infty}(\max_{1\le m\le n} Y_m)/n \ge 0$.

(1) $Y_n/n \xrightarrow{a.s} 0$

(2) $(\max_{1\le m\le n} Y_m)/n \xrightarrow{a.s} 0$

(3) $(\max_{1\le m\le n} Y_m)/n \xrightarrow{P} 0$

(4) $Y_n/n \xrightarrow{P} 0$

60. (Additional Theorem) Suppose $b_n \ge 0$ and $b_n \nearrow \infty$. Let $M_n = \max\{a_1, a_2, \cdots, a_n\}$. Show that $\limsup_{n\to\infty} M_n/b_n = \limsup_{n\to\infty} a_n^+/b_n$.

61. (Exercise 2.3.19) Let X_n be independent Poisson random variables with $E[X_n] = \lambda_n$ and let $S_n = \sum_{m=1}^n X_m$. Show that if $\sum_{n\ge1}\lambda_n = \infty$ then $S_n/E[S_n] \xrightarrow{a.s} 1$.

62. Prove the following lemmas.

(1) (Lemma 2.4.2) In Theorem 2.4.1, let $Y_k(\omega) = X_k \mathbb{I}_{\{|X_k|\le k\}}(\omega)$ and $T_n = Y_1 + \cdots + Y_n$. Show that it is sufficient to prove $T_n/n \to \mu(a.s)$.

(2) (Lemma 2.4.3) In Lemma 2.4.2, show that $\sum_{k=1}^\infty V[Y_k]/k^2 \le 4E|X_1| < \infty$.

(3) (Lemma 2.4.4) Show that if $y > 0$, then $2y \sum_{k>y} k^{-2} \le 4$.

63. (Theorem 2.4.1 Strong law of large numbers) Let $X_1, X_2, \cdots$ be pairwise independent identically distributed variables with $E[X_i] < \infty$. Let $E[X_i] = \mu$ and $S_n = X_1 + \cdots + X_n$. Show that $S_n/n \to \mu(a.s)$ as $n \to \infty$.

64. (Theorem 2.4.5) Let X_1, X_2 be iid with $E[X_1^+] = \infty, E[X_1^-] < \infty$. Show that $S_n/n \to \infty$.

65. (Example 2.4.6 and Theorem 2.4.7) Let $X_1, X_2 ...$ be iid with $0 < X_i < \infty$. Let $T_n = X_1 + ... + X_n$. You may think that T_n is the time of $n - th$ occurence some event. Let $N_t = \sup\{n | T_n \le t\}$. Now suppose $E[X_1] = \mu \le \infty$. Show that when $t \to \infty$, $N_t/t \to 1/\mu$ (a.s)

66. (Example 2.5.7) Verify that the following events are in or not in tail σ-field.

(1) $\{\lim_{n\to\infty} S_n \text{ exists }\} \in \mathcal{T}$

(2) $\{\limsup_{n\to\infty} S_n > 0\} \notin \mathcal{T}$

(3) $\{\limsup_{n\to\infty} S_n/c_n > x\} \in \mathcal{T}$ if $c_n \to \infty$

67. (Theorem 2.5.3 Kolmogorov's 0-1 law) If $X_1, X_2...$ are independent and $A \in \mathcal{T}$ then $P(A) = 0, 1$.

68. (Theorem 2.5.5) State and Prove Kolmogorov's maximal inequality. (hint) Suppose $X_1, X_2, ...$ are independent and $E[X_i] = 0, V[X_i] < \infty, S_n = \sum_{k=1}^{n} X_k$. Then $P(\max_{1\le k\le n} |S_k| \ge x) \le x^{-2}\mathrm{V}[S_n]$.

69. (Theorem 2.5.6) State and Prove Kolmogorov's two series theorem. (hint) $X_1, X_2...$ are independent and $E[X_n] = 0$. If $\sum_{n=1}^{\infty} \mathrm{V}[X_n] < \infty$, then $\sum_{n=1}^{\infty} X_n(\omega)$ converges almost surely.

70. (Theorem 2.5.8) State and Prove Kolmogorov's three series theorem. (hint) Let $X_1, X_2...$ be independent and let $A > 0$. Let $Y_i = X_i \mathbb{I}_{\{|X_i|\le A\}}$. Give three necessary and sufficient conditions for $\sum_{n\in\mathbb{N}} X_n(\omega)$ to converge almost surely.

71. (Theorem 2.5.9) State and Prove Kronecker's lemma. (hint) $a_n \nearrow \infty$ and $\sum_{n\in\mathbb{N}} x_n/a_n$ converges. Then $a_n^{-1} \sum_{m=1}^{n} x_m \to 0$.

72. (Theorem 2.5.10) Give an alternative proof of SLLN using Kolmogorov's three series theorem and kronecker's lemma.

73. (Theorem 2.5.11) Let $X_1, X_2...$ be iid random variables with $E[X_i] = 0$ and $E[X_i^2] = \sigma^2 < \infty$. Let $S_n = X_1 + \cdots + X_n$. Show that for any positive number ϵ, we have $S_n/n^{1/2}(\log(n))^{1/2+\epsilon} \xrightarrow{a.s} 0$.

74. (Exercise 2.5.2) Let $p > 0$. If $S_n/n^{1/p} \to 0$ (a.s.) then $E|X_1|^p < \infty$. (This is converse of Theorem 2.5.12)

75. (Exercise 2.5.3) Let $X_1, X_2, ...$ be iid standard normals. Show that for any t, $\sum_{n=1}^{\infty} X_n \sin(n\pi t)/n$ converges almost surely.

76. (Exercise 2.5.5) Let $X_n \ge 0$ be independent for $n \ge 1$. Show that the following are equivalent.

(1) $\sum_{n=1}^{\infty} X_n < \infty$ (a.s)

(2) $\sum_{n=1}^{\infty} P(X_n > 1) + E[X_n \mathbb{I}_{\{X_n\le 1\}}] < \infty$

(3) $\sum_{n=1} E[X_n/(1 + X_n)] < \infty$.

77. (Exercise 2.5.6) Let $\phi(x) = x^2$ when $|x| \le 1$ and $= |x|$ when $|x| \ge 1$. Show that if $X_1, X_2, ...$ are independent with $E[X_n] = 0$ and $\sum_{n=1}^{\infty} E[\phi(X_n)] < \infty$, then $\sum_{n\ge1}^{\infty} X_n$

converges almost surely.

78. (Exercise 2.5.7) Let $\{X_n\}$ be independent random variables. Suppose $\sum_{n\geq 1}^{\infty} E|X_n|^{p(n)} < \infty$ where $0 < p(n) \leq 2$ for all n and $E[X_n] = 0$ when $p(n) > 1$. Show that $\sum_{n=1}^{\infty} X_n$ converges almost surely.

79. (Exercise 2.5.8) Let $X_1, X_2...$ be iid and not $\equiv 0$. Then radius of convergence of the power series $\sum_{n\geq 1} X_n(\omega)z^n$ is 1 a.s or 0 a.s according as $E[\log^+ |X_1|] < \infty$ or $= \infty$, where $\log^+(x) = \max\{\log(x), 0\}$. The radius of convergence is $r(\omega) = \sup\{c \geq 0 | \sum |X_n(\omega)|c^n < \infty\}$. (hint.) The radius of convergence is equal to $(\limsup_{n\to\infty} |X_n|^{1/n})^{-1}$.

80. (Exercise 2.5.9) Let $X_1, X_2, ...$ be independent and let $S_{m,n} = X_{m+1} + \cdots + X_n$. Show the following inequality. $P(\max_{m<j\leq n} |S_{m,j}| > 2a) \min_{m<k\leq n} P(|S_{k,n}| \leq a) \leq P(|S_{m,n}| > a)$.

81. (Exercise 2.5.10) Use the inequality in Exercise 2.5.9 and prove a theorem of P.Levy. Let $X_1, X_2...$ be independent and let $S_n = X_1 + \cdots X_n$. If $\lim_{n\to\infty} S_n$ exists in probability, then it also exists a.s.

82. (Exercise 2.5.11) Let $X_1, X_2...$ be iid and $S_n = X_1 + X_2 \cdots + X_n$. Use the inequality in Exercise 2.5.9 and prove that if $S_n/n \to 0$ in probability, then $\max_{1\leq m\leq n} S_m/n \to 0$ in probability.

83. (Exercise 2.5.12) Let $X_1, X_2...$ be iid ans $S_n = X_1 + \cdots X_n$. Supose $a(n) \nearrow \infty$ and $a(2^n)/a(2^{n-1})$ is bounded.

(1) Use the inequality in Exercise 2.5.9 and show that if $S_n/a(n) \xrightarrow{P} 0$ and $S_{2^n}/a(2^n) \xrightarrow{a.s} 0$ and then $S_n/a(n) \xrightarrow{a.s} 0$.

(2) Suppose in additiion that $E[X_1] = 0$ and $E[X_1^2] < \infty$. Use the previous exercise and Chebyshev's inequality to show that $S_n/n^{1/2}(\log_2 n)^{1/2+\epsilon} \xrightarrow{a.s} 0$.

Chapter 3. Central Limit Theorem

1. (Resnick. Lemma 8.1) A distribution function $F(x)$ is determined on a dense set. Let D be dense in $\mathbb{R}$. Suppose $F_D(\bullet)$ is defined on D and satisfies the folloinwg:

- $F_D(\bullet)$ is non-decreasing on D.
- $0 \leq F_D(x) \leq 1$ for $\forall x \in D$.
- $\lim_{x\in D, x\to+\infty} F_D(x) = 1, \lim_{x\in D, x\to-\infty} F_D(x) = 0$.

Define for all $x \in \mathbb{R}$, $F(x) := \inf_{y>x, y\in D} F_D(y) = \lim_{y\searrow x, y\in D} F_D(y)$. Show that F is a right continuous probability distribution function. Thus, any two right continuous distribution function's agreeing on a dense set will agree everywhere.

2. (Resnick. Four definitions) State the following four different types of convergence.

- vague convergence
- proper convergence
- weak convergence
- complete convergence

3. (Resnick. Example) Give an example of $F_n(x)$ which does not converge for any x thus whose weak convergence fails, however which vaguely converges.

4. (Resnick. Theorem 8.1.1: Equivalence of the Four Definitions) Show that if F is proper, then the four definitions above are all equivalent.

5. (Resnick. Example 8.1.1) Give an example of a sequence of random variables $\{X_n\}$ which converges in distribution but almost surely nor in probability.

6. (Resnick. 8.2) Regarding a sequence of distribution functions, state the definitions of two types of convergence which are stronger than weak convergence.

7. (Resnick. Example 8.2.1) Show that strong convergence is stronger than weak convergence by giving an example.

8. (Resnick. Lemma 8.2.1) Suppose that $\{F\}_{n\geq 1} \cup \{F\}$ are probability distribution functions with densities $\{f_n\}_{n\geq 1} \cup \{f\}$. Then show the following statements.

(1) $\sup_{B\in\mathcal{B}(\mathbb{R})} |F_n(B) - F(B)| = \frac{1}{2}\int |f_n(x) - f(x)|dx$

(2) In addition, if $f_n(x) \xrightarrow{a.e} f$, then $\int |f_n(x) - f(x)|dx \to 0$.

9. (Exercise 3.1.1) A triangle array of real numbers $c_{j,n}$ satisfies the following three conditions. Show that $\prod_{j=1}^{n}(1 + c_{j,n}) \to e^{\lambda}$.

- $\max_{1\leq j\leq n} |c_{j,n}| \to 0$
- $\sum_{j=1}^{n} c_{j,n} \to \lambda$
- $\sup_n \sum_{j=1}^{n} |c_{j,n}| < \infty$

10. (Theorem 3.2.8 Skorokhod's Representation Theorem) Suppose F_n, F are distribution functions and suppose $F_n \xrightarrow{w} F$. Show that there exists a probability space and random variables $Y_n \sim F_n, Y \sim F, Y_n \xrightarrow{a.s} Y$.

11. (Theorem 3.2.9) Give an equivalent statement with $X_n \xrightarrow{d} X$ involving a bounded continuous function g.

12. (Theorem 3.2.10 Continuous Mapping Theorem)

(1) State what is continuous mapping theorem.

(2) Prove continuous mapping theorem.

13. (Theorem 3.2.11) Fill in the blank so that the following statements are all equivalent.

- $X_n \xrightarrow{d} X$
- For all open sets G, (•) $\geq$ (•).
- For all closed sets K, (•) $\leq$ (•).
- For all borel sets A with (•), (•) = (•).

14. (Theorem 3.2.12 Helly's Selection Theorem) For every sequence $\{F_n\}_{n\geq 1}$ of distribution functions, there is a sub-sequence $F_{n(k)}$ and a right continuous non-decreasing function F (this is not necessarily a distribution function) so that $\lim_{k\to\infty} F_{n(k)} \to F$ at all continuity points of F.

15. (Theorem 3.2.13 An equivalent statement with a sequence of df being tight) Let $\{F_n\}_{n\geq 1}$ be a sequence of distribution functions.

(1) State the definition of $\{F_n\}$ being tight.

(2) Show that if a subsequence of $\{F_n\} : F_{n(k)} \xrightarrow{w} F$, then F is a distribution function $\Leftrightarrow$ $\{F_n\}$ is tight.

16. (Theorem 3.2.14) If there is a $\phi \geq 0$ so that $\phi(x) \to \infty$ as $|x| \to \infty$ and $C = \sup_{n\in\mathbb{N}} \int \phi(x) dF_n(x) < \infty$ then F_n is tight.

17. (Exercise 3.2.1) Give an example of random variables X_n with densities f_n so that $X_n \xrightarrow{d} U \sim \text{Uniform}(0,1)$ but $f_n(x)$ does not converge to 1 for any $x \in [0,1]$.

18. (Exercise 3.2.2 Convergence of maxima) Let $X_1, X_2, \cdots$ be independent with distribution F and let $M_n = \max\{X_1, \cdots X_n\}$. Then $P(M_n \leq x) = F(x)^n$. Prove the following statements.

(1) If $F(x) = 1 - x^{-\alpha}$ for $x \geq 1$ where $\alpha > 0$ then for $y > 0$ we have $P(M_n/n^{1/\alpha} \leq y) \to \exp(-y^{-\alpha})$.

(2) If $F(x) = 1 - |x|^{\beta}$ for $-1 \leq x \leq 0$ where $\beta > 0$ then for $y < 0$ we have $P(n^{1/\beta} M_n \leq y) \to \exp(-|y|^{\beta})$.

(3) If $F(x) = 1 - e^{-x}$ for $x \geq 0$ then for $\forall y \in \mathbb{R}$ we have $P(M_n - \log n \leq y) \to \exp(-e^{-y})$.

19. (Exercise 3.2.4 Fatou's Lemma) Let $g \geq 0$ be continuous. If $X_n \xrightarrow{d} X$, then $\liminf_{n\to\infty} E[g(X_n)] \geq E[g(X)]$.

20. (Exercise 3.2.5 Integration to the limit) Suppose g, h are continuous with $g(x) > 0$ and $|h(x)|/g(x) \to 0$ as $|x| \to \infty$. If $F_n \xrightarrow{w} F$ and $\int gdF_n(x) \leq C < \infty$ then $\int h(x)dF_n(x) \to \int h(x)dF(x)$.

21. (Exercise 3.2.9) If $F_n \xrightarrow{w} F$ and F is continuous then $\sup_x |F_n(x) - F(x)| \to 0$.

22. (Theorem 2.4.9) State and prove Glivenko Cantelli's lemma.

23. (Exercise 3.2.12) Show that $X_n \xrightarrow{P} X$ then $X_n \xrightarrow{d} X$ and conversely if $X_n \xrightarrow{d} c$, where c is a constant then $X_n \xrightarrow{P} c$.

24. (Exercise 3.2.13 Converting together lemma) If $X_n \xrightarrow{d} X, Y_n \xrightarrow{d} c$ where c is a constant then $X_n + Y_n \xrightarrow{d} X + c$. A useful consequence of this result is that if $X_n \xrightarrow{d} X$ and $Z_n - X_n \xrightarrow{d} 0$ then $Z_n \xrightarrow{d} X$.

25. (Exercise 3.2.14) Suppose $X_n \xrightarrow{d} X, Y_n \geq 0$ and $Y_n \xrightarrow{d} c$ where $c > 0$ is a constant. Then $X_nY_n \xrightarrow{d} cX$. This result is true without the assumptions that $Y_n \geq 0$ and $c > 0$.. We have imposed these only to make the proof less tedious. (Note) Actually you may use the conclusion of Exercise 3.2.13. If $X_nY_n - cX_n \xrightarrow{d} 0$ and $cX_n \xrightarrow{d} cX$ then $X_nY_n \xrightarrow{d} cX$. So you may first prove $X_nY_n - cX_n \xrightarrow{P} 0$.

26. (Exercise 3.2.16) Suppose $Y_n \geq 0$, $E[Y_n^\alpha] \to 1$ and $E[Y_n^\beta] \to 1$ for some $0 < \alpha < \beta$. Show that $Y_n \xrightarrow{P} 1$.

27. (Theorem 3.3.11 The inversion formula)

(1) The inversion formula states that $\lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{e^{-ita} - e^{-itb}}{it} \phi(t)dt = (\qquad \bullet \qquad)$. Fill in the blank in the equation.

(2) Prove the inversion formula.

28. (Additional Lemma) Show that $|\int_0^T \frac{\sin x}{x} - \frac{\pi}{2}| \leq \frac{T+1}{T^2}$ and $\sup_{y>0} S(y) < \infty$ where $S(T) = \int_0^T \frac{\sin(x)}{x} dx$. (hint) First you may rewrite $\int_0^T \frac{\sin x}{x} = \int_{x=0}^{x=T} \sin x dx \int_0^\infty e^{-xy} dy$. To verify that Fubini's theorem is applicable, you may use the fact that $|\sin x e^{-xy}| \leq xe^{-xy} (x, y > 0)$.

29. (Theorem 3.3.14) Show that if $\int |\phi(t)|dt < \infty$ then μ has bounded continuous density $f(y) = \frac{1}{2\pi} \int e^{-ity} \phi(t)dt$.

30. (Exercise 3.3.1) Show that ϕ is a chf then $\mathrm{Re}(\phi)$ and $|\phi|^2$ are also characteristic functions.

31. (Exercise 3.3.2) Answer the following questions.

(1) Imitate the proof of Theorem 3.3.11 to show that $\mu(\{a\}) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{-ita} \phi(t)dt$.

(2) Show that if $P(X \in h\mathbb{Z}) = 1$ where $h > 0$, then its chf has $\phi(2\pi/h + t) = \phi(t)$

so $P(X=x) = \frac{h}{2\pi}\int_{-\pi/h}^{\pi/h} e^{-itx}\phi(t)dt$ for $x \in h\mathbb{Z}$.

(3) Show that if $X+b$ then $E[\exp(itX)] = e^{itb}E[\exp(itY)]$. So if $P(X \in b + h\mathbb{Z}) = 1$, the inversion formula in (2) is valid for $x \in b + h\mathbb{Z}$.

32.

(1) (Example 3.3.10, 3.3.16) Find the characteristic function of Lhaplus distribution. Use Theorem 3.3.14 to find the characteristic function of Cauchy distribution.

(2) (Exercise 3.3.6) Use the result in Example 3.3.16 to conclude that if $X_1, X_2 \cdots$ are independent and have the Cauchy distribution, then $\bar{X_n}$ has the same distribution as X_1.

33. (Theorem 3.3.17 Continuity Theorem)

(1) Let $\{\mu_n\} \cup \{\mu\}$ be probability measures with chf $\phi_n(t)$ and $\phi(t)$. Show that if $\mu_n \xrightarrow{d} \mu$ then $\phi_n(t) \to \phi(t)$ for all $t \in \mathbb{R}$.

(2) Let $\{\mu_n\}$ be probability measures with chf $\phi_n(t)$. Suppose that $\phi_n(t) \to \phi(t)$ for all $t \in \mathbb{R}$ and suppose that $\phi(t)$ is continuous at $t = 0$. (This is not necessarily a characteristic function.) Show that $\{\mu_n\}$ is tight and $\mu_n \xrightarrow{d} \mu$ where μ has chf $\phi(t)$.

34. (Exercise 3.3.7) Suppose that $X_n \xrightarrow{d} X$ and X_n has a normal distribution with mean 0 and variances σ_n^2. Prove that $\sigma_n^2 \to \sigma^2 \in [0, \infty)$.

35. (Exercise 3.3.8) Show that if X_n and Y_n are independent for $1 \leq n \leq \infty$, $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{d} Y$, then $X_n + Y_n \xrightarrow{d} X + Y$.

36. (Lemma 3.3.19) Answer the following questions.

(1) Show $|e^{ix} - \sum_{m=0}^{n} \frac{(ix)^m}{m!}| \leq \min\{\frac{|x|^{n+1}}{(n+1)!}, \frac{2|x|^n}{n!}\}$.

(2) If you put $x = tX$, what result do you obtain?

37. (Theorem 3.3.18) Use Lemma 3.3.19 and prove if $\int_{\mathbb{R}} |x|^n \mu(dx) < \infty \Rightarrow$ then $\phi^{(n)}(t) = \int_{\mathbb{R}} (ix)^n e^{itx}\mu(dx)$. (hint) You do not have to use lemma in Appendix. You may find an upper bound of $|(\phi(t+h) - \phi(t))/h|$ which is not related with h by using Lemma 3.3.19. Finally use Lebesgue Dominated Convergence Theorem.

38. (Theorem 3.3.20) Suppose $E[X^2] < \infty$. Show that $\phi(t) = 1 + itE[X] - t^2E[X^2]/2 + o(t^2)$. (Note) $o(t^2)$ means $o(t^2)/t^2 \to 0$ as $t \to 0$. So you must prove that the error term $(\phi(t) - (1 + itE[X] - t^2E[X^2]/2))/t^2 \to 0$ as $t \to 0$.

39. (Theorem 3.3.21) Prove if $\limsup_{h \searrow 0}\{\phi(h) - 2\phi(0) + \phi(-h)\}/h^2 > -\infty$, then $E|X|^2 < \infty$.

40. (Exercise 3.3.12) Use Theorem 3.3.18 and the series expansion for $e^{-t^2/2}$ to show that the standard normal distribution has $E[X^{2n}] = (2n)!/2^n n!$.

41. (Exercise 3.3.13) Answer the following questions.

(1) Suppose that a family of measure $\{\mu_i\}_{i\in I}$ is tight. In other words, $\sup_{i\in I} \mu_i([-M, M]^c) \to 0$ as $M \to \infty$. Use (d) in Theorem 3.3.1 and (3.3.3) with $n = 0$ to show that their characteristic functions ϕ_i are (uniformly) equicontinuous.

(2) Suppose $\mu_n \xrightarrow{d} \mu$. Use Theorem 3.3.17 and equicontinuity to conclude that the $\phi_n(t) \to \phi(t)$ uniformly on compact sets.

42. (Exercise 3.3.14) Let $X_1, X_2 \cdots$ be iid with characteristic function $\phi(t)$ and let $S_n = X_1 + X_2 \cdots X_n$.

(1) Show that $\phi'(0) = ia$, then $S_n/n \xrightarrow{P} a$.

(2) Show that if $S_n/n \xrightarrow{P} a$, then $\phi(t/n)^n \to e^{ita}$ as $n \nearrow \infty$ $(n \in \mathbb{N})$.

(3) Use (2) and the uniform continuity established in (d) of Theorem 3.3.1 to show that $(\phi(h) - 1)/h \to -ia$ as $h \to 0$ through the positive reals. Thus the weak law holds if and only if $\phi'(0)$ exists.

43. (Exercise 3.3.16) Show that if $\lim_{t\searrow 0}(\phi(t) - 1)/t^2 = c > -\infty$ then $E[X] = 0$ and $E[X^2] = -2c < \infty$. In particular, if $\phi(t) = 1 + o(t^2)$ then $\phi(t) = 1$.

44. (Exercise 3.3.17) If Y_n are random variables with characteristic functions $\{\phi_n\}$ then $Y_n \xrightarrow{d} 0$ if and only if there is a $\delta > 0$ so that $\phi_n(t) \to 1$ for $|t| \le \delta$.

45. (Exercise 3.3.18) Let $X_1, X_2 \cdots$ be independent. Show that if $S_n = \sum_{m=1\cdots n} X_m$ converges in distribution, then it converges in probability.

46. (Theorem 3.4.1 Central Limit Theorem) Let $X_1, X_2 \cdots$ be iid with $E[X_i] = \mu, V[X_i] = \sigma^2 \in (0, \infty)$. If $S_n = X_1 + X_2 \cdots X_n$, then $(S_n - n\mu)/\sigma n^{1/2} \xrightarrow{d} \mathcal{N}(0, 1)$.

47. Prove the following lemma and theorem.

(1) (Lemma 3.4.3) Let $z_1, \cdots z_n$ and $w_1 \cdots w_n$ be complex numbers of modulus $\le \theta$. Show that $|\prod_{m=1}^n z_m - \prod_{m=1}^n w_m| \le \theta^{n-1} \sum_{m=1}^n |z_m - w_m|$.

(2) (Lemma 3.4.4) Show that if b is a complex number with $|b| \le 1$ then $|e^b - (1 + b)| \le |b|^2$.

(3) (Theorem 3.4.2) Show that $\{c_n\} \cup \{c\} \subset \mathbb{C}$ and $c_n \to c$ then $(1 + c_n/n)^n \to e^c$.

48. (Example 3.4.9) Though pairwise independent is good enough for the strong law of large numbers, however it is not good enough for the central limit theorem. Give an example to show this statement.

49. (Exercise 3.4.1) Suppose you roll a die 180 times. Use the normal approximation to estimate the probability you will get fewer than 25 sixes.

50. (Exercise 3.4.4) Let $X_1, X_2, \cdots$ be iid with $X_i \geq 0, E[X_i] = 1, V[X_i] = \sigma^2 \in (0, \infty)$. Show that $2(\sqrt{S_n} - \sqrt{n}) \xrightarrow{d} \mathcal{N}(0, \sigma^2)$.

51. (Exercise 3.4.5 self-normalized sums) Let $X_1, X_2 \cdots$ be iid with $E[X_i] = 0, E[X_i^2] = \sigma^2 \in (0, \infty)$. Show that $\sum_{m=1}^n X_m / (\sum_{m=1}^n X_m^2)^{1/2} \xrightarrow{d} \mathcal{N}(0, 1)$.

52. (Exercise 3.4.6) Let $X_1, X_2 \cdots$ be iid with $E[X_i] = 0, E[X_i^2] = \sigma^2 \in (0, \infty)$ and let $S_n = X_1 + X_2 \cdots X_n$. Let N_n be a sequence of non-negative integer-valued random ariables and a_n be a sequence of integers with $a_n \to \infty$ and $N_n / a_n \xrightarrow{P} 1$. Show that $S_{N_n} / \sigma \sqrt{a_n} \xrightarrow{d} \mathcal{N}(0, 1)$.

53. (Exercise 3.4.7 A central limit theorem in renewal theory) Let $Y_1, Y_2 \cdots$ be iid positive random variables with $E[Y_i] = \mu, V[Y_i] = \sigma^2 \in (0, \infty)$. Let $S_n = Y_1 + Y_2 \cdots Y_n$ and let $N_t = \sup\{m | S_m \leq t\}$. Apply the previous exercise to $X_i = Y_i - \mu$ to prove that as $t \to \infty$, $(\mu N_t - t)/(\sigma^2 t/\mu)^{1/2} \xrightarrow{d} \mathcal{N}(0, 1)$.

54. (Theorem 3.4.10 The Lindeberg-Feller theorem) Let $\{X_{n,m}\}_{\{m=1 \cdots n; n \in \mathbb{N}\}}$ be independent random variables with $E[X_{n,m}] = 0$. We suppose the following two conditions.

- $\sum_{m=1}^n E[X_{n,m}^2] \to \sigma^2 > 0$.
- For all $\epsilon > 0$, $\lim_{n \to \infty} \sum_{m=1}^n E[|X_{n,m}|^2 \cdot \mathbb{I}_{\{|X_{n,m}| > \epsilon\}}] = 0$.

Then $S_n = X_1 + X_2 \cdots X_n \xrightarrow{d} \mathcal{N}(0, \sigma^2)$.

55. (Example 3.4.11 Cycles in a random permutation and record values) Let $Y_1, Y_2 \cdots$ be independent with $P(Y_m = 1) = 1/m, P(Y_m = 0) = 1 - 1/m$. Show that $(S_n - \log(n))/(\log(n))^{1/2} \xrightarrow{d} \mathcal{N}(0, 1)$.

56. (Example 3.4.12) Prove the Kolmogorov's three series theorem by applying Lindeberg-Feller's theorem.

57. (Exercise 3.4.9) Let $X_1, X_2 \ldots$ be independent and let $S_n = \sum_{m=1}^n X_m$. Suppose $P(X_m = m) = P(X_m = -m) = m^{-2}/2 (m \geq 2)$ and $P(X_m = 1) = P(X_m = -1) = (1 - m^{-2})/2$. Show that $V[S_n]/n \to 2$ but $S_n / \sqrt{n} \xrightarrow{d} \mathcal{N}(0, 1)$. The trouble here is that $X_{n,m} = X_m / \sqrt{n}$ does not satisfy the second condition of Theorem 3.4.10.

58. (Exercise 3.4.10) Let $X_1, X_2 \ldots$ be independent and let $S_n = \sum_{m=1}^n X_m$. Show that if $|X_i| \leq M$ and $\sum_{n \geq 1} V[X_n] = \infty$,then $(S_n - E[S_n])/\sqrt{V[S_n]} \xrightarrow{d} \mathcal{N}(0, 1)$.

59. (Exercise 3.4.11) Let $X_1, X_2 \ldots$ be independent and let $S_n = \sum_{m=1}^n X_m$. Suppose that $E[X_i] = 0, E[X_i^2] = 1$ and $E[|X_i|^{2+\delta}] \leq C$ for some $\delta > 0$ and $C < \infty$. Show that $S_n / \sqrt{n} \xrightarrow{d} \mathcal{N}(0, 1)$.

60. (Exercise 3.4.12 Lyapunov's theorem) Let $X_1, X_2 \ldots$ be independent and let $S_n = \sum_{m=1}^{n} X_m$. Let $\alpha_n = V[S_n]^{1/2}$. If there is a $\delta > 0$ so that $\lim_{n\to\infty} \alpha_n^{-(2+\delta)} \sum_{m=1}^{n} E|X_m - E[X_m]|^{2+\delta} = 0$, then $(S_n - E[S_n])/\alpha_n \xrightarrow{d} \mathcal{N}(0,1)$. Note that the previous exercise is the special case of this result.

Chapter 4. Martingales

We refer to Durret's Probability Theory ver 4.1 here.

1. (Chapter 4.1 Stopping Times from ver4.1)

(1) Let $N(\omega) : \Omega \to \mathbb{N} \cup \{\infty\}$. State the definition of N being a stopping time or an optimal random variable.

(2) The canonical exmaple of a stopping time is a hitting time. State the definition of the hitting time of A.

2. (Example 4.1.2 from ver4.1) Let $X_i \geq 0$ and $N_t = \sup\{n : S_n \leq t\}$ be the random variable. Give a stopping time using N_t.

3. (Theorem 4.1.3 from ver4.1) Let $X_1, X_2 \cdots$ be iid random variables and let $\mathcal{F}_n = \sigma(X_1, \cdots X_n)$ and N be a stopping time with $P(N < \infty) > 0$. Conditioning on $\{N < \infty\}$, $\{X_{N+n}\}_{n\geq 1}$ is independent of $\mathcal{F}_N$ and has the same distribution as the originial sequences.

4. (Theorem 4.1.5 Wald's equation from ver4.1) Let $X_1, X_2 \cdots$ be iid with $E|X_i| < \infty$. If N is a stopping time with $E[N] < \infty$ then $E[S_N] = E[X_1]E[N]$.

5. (Exercice 4.1.3 from ver 4.1) Show that if S, T are stoppping times then $\min\{S, T\}$ and $\max\{S, T\}$ are also stopping times. Also show that $\min\{S, n\}$, $\max\{S, n\}$ are stopping times.

6. (Exercice 4.1.4 from ver 4.1) Suppose S, T are stopping times. State if $S + T$ is a stopping time. Give a proof or a counter example.

7. (Exercice 4.1.6 from ver 4.1) Show that $M \leq N$ are stopping times then $\mathcal{F}_M \subset \mathcal{F}_N$.

8. (Exercice 4.1.7 from ver 4.1) Show that if $L \leq M$ and $A \in \mathcal{F}_L$ then $N = L(\omega \in A), = M(\omega \in A^c)$ is a stopping time.

From here we refer to Durret's Probability Theory ver 5a.

9. (Definition, Lemma 4.1.1)

(1) Consider a probability space $(\Omega, \mathcal{F}_0, P)$ and a sub σ-algebra $\mathcal{F} \subset \mathcal{F}_0$. Let X be a random variable on $(\Omega, \mathcal{F}_0, P)$ and suppose that $E|X| < \infty$. Give the definition of the conditional expectation of X given $\mathcal{F}$.

(2) Suppose Y satisfies conditions to be $E[X|\mathcal{F}]$. Show that Y is integrable.

(3) If both Y and Y' satisfies conditions to be $E[X|\mathcal{F}]$, then show that they are unique in the sense of almost surely.

10. (Theorem 4.1.2) Show that if $X_1 = X_2$ on $B \in \mathcal{F}$ then $E[X_1|\mathcal{F}] = E[X_2|\mathcal{F}]$ (a.s) on B.

11. (Existence of conditional expectation)

(1) State the definition of absolute continuity of measures.

(2) State Radon-Nikodym Theorem.

(3) Show existence of conditional expectation $E[X|\mathcal{F}]$.

12. (Example 4.1.3) Show that if X is $\mathcal{F}$-measurable ($\mathcal{F} \subset \mathcal{F}_0$), then $E[X|\mathcal{F}] = X$(a.s).

13. (Example 4.1.4) Suppose X is independent of $\mathcal{F}$. Show that $E[X|\mathcal{F}] = E[X]$.

14. (Example 4.1.7) Suppose X, Y are independent. Let ϕ be a function with $E|\phi(X,Y)| < \infty$ and let $g(x) = E\phi(x,Y)$. Show that $E[\phi(X,Y)|X] = g(X)(a.s)$.

15. (Theorem 4.1.9) Show the following statements are true. In the first two parts,we assume that $E|X|, E|Y| < \infty$.

(1) Conditional expectations have linearity.

(2) Conditional expectations have monotonicity.

(3) Suppose $X_n \geq 0, X_n \nearrow X$ with $E[X] < \infty$. Then monotone convergence theorem holds on $E[X_n|\mathcal{F}], E[X|\mathcal{F}]$.

16. (Theorem 4.1.10 Jensen's Inequality) If ϕ is convex and $E|X|, E|\phi(X)| < \infty$ then $\phi(E[X|\mathcal{F}]) \leq E[\phi(X)|\mathcal{F}]$.

17. (Theorem 4.1.11) Show that conditional expectations is a contraction in $L^p(p \geq 1)$.

18. (Theorem 4.1.12) Consider two sub σ-algebras $\mathcal{F} \subset \mathcal{G}$. Suppose $E[X|\mathcal{G}] \in \mathcal{F}$ then show that $E[X|\mathcal{F}] = E[X|\mathcal{G}]$.

19. (Theorem 4.1.13) Suppose $\mathcal{F}_1 \subset \mathcal{F}_2$. Show the following statements.

(1) $E[E[X|\mathcal{F}_1]|\mathcal{F}_2] = E[X|\mathcal{F}_1]$.

(2) $E[E[X|\mathcal{F}_2]|\mathcal{F}_1] = E[X|\mathcal{F}_1]$.

20. (Theorem 4.1.14) Show that if X is $\mathcal{F}$-measurable random variable and $E|Y|, E|XY| < \infty$ then $E[XY|\mathcal{F}] = XE[Y|\mathcal{F}]$.

21. (Theorem 4.1.15) Suppose $EX^2 < \infty$. Show that $E[X|\mathcal{F}]$ is the $\mathcal{F}$-measurable random variable that minimizes mean square error.

22. (Definition 4.1.3 Regular Conditional Probabilities) Let $(\Omega, \mathcal{F}, P)$ be a probability space, $X : (\Omega, \mathcal{F}) \to (S, \mathcal{S})$ a measurable map, and $\mathcal{G}$ a σ-field $\subset \mathcal{F}$. Let $\mu : \Omega \times \mathcal{S} \to [0,1]$ be a regular conditional distribution for X given $\mathcal{G}$. State the conditions for μ to satisfy.

23. (4.2 Martingales) Explain the meanings of the following terms.

(1) $\mathcal{F}_n$ is a filtration

(2) A sequence $\{X_n\}_{n\geq 1}$ is adapted to $\mathcal{F}_n$

(3) X_n is a martingale

(4) X_n is a supermartingale

(5) X_n is a submartingale

24. (Example 4.2.1 Liear martingale) Let $\xi_1 \cdots \xi_n$ be iid random variables and let $S_n = \sum_{j=1}^n \xi_j$ and let $\mathcal{F}_n = \sigma[\xi_1, \cdots \xi_n]$. Show that if $\mu = E\xi_i = 0$ then $S_n (n \geq 0)$ is a martingale with respect to $\mathcal{F}_n$.

25. (Example 4.2.2 Quadratic martingale) Let $\xi_1 \cdots \xi_n$ be iid random variables and let $S_n = \sum_{j=1}^n \xi_j$ and let $\mathcal{F}_n = \sigma[\xi_1, \cdots \xi_n]$. Suppose that if $E\xi_i = 0$ and $\sigma^2 = \mathrm{Var}[\xi_i] < \infty$ then $S_n^2 - n\sigma^2$ is a martingale.

26. (Example 4.2.3 Exponetial martingale) Let $Y_1, Y_2 \cdots$ be nonnegative iid random variables with $E[Y_m] = 1$. If $\mathcal{F}_n = \sigma[Y_1, \cdots, Y_n]$ then $M_n = \prod_{m=1}^n Y_m$ defines a martingale.

27. (Theorem 4.2.4) Show that if X_n is a supermartingale then for $n > m$, $E[X_n|\mathcal{F}_m] \leq X_m$.

28. (Theorem 4.2.6) Show that if X_n is a martingale with regard to $\mathcal{F}_n$ and ϕ is a convex function with $E|\phi(X_n)| < \infty$ for all n, then $\phi(X_n)$ is a submartingale with regard to $\mathcal{F}_n$. Consequently, if $p \geq 1$ and $E|X_n|^p < \infty$ for all n, then $|X_n|^p$ is a submartingale with regard to $\mathcal{F}_n$.

29. (Theorem 4.2.7) Show that if X_n is a submartingale with regard to $\mathcal{F}_n$ and ϕ is an increasing convex function with $E|\phi(X_n)| < \infty$ for all n, then $\phi(X_n)$ is a submartingale with regard to $\mathcal{F}_n$. Also give some examples of $\phi(\bullet)$.

30. Let $\{\mathcal{F}_n\}_{\{n\geq 0\}}$ be a filtration Explain or state the definition of the following terms.

(1) $\{H_n\}_{n\geq 1}$ is a predictable sequence.

(2) $(H \cdot X)_n$

31. (Theorem 4.2.8) Let $X_n (n \geq 0)$ be a supermatrtingale. Suppose $H_n \geq 0$ is a predictable sequence and each H_n is bounded. Show that $(H \cdot X)_n$ is a supermatingale.

32. (Theorem 4.2.9) By using Theorem 4.2.8, show that if N is a stopping time and X_n is a supermatingale, then $X_{N\wedge n}$ is a supermartingale.

33. (Theorem 4.2.10 Upcrossing Inequality) Now we consider upcrossing inequality. If $X_m (m \geq 0)$ is a submartingale then $(b-a)E[U_n] \leq E(X_n - a)^+ - E(X_0 - a)^+$.

(1) State the definition of $N_k (N_{2k-1}, N_{2k}, N_0)$

(2) We define. $H_m = 1$ if $N_{2k-1} < m \leq N_{2k}$ for some k, and 0 otherwise. Explain H_m is a predictable sequence.

(3) State the definition of U_n.

(4) State and prove the upcrossing inequality.

34. (Theorem 4.2.11 Martingale convergence theorem) If X_n is a submartingale with $\sup E[X_n^+] < \infty$, then as $n \to \infty$, X_n converges almost surely to a limit X with $E|X| < \infty$.

35. (Theorem 4.2.12) If $X_n \geq 0$ is a supermartingale then as $n \to \infty$, $X_n \to X$ a.s and $E[X] \leq E[X_0]$.

36. (Example 4.2.13) Explain that Theorem 4.2.12 or 4.2.11 do not guarantee convergence in L^1 by giving an example. ($S_0 = 1, S_n = S_{n-1} + \xi_n, P(\xi_i = 1) = P(\xi_i = -1) = 1/2$ where ξ_i are iid random variables.)

37. (Example 4.2.14) Give an example of a martingale X_n with $X_n \xrightarrow{P} 0$ but not almost surely.

38. (Exercise 4.2.3) Suppose X_n, Y_n are submartingales with regard to $\mathcal{F}_n$. Then show that $X_n \vee Y_n$ is also a submartingale.

39. (Exercise 4.2.6) Let $Y_1, Y_2 \cdots$ be nonnegative iid random variables with $EY_m = 1$ and $P(Y_m = 1) < 1$. We have already shown that $X_n = \prod_{m=1}^n Y_m$ defines a martingale.

(1) Show that $X_n \xrightarrow{a.s} 0$.

(2) Show that $(1/n) \log X_n \xrightarrow{a.s.} c < 0$

40. (Exercise 4.2.8) Let X_n and Y_n be positive integrable and adapted to $\mathcal{F}_n$. Suppose that $E[X_{n+1}|\mathcal{F}_n] \leq (1+Y_n)X_n$ with $\sum_{n\geq 1} Y_n < \infty (a.s.)$. Prove that X_n

converges a.s. to a finite limit by finding a closely related supermartingale to which Theorem 4.2.12 can be applied.

41. (Theorem 4.3.1) Let $X_1, X_2 \cdots$ be a martingale with $|X_{n+1} - X_n| \le M < \infty$. Let $C = \{\lim X_n \text{ exists and is finite}\}$ and let $D = \{\limsup X_n = \infty \text{ and } \liminf X_n = -\infty\}$. Show that $P(C \cup D) = 1$.

42. (Theorem 4.3.2 Doob's decomposition) Any submartingale $X_n (n \ge 0)$ can be written in a unique way $X_n = M_n + A_n$ where M_n is a martingale and A_n is a predictable increasing sequene with A_0 =0. Show the statement above.

43. (Theorem 4.3.4) Let $\mathcal{F}_n (n \ge 0)$ be a filtration with $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $A_n (n \ge 1)$ a sequence of events with $A_n \in \mathcal{F}_n$. Then state and prove the second Borel-Cantelli's lemma.

44. (4.3.4 Branching Process) Let ξ_i^n be iid non-negative integer-valued random variables. Define Z_n which is a Galton-Watson process.

45. (Lemma 4.3.9) Let $\mathcal{F}_n = \sigma[\xi_i^m | i \ge 1, 1 \le m \le n]$ and $\mu = E[\xi_i^m] \in (0, \infty)$. Then show that Z_n/μ^n is a martingale with regard to $\mathcal{F}_n$.

46. (Theorem 4.3.10) If $\mu < 1$ then $Z_n = 0$ for all n sufficiently large. Then show that $Z_n/\mu^n \to 0$.

47. (Theorem 4.3.11) If $\mu = 1$ and $P(\xi_i^m = 1) < 1$, then $Z_m = 0$ for all n sufficiently large.

48. (Theorem 4.4.1) If X_n is a submartingale and N is a stopping time with $P(N \le k) = 1$ then $E[X_0] \le E[X_N] \le E[X_k]$.

49. (Theorem 4.4.2 Doob's inquality) Let X_m be a submartingale and define $\bar{X}_n = \max_{0 \le m \le n} X_m^+$. Let $\lambda > 0$ and $A = \{\bar{X}_n \ge \lambda\}$. Show that $\lambda P(A) \le E[X_n \mathbb{I}_A] \le E[X_n^+]$

50. (Example 4.4.3 random walks) Consider $S_n = \xi_1 + \cdots \xi_n$ where ξ_m are independent and have $E[\xi_m] = 0, E[\xi_m^2] = \sigma_m^2 < \infty$. Let $\lambda = x^2$ and obtain Kolmogorov's maximimal inequality by applying Theorem 4.4.2.

51. (Theorem 4.4.4 L^p maximal inequality) Show that if X_n is a submartingale then $E[\bar{X}_n^p] \le (\frac{p}{p-1})^p E[(X_n^+)^p]$ for $p \in (1, \infty)$

52. (Theorem 4.4.6 L^p convergence theorem) Suppose X_n is a martingale with $\sup E|X_n|^p < \infty$ where $p > 1$. Show that $X_n \xrightarrow{a.s} X$ and $X_n \xrightarrow{L^p} X$.

53. (Exercise 4.4.2 and 4.4.4)

(1) (Generalization of Theorem 4.1) If X_n is a submartingale and $M \le N$, $P(N \le k) = 1$, then show that $E[X_M] \le E[X_N]$.

(2) Strengthen the concolusion of Exercise 4.4.2 to $X_M \le E[X_N | \mathcal{F}_M]$. Hint:

You may consider $L = M(\omega \in A), L = N(\omega \notin A)$ which is a stopping time.

54. (Exercise 4.4.6) Let ξ_m are independent and $E[\xi_m] = 0, E[\xi_m^2] = \sigma_m^2, |\xi_m| \leq K$. Let $S_n = \sum_{m=1}^n \xi_m$ and let $s_n = \sum_{m=1}^n \sigma_m^2$. Then $S_n^2 - s_n^2$ is a martingale. By applying theorem 4.4.1, show that $P(\max_{m=1\cdots n} |S_m| \geq x)(x+K)^2/V[S_n]$.

55. (Exercise 4.4.9) Let X_n and Y_n be martingales with $E[X_n^2], E[Y_n^2] < \infty$. Show that $E[X_n Y_n] - E[X_0 Y_0] = \sum_{m=1}^n E[(X_m - X_{m-1})(Y_m - Y_{m-1})]$.

56. (Uniform Integrability)

(1) State the definition of $\{X_i\}_{\{i\in I\}}$ being uniformly integrable.

(2) Prove that $\{X_i\}_{\{i\in I\}}$ is uniformly integrable $\Leftrightarrow$ L_1-bounded and absolutely continuous. (You may refer to the textbook of Beijing University. theorem 3.4.3)

57. (Theorem 4.6.1) Given a probability space $(\Omega, \mathcal{F}_0, P)$ and an $X \in L^1$. Show that $\{E[X|\mathcal{F}]\}_{\mathcal{F}\subset\mathcal{F}_0}$ ($\mathcal{F}$: σ-algebra) is uniformly integrable.

58. (Theorem 4.6.2) Let $\phi \geq 0$ be any function with $\phi(x)/x \to \infty$ as $x \to \infty$. If $E[\phi(|X_i|)] \leq C$ for all $i \in I$ then $\{X_i\}_{i\in I}$ is uniformly integrable.

59. (Theorem 4.6.3) Show that if $X_n \xrightarrow{P} X$ then the following are equivalent.

(1) $\{X_n\}_{n\geq 0}$ is uniformly integrable

(2) $X_n \xrightarrow{L_1} X$

(3) $E|X_n| \to E|X| < \infty$

60. (Theorem 4.6.4) Show that for a submartingale, the following are equivalent.

(1) It is uniformly integrable

(2) It converges a.s and in L^1

(3) It converges in L^1

61. (Lemma 4.6.5) Show that if integrable random variable $X_n \xrightarrow{L^1} X$ then $E[X_n \mathbb{I}_A] \to E[X\mathbb{I}_A]$.

62. (Lemma 4.6.6) If a martingale $X_n \xrightarrow{L^1} X$ then $X_n = E[X|\mathcal{F}_n]$.

63. (Theorem 4.6.7) For a martingale, the following are equivalent.

(1) It is uniformly integrable

(2) It converges a.s and in L^1

(3) It converges in L^1

(4) There is an integrable random variable X so that $X_n = E[X|\mathcal{F}_n]$

64. (Theorem 4.6.8) Suppose $\mathcal{F}_n \nearrow \mathcal{F}_\infty$ ($\mathcal{F}_n, \mathcal{F}_\infty$ are σ-algebras. Thus $\mathcal{F}_\infty = \sigma[\cup_{n\geq 1}\mathcal{F}_n]$) Show that $E[X|\mathcal{F}_n] \xrightarrow{a.s,L^1} E[X|\mathcal{F}_\infty]$.

65. (Theorem 4.6.9 Levy's 0-1 law) State Levy's 0-1 law.

66. (Theorem 4.6.10) Dominated Convergence Theorem for conditional expectations. Suppose $Y_n \xrightarrow{a.s} Y$ and $|Y_n| \leq Z$ for all n where $E[Z] < \infty$. Show that if $\mathcal{F}_n \nearrow \mathcal{F}_\infty$ ($\mathcal{F}_n, \mathcal{F}_\infty$ are σ-algebras) then $E[Y_n|\mathcal{F}_n] \xrightarrow{a.s} E[Y|\mathcal{F}_\infty]$.

67. (Exercise 4.6.4) Let X_n be random variables taking values in $[0,\infty)$. Let $D = \{\omega | \exists n \in \mathbb{N} \ s.t \ X_n = 0\}$ and assume that $P(D|X_1 \cdots X_n) \geq \delta(x) > 0$ a.s on $\{X_n \leq x\}$. Use theorem 4.6.9 to conclude that $P(D \cup \{\lim_n X_n = \infty\}) = 1$.

68. (Exercise 4.6.7) Show that if $\mathcal{F}_n \nearrow \mathcal{F}_\infty$ ($\mathcal{F}_n, \mathcal{F}_\infty$ are σ-algebras) and $Y_n \xrightarrow{L^1} Y$ then $E[Y_n|\mathcal{F}_n] \xrightarrow{L^1} E[Y|\mathcal{F}_\infty]$.

69. Answer the following questions.

(1) (Definition) State the definition of backwards martingales.

(2) (Thorem 4.7.1) Let $\{X_n\}_{n\leq 0}$ be a backward martingale. Show that $X_{-\infty} = \lim_{n\to-\infty} X_n$ exists a.s and in L^1.

70. (Theorem 4.7.2) Let $\{X_n\}_{n\leq 0}$ be a backward martingale. Show that $X_{-\infty} = E[X_0|\mathcal{F}_{-\infty}]$ where $X_{-\infty} = \lim_{n\to-\infty} X_n$ and $\mathcal{F}_{-\infty} = \cap_{n\leq 0}\mathcal{F}_\infty$.

71. (Theorem 4.7.3) Suppose Y is an integrable random variable. Consider $\{\mathcal{F}_n\}$ where $\mathcal{F}_n \searrow \mathcal{F}_{-\infty}$ as $n \to -\infty$. Show that $E[Y|\mathcal{F}_n] \xrightarrow{a.s/L^1} E[Y|\mathcal{F}_{-\infty}]$.

72. (Theorem 4.8.1) Show that if X_n is a uniformly integrable submartingale then for any stopping time N, $X_{X\wedge n}$ is uniformly integrable.

73. Prove the following theorems.

(1) (Theorem 4.8.2) Show that if X_n is a uniformly integrable submartingale then for any stopping time $N \leq \infty$, we have $E[X_0] \leq E[X_N] \leq E[\lim_{n\to} X_n]$.

(2) (Theorem 4.8.3) Show that if $E|X_N| < \infty$ and $X_n\mathbb{I}_{\{}N > n\}$ is uniformly integrable, then $X_{N\wedge n}$ is uniformly integrable and hence $E[X_0] \leq E[X_N]$. ($\{X_n\}_{n\geq 0}$ is a submartingale).

(3) (Theorem 4.8.4) If X_n is a nonnegative supermartingale and $N \leq \infty$ is a stopping time, then $E[X_0] \geq E[X_N]$.

74. (Theorem 4.8.5) Suppose X_n is a submartingale and $E(|X_{n+1} - X_n| \, |\mathcal{F}_n) \leq B$ a.s. Show that if N is a stopping time with $E[N] < \infty$, then $X_{N\wedge n}$ is uniformly integrable and hence $E[X_N] \geq E[X_0]$.

References

[1] Rick Durrett. Probability: Theory and examples, 2018.

[2] S.I. Resnick. *A Probability Path*. Modern Birkhäuser Classics. Birkhäuser Boston, 2013.