• 資料科學之統計基礎

No.

Date Casella Berger Chapter 1.

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1.5

(a) 有人生了孩子，結果：「她生了雙胞胎 且他們為同卵雙胞胎 且 兩個都是女的」

(b) 為了簡單起見，利用貝氏公式

$$Pr(A\cap B\cap C) = \underbrace{Pr(A\cap B | C)}_{\frac{1}{3}\times\frac{1}{2}} \cdot \underbrace{Pr(C)}_{=\frac{1}{90}}$$

$$= \frac{1}{540}$$

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1.22　　　　　　　　　　註 $aC_b = \binom{a}{b}$ 之意

(a) 每個月都出現15天.

、所有組合… $_{366}C_{180}$ (每種組合出現的機率為相同)

、滿足要求的情況… $_{31}C_{15}$ · $_{29}C_{15}$ · $_{31}C_{15}$ · $_{30}C_{15}$ · … · $_{31}C_{15}$
(1月)　(2月)　(3月)　(4月)　(12月)

$$M \overset{\text{def}}{=} \{31, 29, 31, 30, 31, 30, 31, 31, 30, 31, 30, 31\}$$

$$\therefore \frac{\prod_{m \in M} {}_{m}C_{15}}{{}_{366}C_{180}}$$

(b) 所有組合… $_{366}C_{30}$

不包含9月的組合… $_{336}C_{30}$

$$\therefore \frac{{}_{336}C_{30}}{{}_{366}C_{30}}$$

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1.24.

(a) {正} {反反正} {反反反反正} ……

$\quad\ \ $A $\qquad$ A B A $\qquad$ A B A B A

我們考慮第 $2n-1$ $(n=1,2,\cdots)$ 次時才第一次出現正面的機率(之相加)

Pr(第 $2n-1$ 次時第一次出現正面)

$= \left(\frac{1}{2}\right)^{2n-1}$

$\therefore \sum_{n=1} \left(\frac{1}{2}\right)^{2n-1} = \frac{\frac{1}{2}}{1-\frac{1}{4}} = \frac{1}{2}\cdot\frac{4}{3} = \frac{2}{3}$

$\therefore$ A 贏的機率為 $\frac{2}{3}$

(b) 接著考慮出現正面機率為 p 的情況

Pr(第 $2n-1$ 次才第1次出現正面)

$= (1-p)^{2n-2}\cdot p$

$\therefore$ A 贏的機率 $= \sum_{n=1} (1-p)^{2n-2}\cdot p = \frac{p}{1-(1-p)^2} = \frac{1}{2-p}$

整理

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由此可知 A贏的機率為 $\frac{1}{2-p}$ $(\geq \frac{1}{2})$ when $(0 \leq p \leq 1)$

∴ 證明完成

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1.32. 題意有點不清楚，因此依照以下意思解題

- 每個人的評分不重複
- 若第一個人的分數最高 ⇒ 錄取第一個人

該求的機率：Pr(第i個人的評分最高 | 錄取第i個人)

= 看到 $i-1$ 個人的時候，從剩下的 $N-(i+1)$ 個人中

選擇評分最高的人 ∴ $\dfrac{1}{N-(i+1)}$ $(i\geq 2)$

$i=1$ 時：$\dfrac{1}{N}$ = 第一個人抽到評分最高的人的機率

∴ $i=1$ 依然成立

∴ $\dfrac{1}{N-(i+1)}$

1.36

① $Pr(\text{成功两次} \cup \text{成功三次} \cup \cdots \cup \text{成功十次})$

$= 1 - Pr(\text{成功零次} \cup \text{成功一次})$

$= 1 - Pr(\text{成功零次}) - Pr(\text{成功一次})$

$= 1 - \left(\frac{4}{5}\right)^{10} - \left(\frac{1}{5}\right)^{1}\left(\frac{4}{5}\right)^{9} \cdot {}_{10}C_1$

② $Pr(\text{成功两次以上} \mid \text{成功一次以上})$

$$= \frac{Pr(\text{成功2次以上})}{Pr(\text{成功1次以上})} = \frac{1 - \left(\frac{4}{5}\right)^{10} - \left(\frac{1}{5}\right)\left(\frac{4}{5}\right)^{9} \cdot {}_{10}C_1}{1 - \left(\frac{4}{5}\right)^{10}}$$

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(a)

	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
Prisoner pardoned:	A	B	C
Warden tells	$\frac{r}{3}$ B $\frac{1-r}{3}$ C	$\frac{1}{3}$ C	$\frac{1}{3}$ B

Pr(A is pardoned | Warden tells B would die)

$$= \frac{Pr(\text{A is pardoned, Warden tells B would die})}{Pr(\text{Warden tells B would die})}$$

$$= \frac{\frac{r}{3}}{\frac{1}{3}+\frac{r}{3}} = \frac{r}{1+r}$$

$\because \frac{r}{1+r} \leq \frac{1}{3} \Rightarrow 3r \leq 1+r \Rightarrow 0 \leq r \leq \frac{1}{2}$

$(0 \leq r \leq 1)$

(b) 這跟 Monty Hall Problem 相同…

$r=\frac{1}{2}$ 時，$Pr(A|W)=\frac{1}{3}$

$Pr(C|W)=\frac{2}{3}$

∴ C生存的機率較高 $\left(\frac{2}{3}\right)$ ∴ 跟C交換比較好

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1.39

(a) $A\cap B=\phi$ 成立

若事件A與B獨立，則 $Pr(A)\cdot Pr(B)=Pr(A\cap B)$

但 $Pr(A)\cdot Pr(B)>0$，$Pr(A\cap B)=Pr(\phi)=0$

∴ 矛盾 ⇒ 證明完成

(b) $Pr(A)Pr(B)=Pr(A\cap B)$ 成立

若事件A與B互斥，則 $Pr(A\cap B)=Pr(\phi)=0$

但 $Pr(A)Pr(B)>0$ ∴ 跟(a)同理，矛盾

∴ 證明完成

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$Pr(- \mid \bullet) = \frac{1}{4}$; $Pr(\bullet \mid -) = \frac{1}{3}$

接收— 發出•　　　接 發

(a) $Pr(\text{接收「—」}) = Pr(\text{接收「—」, 發出「•」}) + Pr(\text{接收「—」, 發出「—」})$

$= Pr(- \mid \bullet)\, Pr(\bullet) + Pr(- \mid -) \cdot Pr(-)$

$= \frac{1}{4} \cdot \frac{3}{7} + \left(1-\frac{1}{3}\right) \cdot \frac{4}{7} = \frac{3}{28} + \frac{8}{21} = \frac{9+32}{84} = \frac{41}{84}$

所求的機率 $= Pr(\text{發出「—」} \mid \text{接收「—」})$

$= \dfrac{Pr(\text{發出「—」, 接收「—」})}{Pr(\text{接收「—」})} = \dfrac{Pr(- \mid -)\, Pr(-)}{\frac{41}{84}}$

$= \dfrac{\left(1-\frac{1}{3}\right) \cdot \frac{4}{7}}{\frac{41}{84}} = \dfrac{\frac{8}{21}}{\frac{41}{84}} = \frac{32}{41}$

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(b) 所求的機率：
① $\Pr($發「・」「・」 $|$ 接收「・」「・」$)$
② $\Pr($發「・」「—」 $|$ 接收「・」「・」$)$
③ $\Pr($發「—」「・」 $|$ 接收「・」「・」$)$
④ $\Pr($發「—」「—」 $|$ 接收「・」「・」$)$

$\Pr($接收「・」「・」$) = \Pr($接收「・」$)^2$ (∵獨立性)

$$= \left(1-\frac{41}{84}\right)^2 = \left(\frac{43}{84}\right)^2$$

$\Pr($發出「・」「・」, 接收「・」「・」$) = \Pr($發「・」, 接收「・」$)^2$

$$= \left\{\left(1-\frac{1}{4}\right)\cdot\frac{3}{7}\right\}^2 = \left(\frac{9}{28}\right)^2$$

∴ ① … $\left\{\dfrac{\left(\frac{9}{28}\right)}{\left(\frac{43}{84}\right)}\right\}^2 = \left(\dfrac{27}{43}\right)^2$

$\Pr($發出「・」「—」, 接收「・」「・」$) = \frac{9}{28}\cdot\Pr(\cdot\,|\,-)\cdot\Pr(-) = \frac{9}{28}\cdot\frac{4}{7}\cdot\frac{1}{3}$

$$②\ (=③) = \frac{\frac{9}{28}\cdot\frac{4}{21}}{\left(\frac{43}{84}\right)^2} = \frac{84^2}{43^2}\cdot\frac{9}{28}\cdot\frac{4}{21} = \frac{432}{43^2}$$

$\Pr($發出「—」「—」, 接收「・」「・」$) = \left\{\Pr(\cdot\,|\,-)\cdot\Pr(-)\right\}^2 = \left(\frac{4}{21}\right)^2$

$$∴ ④ = \frac{\left(\frac{4}{21}\right)^2}{\left(\frac{43}{84}\right)^2} = \left(\frac{16^2}{43^2}\right)$$

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15| 服從超幾何分佈

$$HG(\underset{\text{整體個數}}{\underbrace{30}},\ 5,\ \underset{\text{抽出的次數}}{\underbrace{4}})$$

$$Pr(X=x) = \frac{{}_{25}C_{4-x} \cdot {}_{5}C_{x}}{{}_{30}C_{4}}$$

$$Pr(X=0) = \frac{{}_{25}C_{4} \cdot {}_{5}C_{0}}{{}_{30}C_{4}} = \frac{{}_{25}C_{4}}{{}_{30}C_{4}} = 0.46159$$

$$Pr(X=1) = \frac{{}_{25}C_{3} \cdot {}_{5}C_{1}}{{}_{30}C_{4}} = 0.41963\cdots$$

$$Pr(X=2) = \frac{{}_{25}C_{2} \cdot {}_{5}C_{2}}{{}_{30}C_{4}} = 0.109469\cdots$$

$$Pr(X=3) = \frac{{}_{25}C_{1} \cdot {}_{5}C_{3}}{{}_{30}C_{4}} = 0.009122$$

$$Pr(X=4) = \frac{{}_{25}C_{0} \cdot {}_{5}C_{4}}{{}_{30}C_{4}} = 0.0001824$$

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- cdf ... $Pr(X \leq x) = \sum_{k=0}^{x} Pr(X=k)$

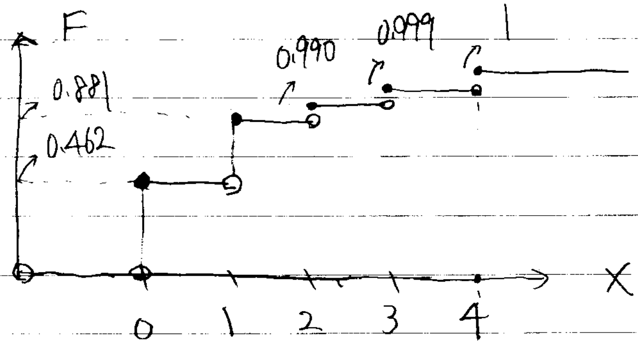
$x=0$ ... 0.46159

$x=1$ ... $0.46159 + 0.41963 = 0.881$

$x=2$ ... $= 0.990$

$x=3$ ... $= 0.999$

$x=4$ ... $= 1$



1.55

$$V \overset{def}{=} 5\cdot I_{\{T<3\}} + 2T\cdot I_{\{T\geq 3\}}$$

我們注意 $2T\cdot I_{\{T\geq 3\}} \in \{0\}\cup[6,\infty)$.

$Pr(V \leq v)$:

① $v<5$ … $Pr(V\leq v)=0$

② $5\leq v<6$ … $Pr(V\leq v)=Pr(V=5)=Pr(T<3)$

$$=\int_0^3 \frac{1}{1.5}\exp\left(\frac{-t}{1.5}\right)dt = \left[-\exp\left(\frac{-t}{1.5}\right)\right]_0^3$$

$$=1-\exp(-2)$$

③ $v\geq 6$ … $Pr(V\leq v)= \underbrace{Pr(V=5)}_{1-\exp(-2)} + Pr(6\leq V\leq v)$

$$=(1-\exp(-2)) + Pr(6\leq 2T\cdot I_{\{T\geq 3\}}\leq v)$$

$$=(1-\exp(-2)) + Pr\left(3\leq T\leq \frac{v}{2}\right)$$

$$=(1-\exp(-2)) + \left[-\exp\left(\frac{-t}{1.5}\right)\right]_3^{\frac{v}{2}}$$

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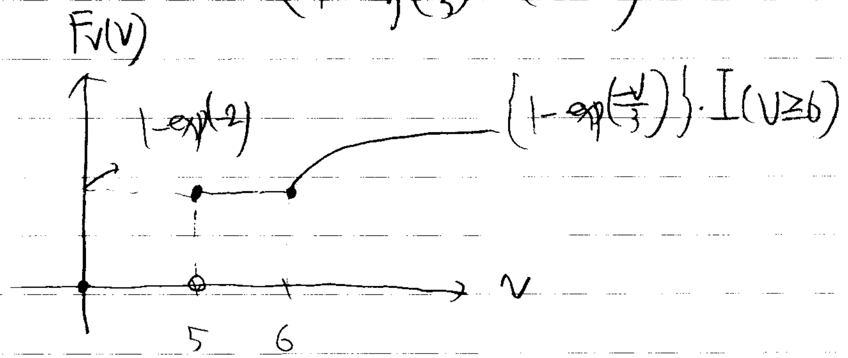
$$= 1-\exp\left(\frac{-v}{3}\right)$$

综合①，②，③

$$Pr(V\le v)=\begin{cases}0 & (v<5)\\ 1-\exp(-2) & (5\le v<6)\\ 1-\exp\left(\frac{-v}{3}\right) & (v\ge 6)\end{cases}$$



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Date 資料科學之統計基礎(I) 作業2 Ch2

2.6.

(a) $\int_{x=-\infty}^{x=\infty} \frac{1}{2}\exp(-|x|)\,dx = 1$ ($\because$ 全機率=1)

$y = |x|^3$ ① $x \geq 0 \dots$ $y = x^3$ $\frac{dy}{dx} = 3x^2 = 3y^{\frac{2}{3}}$

$\therefore dx = \frac{1}{3}y^{\frac{-2}{3}}dy$

② $x < 0 \dots$ $y = -x^3$ $\frac{dy}{dx} = -3x^2 = -3 \cdot y^{\frac{2}{3}}$

$dx = \frac{-1}{3}y^{\frac{-2}{3}}$

$$1 = \int_0^{\infty} \frac{1}{2}\exp(-|x|)\,dx + \int_{-\infty}^{0} \frac{1}{2}\exp(-|x|)\,dx$$

$$= \int_0^{\infty} \frac{1}{2}\exp(-y^{\frac{1}{3}}) \cdot \frac{1}{3}y^{\frac{-2}{3}}dy + \int_{\infty}^{0} \frac{1}{2}\exp(-y^{\frac{1}{3}}) \cdot \frac{-1}{3}y^{\frac{-2}{3}}dy$$

$\begin{pmatrix} x: 0 \to \infty \\ y: 0 \to \infty \end{pmatrix}$ $\begin{pmatrix} x: -\infty \to 0 \\ y: \infty \to 0 \end{pmatrix}$

$$= \int_0^{\infty} \frac{1}{6}y^{\frac{-2}{3}}\exp(-y^{\frac{1}{3}})\,dy + \int_0^{\infty} \frac{1}{6}y^{\frac{-2}{3}}\exp(-y^{\frac{1}{3}})\,dy$$

$$= \int_0^{\infty} \frac{1}{3}y^{\frac{-2}{3}}\exp(-y^{\frac{1}{3}})\,dy = 1$$

由此可知… $f_Y(y) = \frac{1}{3} \cdot y^{\frac{-2}{3}} \cdot \exp(-y^{\frac{1}{3}}) \cdot I_{(0,\infty)}(y)$

且 $\int_0^{\infty} f_Y(y) = 1$ (Weibull 分布)

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(b) 全確率 $= 1 = \int_{-1}^{1} \frac{3}{8}(x+1)^2 dx = \int_{-1}^{0} \frac{3}{8}(x+1)^2 dx + \int_{0}^{1} \frac{3}{8}(x+1)^2 dx$ ① ②

① $x: -1 \to 0$ $\quad (x^2 = 1-y \quad x = -\sqrt{1-y} \ (<0))$

$y: 0 \to 1$

$\frac{dy}{dx} = -2x = 2\sqrt{1-y} \quad \therefore \quad dx = \frac{dy}{2\sqrt{1-y}}$

② $x: 0 \to 1$ $\quad (x^2 = 1-y \quad x = +\sqrt{1-y} \geq 0)$

$y: 1 \to 0$

$\frac{dy}{dx} = -2x = -2\sqrt{1-y} \quad dx = \frac{-dy}{2\sqrt{1-y}}$

$\therefore \ 1 = ① + ② = \int_0^1 \frac{3}{8}(1-\sqrt{1-y})^2 \cdot \frac{dy}{2\sqrt{1-y}} + \int_1^0 \frac{3}{8}(1+\sqrt{1-y})^2 \cdot \frac{-1}{2\sqrt{1-y}} dy$

$= \int_0^1 \frac{3}{16\sqrt{1-y}}(1-\sqrt{1-y})^2 dy + \int_0^1 \frac{3}{16\sqrt{1-y}}(1+\sqrt{1-y})^2 dy$

$= \int_0^1 \frac{3}{16\sqrt{1-y}} \{1 - \cancel{2\sqrt{1-y}} + (1-y) + 1 + \cancel{2\sqrt{1-y}} + (1-y)\} dy$

$= \int_0^1 \frac{3}{16\sqrt{1-y}}(2-y) \times 2 \, dy$

$= \int_0^1 \frac{3(2-y)}{8\sqrt{1-y}} dy = 1$

$\therefore \ f(y) = \frac{3}{8} \frac{2-y}{\sqrt{1-y}} I_{(0,1)}(y)$

且 $\int_0^1 f(y) dy = 1$

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(C) 全機率 $= 1 = \int_0^1 \frac{3}{8}(x+1)^2 dx + \int_{-1}^0 \frac{3}{8}(x+1)^2 dx$

① $x: 0 \to 1$ $(y = 1-x)$

$y: 1 \to 0$

$\frac{dy}{dx} = -1$

② $x: -1 \to 0$ $(y = 1-x^2$ $x = -\sqrt{1-y})$

$y: 0 \to 1$

$\frac{dy}{dx} = -2x = 2\sqrt{1-y}$

全機率

$\therefore 1 = ① + ② = \int_1^0 \frac{3}{8}(2-y)^2 \cdot (-1)\, dy + \int_0^1 \frac{3}{8}(1-\sqrt{1-y})^2 \cdot \frac{1}{2\sqrt{1-y}} dy$

$= \int_0^1 \frac{3}{8}(2-y)^2 dy + \int_0^1 \frac{3}{16\sqrt{1-y}}(1-\sqrt{1-y})^2 dy$

$= \int_0^1 \frac{3}{16}\left\{2(2-y)^2 + \frac{1}{\sqrt{1-y}}(1-\sqrt{1-y})^2\right\} dy$

$\therefore f(y) = \frac{3}{16}\left\{2(2-y)^2 + \frac{1}{\sqrt{1-y}}(1-\sqrt{1-y})^2\right\} \cdot I_{(0,1)}(y)$

$\int_0^1 f(y)\, dy = 1$

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2.11.

(a) $\int_{-\infty}^{\infty} x^2 \cdot \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right) dx = \int_{-\infty}^{\infty} (-x)\,(-x)\, \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx$

$$= \underbrace{\left[(-x)\cdot \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)\right]_{-\infty}^{\infty}}_{=0 \ (\exp \text{収斂勝快})} + \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx}_{=1 \ (\text{全確率})}$$

$\therefore E[X^2] = 1$

(or $X^2 \sim \chi^2(1)$ $\therefore E[X^2] = 1$)

接著考慮 $X^2 = Y$ 的 pdf　　$\frac{dy}{dx} = 2x = \pm 2\sqrt{y}$　if $x \geq 0$ 「+」, if $x < 0$ 「−」

$$1 = \int_0^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx + \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx$$

$$= \int_{y=0}^{y=\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y}{2}\right) \cdot \frac{1}{2\sqrt{y}} dy + \int_{y=\infty}^{y=0} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y}{2}\right) \cdot \frac{-1}{2\sqrt{y}} dy$$

$$= \int_{y=0}^{y=\infty} \frac{1}{\sqrt{2\pi y}} \exp\left(-\frac{y}{2}\right) dy$$

$$\therefore f(y) = \frac{1}{\sqrt{2\pi}\sqrt{y}} \exp\left(-\frac{y}{2}\right) \cdot I_{(0,\infty)}(y)$$

$E(Y) = \int_0^{\infty} \frac{\sqrt{y}}{\sqrt{2\pi}} \exp\left(-\frac{y}{2}\right) dy$　　$z = \frac{y}{2}$　　$\frac{dz}{dy} = \frac{1}{2}$

$$= \int_0^{\infty} \frac{\sqrt{2z}}{\sqrt{2\pi}} \exp(-z) \cdot 2dz = \int \frac{1}{\sqrt{\pi}} \cdot 2 z^{\frac{1}{2}} \exp(-z) dz$$

$$= \frac{2}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}\right) = \frac{2}{\sqrt{\pi}} \left(\frac{1}{2}\right)\left(\sqrt{\pi}\right) = 1 \quad (\because \text{Gamma函数})$$

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(b) $Y = |X|$ 全確率 $= 1 = \int_0^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx + \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx$

$= \int_{y=0}^{y=\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) \cdot (1)\, dy + \int_{y=\infty}^{y=0} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) (-1)\, dy$

$= \int_{y=0}^{y=\infty} \frac{\sqrt{2}}{\sqrt{\pi}} \exp\left(-\frac{y^2}{2}\right) dy$ $\quad \therefore f(y) = \frac{\sqrt{2}}{\sqrt{\pi}} \exp\left(-\frac{y^2}{2}\right) I_{(0,\infty)}(y)$

$E[Y] = \int_0^{\infty} \frac{\sqrt{2}}{\sqrt{\pi}}\, y \exp\left(-\frac{y^2}{2}\right) dy = \left[\frac{\sqrt{2}}{\sqrt{\pi}} \left(-\exp\left(-\frac{y^2}{2}\right)\right)\right]_0^{\infty} = \frac{\sqrt{2}}{\sqrt{\pi}}$

$E[Y^2] = E[X^2] = 1$ ($\because$ (a))

$\therefore V[Y] = E[Y^2] - E[Y]^2 = 1 - \frac{2}{\pi}$

mean… $\frac{\sqrt{2}}{\sqrt{\pi}}$ variance $1 - \frac{2}{\pi}$

pdf $\frac{\sqrt{2}}{\sqrt{\pi}} \exp\left(-\frac{y^2}{2}\right) I_{(0,\infty)}(y)$

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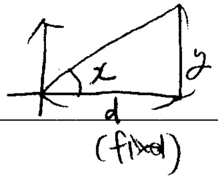

$\tan x = \frac{y}{d}$

2.12 $X \sim \mathrm{Uni}(0, \frac{\pi}{2})$

$\tan X = \frac{Y}{d} \quad \therefore Y = d\tan X$

求Y的機率分佈： $X = \arctan(\frac{Y}{d})$ $\begin{cases} x: 0 \to \frac{\pi}{2} \\ y: 0 \to \infty \end{cases}$

$\therefore \frac{dx}{dy} = \frac{\frac{1}{d}}{1 + \frac{y^2}{d^2}} = \frac{d}{d^2+y^2}$

$\therefore \int_{x=0}^{x=\frac{\pi}{2}} \frac{2}{\pi} dx = 1$ （全機率）

$= \int_0^{\infty} \frac{2}{\pi} \frac{d}{d^2+y^2} dy = 1$

$\therefore f(y) = \frac{2}{\pi} \cdot \frac{d}{d^2+y^2} \cdot I_{(0,\infty)}(y)$

$E(Y) = E[d\tan x] = \int_0^{\frac{\pi}{2}} d \cdot \tan x \cdot \frac{2}{\pi} dx$

$= \frac{2d}{\pi} \cdot \int_0^{\frac{\pi}{2}} \tan x \, dx$

$= \frac{2d}{\pi} \cdot \left[-\log \cos x \right]_0^{\frac{\pi}{2}}$

$= \frac{2d}{\pi} (-\log 0)$ ∴ 不存在 (∞)

∴ 不存在

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2.13. 連續X次出現H或T後又出現另外一面.

$$\Pr(X=x) = p^x\cdot(1-p) + (1-p)^x\cdot p \quad (x\in\mathbb{N})$$

$$E[X] = \sum_{x\in\mathbb{N}} \{p^x(1-p) + (1-p)^x p\}\,x$$

$$= \underbrace{\sum_{x\in\mathbb{N}} x\cdot p^x(1-p)}_{①} + \underbrace{\sum_{x\in\mathbb{N}} x(1-p)^x\cdot p}_{②}$$

利用 Moment Generating Function

$$E[e^{tX}] = \sum_{x\in\mathbb{N}} (pe^t)^x(1-p) + \sum_{x\in\mathbb{N}} ((1-p)e^t)^x p.$$

$$= (1-p)\cdot\frac{1}{1-pe^t} + p\cdot\frac{1}{1-(1-p)e^t}$$

$$\therefore M_X(t) = (1-p)\cdot\frac{1}{1-pe^t} + p\cdot\frac{1}{1-(1-p)e^t}$$

$$\frac{dM_X(t)}{dt} = (1-p)\cdot\frac{pe^t}{(1-pe^t)^2} + p\cdot\frac{(1-p)e^t}{(1-(1-p)e^t)^2}$$

$$M_X'(0) = \frac{(1-p)\cdot p}{(1-p)^2} + \frac{p}{p^2}(1-p) = \frac{p}{1-p} + \frac{1-p}{p} = \frac{p^2+(1-p)^2}{p(1-p)}$$

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$\therefore E[X] = \dfrac{2p^2-2p+1}{p(1-p)}$

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2.14

$(xf(x) \geq 0)$

(1) 利用Fubini定理 保證 $\int dx$; $\int dy$ 可替換

$$\int_0^\infty x f(x)\,dx = \int_0^\infty f(x)\,dx \int_0^x dy \quad \text{替換 } \int dx \int dy$$

$$= \int_0^\infty dy \int_{x=y}^{x=\infty} f(x)\,dx = \int_0^\infty [F(x)]_y^\infty\,dy$$

$$= \int_0^\infty (1-F(y))\,dy \quad \therefore \text{證明完成}$$

(2) 同樣 $E[X] = \sum_{x=0}^{\infty} x \cdot \Pr(X=x) = \sum_{x=1}^{\infty} x \cdot \Pr(X=x)$

(非負的 double-series 可替換 $\sum_x \sum_y$)

$$= \sum_{x=1}^{\infty} \left(\sum_{y=1}^{x} \right) \Pr(X=x) = \sum_{x=1}^{\infty} \sum_{y=1}^{x} \Pr(X=x)$$

$$\sum_{y=1} \sum_{x \geq y} \Pr(X=x) = \sum_{y=1} \Pr(X \geq y) = \sum_{y=1} \{1-\Pr(X \leq y-1)\}$$

$$= \sum_{y=1}^{\infty} \{1-F_X(y-1)\} = \sum_{y=0}^{\infty} (1-F_X(y))$$

$$\therefore = \sum_{k=0}^{\infty} (1-F_X(k))$$

$\therefore$ 證明完成

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217.

(a) $\int_0^m 3x^2 dx = [x^3]_0^m = m^3 = \frac{1}{2}$

$m = \left(\frac{1}{2}\right)^{\frac{1}{3}}$ （理所當然地 $\int_m^1 3x^2 dx = \frac{1}{2}$）

(b) $\int_{-\infty}^{m} \frac{dx}{\pi(1+x^2)} = \left[\frac{1}{\pi} \cdot \arctan x\right]_{-\infty}^{m}$

$= \frac{1}{\pi}\arctan(m) + \frac{1}{\pi} \cdot \frac{\pi}{2} = \frac{1}{2}$

$\therefore \frac{1}{\pi}\arctan(m) = 0 \quad \therefore m = 0$

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218. X: 連續型 隨機變數 (設 $f(x)$ 為其pdf)

$g(a) \overset{def}{=} E[|X-a|] = \int_{-\infty}^{\infty} |x-a| \cdot f(x) dx$

$= \int_a^{\infty} (x-a) f(x) dx + \int_{-\infty}^{a} (a-x) f(x) dx$

現在 $g(a)$ 包含不可微的點.

令 m 為 X 的 median : $\int_{-\infty}^{m} f(x) dx = \frac{1}{2} = \int_m^{\infty} f(x) dx$

↓證明 $g(a) - g(m) \geq 0$ (for all a)

case I: $a = m$ 時 ∴ 顯然成立

case II. $a > m$ 時

分成 $(-\infty, m] \cup (m, a]$

$g(a) - g(m) = \int_a^{\infty} (x-a) f(x) dx + \int_{-\infty}^{a} (a-x) f(x) dy$

$- \int_m^{\infty} (x-m) f(x) dx + \int_{-\infty}^{m} (m-x) f(x) dy$

↳ 分成 $(\infty, a) \cup (a, m)$

$= \int_a^{\infty} (x-a) f(x) dx + \int_{-\infty}^{m} (a-x) f(x) dx + \int_m^{a} (a-x) f(x) dx$

$- \{ \int_a^{\infty} (x-m) f(x) dx + \int_m^{a} (x-m) f(x) dx + \int_{-\infty}^{m} (m-x) f(x) dy \}$

$= \int_a^{\infty} (m-a) f(x) dx + \int_m^{a} (m+a-2x) f(x) dy + \int_{-\infty}^{m} (a-m) f(x) dy$

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$$= \int_a^{\infty} (m-a) f(x)\,dx + \int_m^a \underbrace{(m+a-2x)\, f(x)}_{\substack{\geq (m+a-2a) f(x) \\ = (m-a) f(x)}}\,dy + \int_{-\infty}^m (a-m) f(x)\,dy$$

$$\geq \int_m^{\infty} (m-a) f(x)\,dx + \int_{-\infty}^m (a-m) f(x)\,dy$$

$$= (m-a)\left\{\int_m^{\infty} f(x)\,dx - \int_{-\infty}^m f(x)\,dx\right\} = (m-a)\left(\frac{1}{2}-\frac{1}{2}\right) = 0$$

∴ 證明完成

case II, $a<m$ 時，可以用 case I 一樣方法

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219.

$$E[(X-a)^2] = E[X^2] - 2a\,E[X] + E[a^2]$$

$$= E[X^2] - 2a\,E[X] + a^2$$

$\because \frac{d}{da} E[(X-a)^2] = 2a - 2E[X] = 0 \Rightarrow a = E[X]$

a		$E[X]$	
$\frac{d}{da} E[(X-a)^2]$	$-$	0	$+$
$E[(X-a)^2]$	$\searrow$	min	$\nearrow$

$\therefore a = E[X]$ 使得 $E[(X-a)^2]$ 最小.

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2.33.

(a) $E[e^{tX}] = \sum_{x=0} e^{tx} \cdot e^{-\lambda} \cdot \frac{\lambda^x}{x!}$

$= \sum_{x=0} e^{-\lambda} \frac{1}{x!} (\lambda e^t)^x$

利用 Taylor 展開 $e^x = 1 + x + \frac{x^2}{2!} + \cdots$

$\Rightarrow e^{(\lambda e^t)} = 1 + (\lambda e^t) + \frac{(\lambda e^t)^2}{2!}$

$\therefore \sum_{x=0} e^{-\lambda} \frac{(\lambda e^t)^x}{x!} = e^{(\lambda e^t)} \cdot e^{-\lambda} = \exp(\lambda(e^t - 1))$

$\therefore M_X(t) = \exp(\lambda(e^t - 1))$

利用 Cumulant $K_X(t) = \ln M_X(t) = \lambda(e^t - 1)$

$\frac{d}{dt} K_X(t) = \lambda e^t \qquad K_X'(0) = \lambda = E(X)$

$\frac{d^2}{dt^2} K_X(t) = \lambda e^t \qquad K_X''(0) = \lambda = V(X)$

$\therefore E(X) = V(X) = \lambda$

(b) $\Pr(X=x) = p(1-p)^x \quad E[e^{tX}] = \sum_{x=0}^{\infty} p(1-p)^x \cdot e^{tx}$

$= \sum_{x=0} p\{(1-p)e^t\}^x = p \cdot \frac{1}{1-(1-p)e^t}$

$\therefore M_X(t) = \frac{p}{1-(1-p)e^t}$

$K_X(t) = \ln M_X(t) = \ln p - \ln(1-(1-p)e^t)$

$\frac{\partial K_X(t)}{\partial t} = \frac{(1-p)e^t}{1-(1-p)e^t} \qquad K_X'(0) = \frac{1-p}{p} = E(X)$

$\frac{\partial^2 K}{\partial t^2} = \frac{qe^t(1-qe^t)+q^2e^{2t}}{(1-qe^t)^2} = \frac{qe^t}{(1-qe^t)^2} = \frac{(1-p)e^t}{(1-(1-p)e^t)^2}$

$(q=1-p) \qquad \therefore K_X''(0) = \frac{1-p}{p^2} = V(X)$

$\therefore E[X] = \frac{1-p}{p} \quad V(X) = \frac{1-p}{p^2}$

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(c) $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-1}{2\sigma^2}(x-\mu)^2\right)$

$$E[e^{tX}] = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-1}{2\sigma^2}(x-\mu)^2 + tx\right) dx$$

$z \overset{def}{=} \frac{x-\mu}{\sigma} \qquad \frac{dz}{dx} = \frac{1}{\sigma}$

$$= \int \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-z^2}{2} + t(\sigma z + \mu)\right) \cdot \sigma dz$$

$$= \int \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-z^2}{2} + \sigma t z + \mu t\right) dz$$

$$= \int \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-1}{2}(z-\sigma t)^2 + \frac{\sigma^2 t^2}{2} + \mu t\right) dz$$

$$= \underbrace{\int \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-1}{2}(z-\sigma t)^2\right) dz}_{=1} \cdot \exp\left(\frac{\sigma^2 t^2}{2} + \mu t\right)$$

$\therefore M_X(t) = \exp\left(\frac{\sigma^2 t^2}{2} + \mu t\right)$

$K_X(t) = \ln M_X(t) = \frac{\sigma^2 t^2}{2} + \mu t$

$K_X'(t) = \mu + \sigma^2 t \qquad K_X'(0) = \mu$

$K_X''(t) = \sigma^2 \qquad K_X''(0) = \sigma^2$

$\therefore E(X) = \mu \quad V(X) = \sigma^2$

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2.38.

(a) 我們利用以下Fact.

「$X_1, \dots X_n \overset{iid}{\sim} Geo(p) \Rightarrow X_1+\dots+X_n \sim NB(r,p)$」

$\because \{X_i\}_{i=1}^{n}$ 為在成功一次前失敗的次數 (≥ 0)

$\Rightarrow X_1+\dots+X_n$ 為在成功 r 次前失敗的次數 (≥ 0)

($X_1 \sim X_n$ 為獨立)

$\therefore X_1+\dots+X_n \sim NB(r,p)$

$$\therefore E\left[e^{t(X_1+\dots+X_n)}\right] = \left(E\left[e^{tX_1}\right]\right)^n \quad (\because iid)$$

$$E\left[e^{tX_1}\right] = \sum_{x=0} e^{tx} \cdot p(1-p)^x = \sum_{x=0} p\left\{(1-p)e^t\right\}^x$$

$$= p \cdot \frac{1}{1-(1-p)e^t}$$

$$\therefore M_{X_1}(t) = \frac{p}{1-(1-p)e^t}$$

$$\therefore M_{X_1+\dots X_n}(t) = \left\{\frac{p}{1-(1-p)e^t}\right\}^n$$

$$\therefore M_X(t) = \left(\frac{p}{1-(1-p)e^t}\right)^n$$

$(X = X_1+\dots+X_n)$

(b) $M_X(t) = \left(\frac{p}{1-(1-p)e^t}\right)^r = E[e^{tX}]$

$\therefore E[e^{t(2pX)}] = E[e^{(2pt)X}] = M_X(2pt)$

$\therefore M_X(2pt) = \left\{\frac{p}{1-(1-p)e^{2pt}}\right\}^r$

$\lim_{p\to 0} M_X(2pt) = \lim_{p\to 0}\left\{\frac{p}{1-(1-p)e^{2pt}}\right\}^r$

利用 L'Hôpital's Rule $\lim_{p\to 0}\left(\frac{p}{1-(1-p)e^{2pt}}\right)^r = \lim_{p\to 0}\left(\frac{1}{e^{2pt}-2t(1-p)e^{2pt}}\right)^r$

$= \lim_{p\to 0}\left(\frac{1}{1-2t}\right)^r$

注意 Gamma(α,β) 的 mgf 為 $(1-\beta t)^{-\alpha}$

$p\to\infty$ 時 $2pX \xrightarrow{d} \text{Gamma}(r,2) = \chi^2(2r)$

(分佈收斂)

(∵ Levy 連續性定理)

∴ 證明完成

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資料科學之統計基礎(II) HW3 R05246013 森元俊成

$\boxed{3.2}$ $HG(\underbrace{100}_{\text{總數}}; \underbrace{M}_{\text{defective}}; \underbrace{100-M}_{\text{normal}}; \underbrace{K}_{\text{抽樣數}})$

(a) $Pr(\text{未抽到 defective 樣品} \mid M, K)$

$$= \frac{{}_M C_0 \cdot {}_{100-M} C_k}{{}_{100} C_k} = \frac{k!(100-k)!}{100!} \cdot \frac{(100-M)!}{k!(100-M-k)!}$$

$$= \frac{(100-M)!(100-k)!}{100!(100-M-k)!}$$

求:

$$\underset{K}{\operatorname{argmin}} \left\{ \sup_{M \ge 6} \left\{ \frac{(100-M)!(100-k)!}{100!(100-M-k)!} \right\} \le 0.1 \right\}$$

$$P(M) \stackrel{def}{=} \frac{(100-M)!(100-k)!}{100!(100-M-k)!}$$

$$\frac{P(M)}{P(M-1)} = \frac{\cancel{(100-M)!}\cancel{(100-k)!}}{\cancel{100!}\cancel{(100-M-k)!}} \cdot \frac{\cancel{100!}\overset{(101-M-k)}{\cancel{(101-M-k)!}}}{\underset{(101-M)}{\cancel{(101-M)!}}\cancel{(100-k)!}}$$

$$= \frac{101-M-k}{101-M} < 1 \Rightarrow P(6) > P(7) > P(8) > \cdots$$

$$\therefore \sup_{M \ge 6}\{P(M)\} = P(6) = \frac{94!(100-k)!}{100!(94-k)!}$$

我們求 $\underset{K}{\operatorname{argmin}}\{P(6)_k \le 0.1\}$

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∴求最小的 K 使得 $\frac{94!\,(100-K)!}{100!\,(94-K)!} \leq 0.1$

$(K \geq 32)$ $\left(\begin{array}{l} k=31 \ldots \ 0.1006 \\ k=32 \ldots \ 0.0918 \end{array}\right)$

(b) Pr(抽到 defective 樣品 至多 1 $| M, K$)

$$= \frac{(100-M)!\,(100-K)!}{100!\,(100-M-K)!} + \frac{{}_MC_1 \cdot {}_{100-M}C_{K-1}}{{}_{100}C_K}$$

$$= \frac{(100-M)!\,(100-K)!}{100!\,(100-M-K)!} + \frac{MK\,(100-K)!\,(100-M)!}{100!\,(101-M-K)!}$$

同樣求 $\underset{K}{\operatorname{argmin}} \left\{ \sup_{M \geq 6} \left\{ \frac{(100-M)!\,(100-K)!}{100!\,(100-M-K)!} + \frac{MK(100-K)!\,(100-M)!}{100!\,(101-M-K)!} \right\} \leq 0.1 \right\}$

M=6 時,「抽到 defective 樣品 至多 1」的 機率 最大

∴求 $\underset{K}{\operatorname{argmin}} \left\{ \frac{94!\,(100-K)!}{100!\,(94-K)!} + \frac{6K\,(100-K)!\,94!}{100!\,(95-K)!} \leq 0.1 \right\}$

$(K \geq 51)$ $\left(\begin{array}{l} k=50 \ldots \ 0.1022 \\ k=51 \ldots \ 0.0933 \end{array}\right)$

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$\boxed{3.4}$ 令X為嘗試次數

(a) $Pr(X=x) = \underbrace{\left(\frac{1}{n}\right)}_{\text{成功}} \cdot \underbrace{\left(\frac{n-1}{n}\right)^{x-1}}_{\text{失敗}x-1\text{次}} \sim \overset{\text{幾何分佈}}{Geo}\left(\frac{1}{n}\right)$ $\therefore E[X] = \frac{1}{\left(\frac{1}{n}\right)} = n$

(b) $Pr(X=x) = \frac{(n-1)!}{n!} = \frac{1}{n}$ (離散均勻分佈)

$\Rightarrow$ 考慮 n個鑰匙的排列：$n!$

第x個鑰匙是正確的鑰匙的排列：$(n-1)!$
($\because$ 第x個是固定的)

$\therefore \frac{(n-1)!}{n!} = \frac{1}{n}$

$\therefore Pr(X) = \sum_{x=1}^{n} \frac{x}{n} = \frac{1}{n} \cdot \frac{n(n+1)}{2} = \frac{n+1}{2}$

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(a) 生雙包胎的機率 $= \frac{1}{90}$

$X \sim Bin(60, \frac{1}{90})$ $\quad Pr(X=x) = \left(\frac{1}{90}\right)^x \left(\frac{89}{90}\right)^{60-x} \cdot {}_{60}C_x$

$Pa = Pr(X \geq 5) = 1 - Pr(X \in \{0,1,2,3,4\})$

$= 1 - \left(\frac{89}{90}\right)^{60} - \left(\frac{1}{90}\right)^1 \cdot \left(\frac{89}{90}\right)^{59} \cdot {}_{60}C_1 - \left(\frac{1}{90}\right)^2 \cdot \left(\frac{89}{90}\right)^{58} {}_{60}C_2$

$- \left(\frac{1}{90}\right)^3 \cdot \left(\frac{89}{90}\right)^{57} \cdot {}_{60}C_3 - \left(\frac{1}{90}\right)^4 \left(\frac{89}{90}\right)^{56} \cdot {}_{60}C_4$

(b) New York State 有 310 所幼稚園 (62×5)

$\therefore Pb = Pr(X_1 \geq 5 \cup X_2 \geq 5 \cup \cdots X_{310} \geq 5)$

$\therefore Pb = 1 - Pr(X_1 \leq 4 \cap X_2 \leq 4 \cap \cdots X_{310} \leq 4)$

$= 1 - (1 - Pa)^{310}$

(c) 全美國有 15500 所 (310×50) 幼稚園

$Pc = 1 - (1 - Pa)^{15500}$

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計算結果

(a) $P_a = 5.5663 \cdot 10^{-4}$

(b) $P_b = 0.1585$

(c) $P_c = 0.9998$

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$\boxed{3.10}$ N個…… 有毒品
M個… 無毒品 $(M+N=496$ 個$)$

(a)
先抽4個,再抽2個,先抽的4個是有毒品的,然後後面2個是無毒品的。這樣子的機率是

$$\left(\frac{{}_N C_4}{{}_{N+M} C_4}\right)\cdot\left(\frac{{}_M C_2}{{}_{N+M-4} C_2}\right)$$

這個也同樣道理

所有可能的抽樣方式… ${}_{N+M}C_4$
滿足條件(有毒品)的抽樣方式 ${}_N C_4$

(註) 這個事件考慮前後順序 所以跟「超幾何分佈」不一樣。

(b) $M+N=496$,可寫成 N的函數

$$Pr(N)=\frac{{}_N C_4}{{}_{496} C_4}\cdot\frac{{}_{496-N} C_2}{{}_{492} C_2}$$

考慮: $$\frac{Pr(N)}{Pr(N-1)}=\frac{{}_N C_4\cdot{}_{496-N}C_2}{{}_{N-1}C_4\cdot{}_{497-N}C_2}$$

$$=\frac{N!}{4!(N-4)!}\cdot\frac{(496-N)!}{2!(494-N)!}\times\frac{4!(N-5)!\,\overset{2!}{\vee}(495-N)!}{(N-1)!(497-N)!}$$

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$$\therefore \frac{Pr(N)}{Pr(N-1)} = \frac{N}{N-4} \cdot \frac{495-N}{497-N}$$

$$\therefore \frac{Pr(N)}{Pr(N-1)} \geq 1 \Leftrightarrow 「495N - N^2 \geq 501N - N^2 - 1988」$$

$$\Rightarrow 「6N \leq 1988」 \Rightarrow N \leq 331.33\cdots$$

$$\therefore \frac{Pr(N=2)}{Pr(N=1)} > 1,\ \frac{Pr(N=3)}{Pr(N=2)} > 1, \cdots \frac{Pr(N=331)}{Pr(N=330)} > 1,\ \frac{Pr(N=332)}{Pr(N=331)} < 1.$$

$$\Rightarrow Pr(N=1) < Pr(N=2) < \cdots < \underbrace{Pr(N=331)}_{max} > Pr(N=332) \geq \cdots$$

$\therefore$ $N = 331$ $(M=165)$ 時最大

$Pr(N=331) \approx 0.22$ (代入即可)

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$$\frac{\lambda}{1-e^{-\lambda}}$$

3.13

(a) $X \sim Po(\lambda)$ 時求「$X \mid X \geq 1$」的條件

$$\underset{(x \geq 1)}{Pr(X=x \mid X \geq 1)} = \frac{Pr(X=x \cap X \geq 1)}{Pr(X \geq 1)} = \frac{e^{-\lambda} \cdot \frac{\lambda^x}{x!} \cdot I(x \geq 1)}{1-e^{-\lambda}}$$

$$\therefore Pr(X=x \mid X \geq 1) = \frac{e^{-\lambda}}{1-e^{-\lambda}} \cdot \frac{\lambda^x}{x!} \cdot I(x \geq 1)$$

(x: 整數)

考慮動差母函數

$$E[e^{tX} \mid X \geq 1] = \sum_{x \geq 1} e^{tx} \cdot \frac{e^{-\lambda}}{1-e^{-\lambda}} \frac{\lambda^x}{x!}$$

$$= \sum_{x \geq 1} \frac{e^{-\lambda}}{1-e^{-\lambda}} \cdot \frac{(\lambda e^t)^x}{x!} = \frac{e^{-\lambda}}{1-e^{-\lambda}} \cdot \sum_{x \geq 1} \frac{(\lambda e^t)^x}{x!}$$

$$= \frac{e^{-\lambda}}{1-e^{-\lambda}} \cdot \left(\exp(\lambda e^t) - 1 \right)$$

(註 $e^t = 1 + t + \frac{t^2}{2!} + \cdots \Rightarrow e^t - 1 = t + \frac{t^2}{2!} + \cdots$)

由此可知 $M_{X|X\geq 1}(t) = \frac{e^{-\lambda}}{1-e^{-\lambda}} \left(\exp(\lambda e^t) - 1 \right)$

$$M'_{X|X\geq 1}(t) = \frac{e^{-\lambda}}{1-e^{-\lambda}} \cdot \left(\lambda e^t \cdot \exp(\lambda e^t) \right)$$

$$M''_{X|X\geq 1}(t) = \frac{e^{-\lambda}}{1-e^{-\lambda}} \left((\lambda e^t)^2 \exp(\lambda e^t) + \lambda e^t \exp(\lambda e^t) \right)$$

$$E[X|X\geq 1] = M'_{X|X\geq 1}(0) = \frac{e^{-\lambda}}{1-e^{-\lambda}} \cdot \lambda \cdot \exp(\lambda) = \frac{\lambda}{1-e^{-\lambda}}$$

$$E[X^2|X\geq 1] = M''_{X|X\geq 1}(0) = \frac{e^{-\lambda}}{1-e^{-\lambda}} \left(\lambda^2 \exp(\lambda) + \lambda \exp(\lambda) \right)$$

$$= \frac{\lambda^2+\lambda}{1-e^{-\lambda}}$$

$$\therefore V[X|X\geq 1] = \left(\frac{\lambda^2+\lambda}{1-e^{-\lambda}}\right) - \left(\frac{\lambda}{1-e^{-\lambda}}\right)^2$$

(b) $X \sim NB(r,p)$ … X代表失敗次數

$$Pr(X=x) = p^{r-1} \cdot (1-p)^x \cdot {}_{x+r-1}C_r \cdot p = p^r (1-p)^x \cdot {}_{x+r-1}C_x$$

「 p 成功 ($r-1$次) $1-p$ 失敗 (x次) ‖ p 成功 (1次) 」

可換順序

$$\therefore Pr(X=x|X\geq 1) = \frac{Pr(X=x, X\geq 1)}{1-Pr(X=0)} = \frac{p^r(1-p)^x \cdot {}_{x+r-1}C_x \cdot I_{(x\geq 1)}}{1-p^r}$$

接著考慮 $E[X|X\geq 1]$ 及 $V[X|X\geq 1]$

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※ 3.13

雖然直接求 $E[X|X\geq 1]$ or $E[X^2|X\geq 1]$ 有點麻煩

但 $E[X]=r\left(\frac{1-p}{p}\right)$ $V[X]=\frac{r(1-p)}{p^2}$ 是已知的，利用這件事.

為了方便起見，設 $P(x)=Pr(X=x)$．$\left(P(x)=p^r(1-p)^x \cdot {}_{x+r-1}C_x\right)$

① $E[X|X\geq 1]=\sum_{x\geq 1}\frac{xP(x)}{1-p^r}=\frac{1}{1-p^r}\sum_{x\geq 1}xP(x)$

$=\frac{1}{1-p^r}\sum_{x\geq 0}xP(x)=\frac{1}{1-p^r}E[X]=\frac{1}{1-p^r}\frac{r(1-p)}{p}$

($x=0$ 時 $xP(x)=0$
∴加不加一樣)

② $E[X^2|X\geq 1]=\sum_{x\geq 1}\frac{x^2P(x)}{1-p^r}=\frac{1}{1-p^r}\sum_{x\geq 1}x^2P(x)=\frac{1}{1-p^r}\sum_{x\geq 0}x^2P(x)$

$=\frac{1}{1-p^r}E[X^2]=\frac{1}{1-p^r}\left(\frac{r(1-p)}{p^2}+\frac{r^2(1-p)^2}{p^2}\right)$

∴ $V[X^2|X\geq 1]=E[X^2|X\geq 1]-E[X|X\geq 1]^2$

$=\frac{1}{1-p^r}\left(\frac{r(1-p)}{p^2}+\frac{r^2(1-p)^2}{p^2}\right)-\left\{\frac{1}{1-p^r}\cdot\frac{r(1-p)}{p}\right\}^2$

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若取「失敗次數」
會有不一樣的結果

$\boxed{3.18}$ 利用動差母函數

令 $\{X_j\}_{j=1\sim r} \overset{iid}{\sim} NB(1,p) = Geo(p)$（嘗試次數）

$Pr(X_j=x) = (1-p)^{x-1}\cdot p \quad (x\geq 1)$

$\therefore E[e^{tX_j}] = \sum_{x\geq 1} e^{tx}(1-p)^{x-1}\cdot p = \sum_{x\geq 1} \frac{p}{1-p}\{(1-p)e^t\}^x$

$= \dfrac{pe^t}{1-(1-p)e^t} \quad \therefore M_{X_j}(t) = \dfrac{pe^t}{1-(1-p)e^t}$

$Y \overset{def}{=} \sum_{j=1}^{r} X_j \sim NB(r,p)$

$M_Y(t) = E[e^{t(X_1+\cdots+X_r)}] = M_{X_j}(t)^r = \left\{\dfrac{pe^t}{1-(1-p)e^t}\right\}^r$

$\therefore pY$ 之動差母函數 $= E[e^{t(pY)}] = M_Y(pt)$

$= \left\{\dfrac{pe^{pt}}{1-(1-p)e^{pt}}\right\}^r$

$\therefore M_{pY}(t) \overset{def}{=} \left(\dfrac{pe^{pt}}{1-(1-p)e^{pt}}\right)^r$

$\lim_{p\to 0} M_{pY}(t) = \lim_{p\to 0}\left(\dfrac{pe^{pt}}{1-(1-p)e^{pt}}\right)^r$

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L'Hôpital's rule

$$\lim_{p\to 0}\left(\frac{e^{pt}+pte^{pt}}{e^{pt}-t(1+p)e^{pt}}\right)^r = (1-t)^{-r}$$

$\Gamma(\alpha,\beta)$之動差母函數 $(1-\beta t)^{-\alpha}$

$\therefore \beta=1 \quad \alpha=r$

根據 Lévy 連續性定理，$\{pY\}_p$: $pY \xrightarrow{d} \Gamma(r,1)$.

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$\boxed{3.22}$

(a) $E[X] = \sum_{x=0} e^{-\lambda} \cdot \frac{x\lambda^x}{x!} = \sum_{x=1} e^{-\lambda} \frac{\lambda^x}{(x-1)!}$

$= \sum_{x=0} e^{-\lambda} \frac{\lambda^{x+1}}{x!} = \lambda e^{-\lambda} \underbrace{\sum_{x=0} \frac{\lambda^x}{x!}}_{=e^{\lambda}} = \lambda$

$\left(\because e^t = 1 + t + \frac{t^2}{2!} + \cdots\right)$

$E[X^2] = \sum_{x=0} e^{-\lambda} \cdot \frac{x^2\lambda^x}{x!} = \sum_{x=1} e^{-\lambda} \frac{x^2}{x!} \lambda^x$

$= \sum_{x=1} e^{-\lambda} \cdot \frac{x\lambda^x}{(x-1)!} = \sum_{x=0} e^{-\lambda} \cdot \frac{(x+1)}{x!} \lambda^{x+1}$

$= \sum_{x=0} e^{-\lambda} \cdot \frac{x \cdot \lambda^{x+1}}{x!} + \underbrace{\sum_{x=0} e^{-\lambda} \cdot \frac{\lambda^{x+1}}{x!}}_{\lambda e^{-\lambda}(e^{\lambda}) = \lambda}$ ($\uparrow$ 一樣)

$= \sum_{x=1} e^{-\lambda} \cdot \frac{x\lambda^{x+1}}{x!} + \lambda$

$= \sum_{x=1} e^{-\lambda} \frac{\lambda^{x+1}}{(x-1)!} + \lambda$

$= \sum_{x=0} \frac{e^{-\lambda} \cdot \lambda^{x+2}}{x!} + \lambda$

$= \lambda^2 e^{-\lambda} (e^{\lambda}) + \lambda = \lambda^2 + \lambda$

$\therefore V[X] = (\lambda^2 + \lambda) - (\lambda)^2 = \lambda$

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(c) $E[X] = \int_0^\infty x \cdot \frac{x^{\alpha-1}}{\Gamma(\alpha)\beta^\alpha} \exp\left(\frac{-x}{\beta}\right) dx \qquad \frac{x}{\beta} = y \qquad \frac{dy}{dx} = \frac{1}{\beta}$

$= \int_0^\infty \frac{(\beta y)^\alpha}{\Gamma(\alpha)\beta^\alpha} \exp(-y) \cdot \beta dy$

$= \int_0^\infty \frac{\beta}{\Gamma(\alpha)} y^\alpha \exp(-y) dy = \frac{\beta}{\Gamma(\alpha)} \cdot \Gamma(\alpha+1) = \underline{\underline{\alpha\beta}}$

$E[X^2] = \int_0^\infty x^2 \cdot \frac{x^{\alpha-1}}{\Gamma(\alpha)\beta^\alpha} \exp\left(\frac{-x}{\beta}\right) dx \qquad \frac{x}{\beta} = y \qquad \frac{dy}{dx} = \frac{1}{\beta}$

$= \int_0^\infty \frac{(\beta y)^{\alpha+1}}{\Gamma(\alpha)\beta^\alpha} \exp(-y) \beta dy$

$= \int_0^\infty \beta^2 \cdot \frac{y^{\alpha+1}}{\Gamma(\alpha)} \exp(-y) dy$

$= \beta^2 \cdot \Gamma(\alpha+2)/\Gamma(\alpha) = \beta^2 \cdot \alpha(\alpha+1)$

$\therefore V[X] = \beta^2 \cdot \alpha(\alpha+1) - (\alpha\beta)^2 = \underline{\underline{\alpha\beta^2}}$

↓ (下一頁)

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※ 3.22

(e) $\int_{-\infty}^{\infty} \frac{1}{2\sigma} \exp\left(\frac{-|x-\mu|}{\sigma}\right) dx$

$$E[X] = \int_{-\infty}^{\infty} \frac{x}{2\sigma} \exp\left(\frac{-|x-\mu|}{\sigma}\right) dx \qquad y = x-\mu$$

$$= \int_{-\infty}^{\infty} \frac{(y+\mu)}{2\sigma} \exp\left(\frac{-|y|}{\sigma}\right) dy$$

$$= \underbrace{\int_{-\infty}^{\infty} \frac{y}{2\sigma} \exp\left(\frac{-|y|}{\sigma}\right) dy}_{\to 0\ (\because 奇函数)} + \mu \underbrace{\int_{-\infty}^{\infty} \frac{1}{2\sigma} \exp\left(\frac{-|y|}{\sigma}\right) dy}_{\to 1\ (\because \int pdf = 1)}$$

$$= \mu$$

$$E[X^2] = \int_{-\infty}^{\infty} \frac{x^2}{2\sigma} \exp\left(\frac{-1}{\sigma}|x-\mu|\right) dx \qquad y = x-\mu$$

$$= \int_{-\infty}^{\infty} \frac{(y+\mu)^2}{2\sigma} \exp\left(\frac{-1}{\sigma}|y|\right) dy$$

$$= \int_{-\infty}^{\infty} \frac{y^2}{2\sigma} \exp\left(\frac{-1}{\sigma}|y|\right) dy + 2\mu \underbrace{\int \frac{y}{2\sigma} \exp\left(\frac{-1}{\sigma}|y|\right) dy}_{\to 0}$$

$$+ \mu^2 \underbrace{\int_{-\infty}^{\infty} \frac{1}{2\sigma} \exp\left(\frac{-1}{\sigma}|y|\right) dy}_{\to 1}$$

$$= \mu^2 + \int_{-\infty}^{\infty} \frac{y^2}{2\sigma} \exp\left(\frac{-1}{\sigma}|y|\right) dy$$

↗ $z = |y|$

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$z = |y|$

$$= \mu^2 + \int_0^\infty \frac{y^2}{2\sigma} \exp\left(-\frac{|y|}{\sigma}\right) dy + \int_{-\infty}^0 \frac{y^2}{2\sigma} \exp\left(-\frac{|y|}{\sigma}\right) dy$$

$$= \mu^2 + \int_0^\infty \frac{z^2}{2\sigma} \exp\left(\frac{-z}{\sigma}\right) dz + \int_\infty^0 \frac{z^2}{2\sigma} \exp\left(\frac{-z}{\sigma}\right)(-1)\, dz$$

$$= \mu^2 + \int_0^\infty \frac{z^2}{\sigma} \exp\left(\frac{-z}{\sigma}\right) dz$$

(利用(c)的結果: $\alpha = 1 \quad \beta = \sigma$
$E[X^2] = \beta^2(\alpha)(\alpha+1) = 2\sigma^2$)

$$= \mu^2 + 2\sigma^2$$

$$\therefore V[X] = (\mu^2 + 2\sigma^2) - (\mu)^2 = 2\sigma^2$$

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$\boxed{3.24}$ ① 求pdf

(a) 全機率 $=1=\int_0^\infty \frac{1}{\beta}\exp\left(-\frac{1}{\beta}x\right)dx$

$x=y^\gamma$

$\frac{dx}{dy}=\gamma y^{\gamma-1}$

$x: 0\to\infty$

$y: 0\to\infty$

$=\int_{y=0}^{y=\infty}\frac{1}{\beta}\exp\left(-\frac{1}{\beta}y^\gamma\right)\cdot\left(\gamma y^{\gamma-1}\right)dy$

$=\int_{y=0}^{y=\infty}\frac{\gamma}{\beta}y^{\gamma-1}\exp\left(-\frac{y^\gamma}{\beta}\right)dy$

$\therefore f_Y(y)=\frac{\gamma}{\beta}y^{\gamma-1}\exp\left(-\frac{y^\gamma}{\beta}\right)\cdot I_{(y>0)}$

$\int f_Y(y)dy=1$ (定積分的變數轉換不會改變結果)

(c) 全機率 $=1=\int_0^\infty\frac{1}{\Gamma(\alpha)\beta^\alpha}x^{\alpha-1}\exp\left(-\frac{x}{\beta}\right)dx$

$x=\frac{1}{y}$

$\frac{dx}{dy}=-\frac{1}{y^2}$

$x: 0^+\to\infty$

$y: \infty\to 0$

$=\int_{y=\infty}^{y=0}\frac{1}{\Gamma(\alpha)\beta^\alpha}\left(\frac{1}{y}\right)^{\alpha-1}\exp\left(-\frac{1}{\beta y}\right)\cdot\frac{-1}{y^2}dy$

$=\int_0^\infty\frac{1}{\Gamma(\alpha)\beta^\alpha}\cdot\frac{1}{y^{\alpha+1}}\exp\left(-\frac{1}{\beta y}\right)dy$

$\therefore f_Y(y)=\frac{1}{\Gamma(\alpha)\beta^\alpha}y^{-(\alpha+1)}\exp\left(-\frac{1}{\beta y}\right)\cdot I_{(y>0)}$

$\int f_Y(y)dy=1$ ($\because$ ↑)

(e) 全確率 $= 1 = \int_0^{\infty} e^{-x} dx$

$y = \alpha - \gamma \ln x$

$x = \exp\left(-\frac{1}{\gamma}(y-\alpha)\right)$

$\frac{dx}{dy} = -\frac{1}{\gamma} \exp\left(-\frac{1}{\gamma}(y-\alpha)\right)$

$x : 0 \to \infty$

$y : \infty \to -\infty$

$$= \int_{y=\infty}^{y=-\infty} \exp\left(-\exp\left(-\frac{1}{\gamma}(y-\alpha)\right)\right) \cdot -\frac{1}{\gamma} \exp\left(-\frac{1}{\gamma}(y-\alpha)\right) dy$$

$$= \int_{y=-\infty}^{y=\infty} \frac{1}{\gamma} \exp\left(-\exp\left(-\frac{1}{\gamma}(y-\alpha)\right) - \frac{1}{\gamma}(y-\alpha)\right) dy$$

$\therefore f(y) = \frac{1}{\gamma} \exp\left(-\exp\left(-\frac{1}{\gamma}(y-\alpha)\right) - \frac{1}{\gamma}(y-\alpha)\right)$

$\int f(y) dy = 1 \quad (\because \uparrow)$

② 下一頁：（$E[Y]$, $V[Y]$）

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$\boxed{3.25}$

$$h_T(t) = \lim_{\delta \downarrow 0} \frac{\Pr(t \leq T < t+\delta \mid T \geq t)}{\delta}$$

$$= \lim_{\delta \downarrow 0} \frac{\Pr(t \leq T < t+\delta)}{\delta \cdot \Pr(T \geq t)}$$

$$= \lim_{\delta \downarrow 0} \frac{\Pr(T < t+\delta) - \Pr(T < t)}{\delta \cdot [1 - \Pr(T < t)]}$$

$$= \lim_{\delta \downarrow 0} \frac{\Pr(T < t+\delta) - \Pr(T < t)}{\delta} \cdot \frac{1}{1 - \Pr(T < t)}$$

現在T為連續型隨機變數、$F_T(t)$為連續函數

故 $\Pr(T<t) = \lim_{x \uparrow t} \Pr(T \leq x) = F_T(t)$.
(左連續)

$$\therefore h_T(t) = \lim_{\delta \downarrow 0} \frac{F_T(t+\delta) - F_T(t)}{\delta} \cdot \frac{1}{1 - F_T(t)}$$

$$= \frac{f_T(t)}{1 - F_T(t)} \quad (\because \text{cdf 的微分} = \text{pdf})$$

$$-\frac{d}{dt} \ln(1 - F_T(t)) = -\frac{\frac{d}{dt}(1 - F_T(t))}{1 - F_T(t)} = \frac{f_T(t)}{1 - F_T(t)}$$

$\therefore$ 證明完成

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$\boxed{3.28}$ Exponetial Family: $f(x|\vec{\theta}) = h(x)\cdot \exp\left(c_1(\vec{\theta})T_1(x)+\cdots+c_k(\vec{\theta})T_k(x)\right)$

(a) $f(x|\mu,\sigma^2) = \dfrac{1}{\sqrt{2\pi\sigma^2}}\exp\left(\dfrac{-1}{2\sigma^2}(x-\mu)^2\right)$

① μ is known ... (μ可以當常數)

$$\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)\exp\left(\frac{-1}{2\sigma^2}(x-\mu)^2\right)$$

$$= \exp\left(\frac{-1}{2\sigma^2}(x-\mu)^2 - \frac{1}{2}\ln(2\pi\sigma^2)\right)$$

$$= \exp\left(c_1(\sigma^2)T_1(x) + d(\sigma^2)\right)$$

• where $c_1(\sigma^2) = \dfrac{-1}{2\sigma^2}$

$T_1(x) = (x-\mu)^2$

$d(\sigma^2) = \dfrac{-1}{2}\ln(2\pi\sigma^2)$

∴ 指數族

② σ^2 is known

$$f = \exp\left(\frac{-1}{2\sigma^2}x^2 + \frac{\mu x}{\sigma^2} - \frac{\mu^2}{2\sigma^2} - \frac{1}{2}\ln(2\pi\sigma^2)\right)$$

$$= \underbrace{\exp\left(\frac{-x^2}{2\sigma^2}\right)}_{h(x)}\cdot\exp\left(\underbrace{\mu}_{c_1(\mu)}\underbrace{\left(\frac{x}{\sigma^2}\right)}_{T_1(x)} \underbrace{- \frac{\mu^2}{2\sigma^2} - \frac{1}{2}\ln(2\pi\sigma^2)}_{d_1(\mu)}\right)$$

∴ 指數族

(b) $$f(x) = \frac{x^{\alpha-1}}{\Gamma(\alpha)\cdot\beta^{\alpha}} \exp\left(\frac{-x}{\beta}\right) I(x>0)$$

$I(x>0)$

(I) α: known ; β: unknown

$$f = \underbrace{\frac{x^{\alpha-1}}{\Gamma(\alpha)}}_{h(x)} \exp\Big(\underbrace{\frac{-1}{\beta}}_{c_1(\beta)}\cdot \underbrace{x}_{T_1(x)} + \underbrace{-\alpha\ln\beta}_{d(\beta)}\Big)$$

∴ 指數族

(II) α: unknown ; β: known

$$f = \underbrace{I(x>0)\cdot\frac{1}{x}\cdot\exp\left(\frac{-x}{\beta}\right)}_{h(x)} \cdot \exp\Big(\underbrace{\alpha}_{c_1(\alpha)}\underbrace{\ln x}_{T_1(x)} + \underbrace{-\ln\Gamma(\alpha)-\alpha\ln\beta}_{d(\alpha)}\Big)$$

∴ 指數族

(III) α, β: unknown

$$f = \underbrace{I(x>0)\cdot\frac{1}{x}}_{h(x)}\cdot \exp\Big(\underbrace{\alpha}_{c_1(\alpha,\beta)}\cdot\underbrace{\ln x}_{T_1(x)} + \underbrace{\left(\frac{-1}{\beta}\right)}_{c_2(\alpha,\beta)}\cdot\underbrace{x}_{T_2(x)} + \underbrace{-\ln\Gamma(\alpha)-\alpha\ln\beta}_{d(\alpha,\beta)}\Big)$$

∴ 指數族

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(*) 3.28

(c) $f(x) = \frac{1}{Be(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1} \cdot I_{(0,1)}(x)$

(I) : α : known , β : unknown

$$f = \underbrace{x^{\alpha-1} \cdot I_{(0,1)}(x) \cdot \frac{1}{1-x}}_{h(x)} \cdot \exp\Big(\underset{c(\beta)}{\underset{\Downarrow}{\beta}} \cdot \ln \underset{T(x)}{\underset{\Downarrow}{(1-x)}} \underbrace{- \ln Be(\alpha,\beta)}_{d(\beta)} \Big)$$

(II) α : unknown β : known

$$f = \underbrace{\frac{1}{x}(1-x)^{\beta-1} \cdot I_{(0,1)}(x)}_{h(x)} \cdot \exp\Big(\underset{c(\alpha)}{\underset{\Downarrow}{\alpha}} \cdot \underset{T(x)}{\underset{\Downarrow}{\ln x}} \underset{d(\alpha)}{\underset{\Downarrow}{- \ln Be(\alpha,\beta)}} \Big)$$

(III) α : unknown β : unknown

$$f = \underset{h(x)}{\underset{\Downarrow}{\frac{I_{(0,1)}(x)}{x(1-x)}}} \cdot \exp\Big(\underset{c_1\binom{\alpha}{\beta}}{\underset{\Downarrow}{\alpha}} \cdot \underset{T_1(x)}{\underset{\Downarrow}{\ln(x)}} + \underset{c_2\binom{\alpha}{\beta}}{\underset{\Downarrow}{\beta}} \cdot \underset{T_2(x)}{\underset{\Downarrow}{\ln(1-x)}} \underset{d\binom{\alpha}{\beta}}{\underset{\Downarrow}{- \ln Be(\alpha,\beta)}} \Big)$$

$$c_1\binom{\alpha}{\beta} \cdot T_1(x) + c_2\binom{\alpha}{\beta} \cdot T_2(x) + d\binom{\alpha}{\beta}$$

(d) $Pr(X=x) = e^{-\lambda} \cdot \frac{\lambda^x}{x!} \cdot I(x \in \mathbb{N} \cup \{0\})$

$$= \underbrace{\frac{I(x \in \mathbb{N} \cup \{0\})}{x!}}_{h(x)} \cdot \exp\Big(\underset{c(\lambda)}{\underset{\Downarrow}{\ln \lambda}} \cdot \underset{T(x)}{\underset{\Downarrow}{x}} \underset{d(\lambda)}{\underset{\Downarrow}{- \lambda}} \Big)$$

$$c(\lambda) \cdot T(x) + d(\lambda)$$

(e) X … 失敗次数 $\sim NB(r, p)$

「$\underset{r-1\text{次}}{(\text{成功})}\ \underset{x\text{次}}{(\text{失敗})}\ \Big\|\ \underset{1\text{回}}{(\text{成功})}$」

$$Pr(X=x) = p^{r-1} \cdot (1-p)^{x} \cdot {}_{r+x-1}C_{x} \cdot p \cdot I(x \in \mathbb{N} \cup \{0\})$$

$$= p^{r} (1-p)^{x}\, {}_{r+x-1}C_{x} \cdot I(x \in \mathbb{N} \cup \{0\})$$

$$= \underbrace{{}_{r+x-1}C_{x} \cdot I(x \in \mathbb{N} \cup \{0\})}_{h(x)} \cdot \exp\Big(\underbrace{(1-p)}_{c(\phi)} \cdot \underbrace{\ln x}_{T(x)} + \underbrace{r \ln p}_{d(\phi)} \Big)$$

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⊛ 3.24

② (a) 期望值，變異數

$$E[Y]=E[X^{\frac{1}{r}}]=\int_0^\infty \frac{1}{\beta}\exp\left(-\frac{x}{\beta}\right)\cdot x^{\frac{1}{r}}\,dx \qquad \frac{x}{\beta}=y \quad \frac{dy}{dx}=\frac{1}{\beta}$$

$$=\int_0^\infty \frac{1}{\beta}\exp(-y)\,(\beta y)^{\frac{1}{r}}\cdot \beta\,dy$$

$$=\int_0^\infty \beta^{\frac{1}{r}}\cdot y^{(1+\frac{1}{r})-1}\exp(-y)\,dy=\beta^{\frac{1}{r}}\cdot\Gamma\left(1+\frac{1}{r}\right)$$

$$E[Y^2]=\int_0^\infty \frac{1}{\beta}\exp\left(-\frac{x}{\beta}\right)\cdot x^{\frac{2}{r}}\,dx$$

$$=\int_0^\infty \frac{1}{\beta}\exp(-y)\cdot(\beta y)^{\frac{2}{r}}\cdot\beta\,dy$$

$$=\int_0^\infty \beta^{\frac{2}{r}}y^{(1+\frac{2}{r})-1}\exp(-y)\,dy=\beta^{\frac{2}{r}}\cdot\Gamma\left(1+\frac{2}{r}\right)$$

$$\therefore V[Y]=\beta^{\frac{2}{r}}\Gamma\left(1+\frac{2}{r}\right)-\beta^{\frac{2}{r}}\left(\Gamma\left(1+\frac{1}{r}\right)\right)^2$$

(c) 期望值，變異數 (假設 $\alpha>2$)

$$E[Y]=E\left[\frac{1}{X}\right]=\int_0^\infty \frac{x^{\alpha-2}}{\Gamma(\alpha)\beta^\alpha}\exp\left(-\frac{x}{\beta}\right)dx \qquad y=\frac{x}{\beta} \quad \frac{dy}{dx}=\frac{1}{\beta}$$

$$=\int_0^\infty \frac{(\beta y)^{\alpha-2}}{\Gamma(\alpha)\beta^\alpha}\exp(-y)\,\beta\,dy$$

$$=\int_0^\infty \frac{y^{\alpha-2}}{\beta\cdot\Gamma(\alpha)}\exp(-y)\,dy=\frac{\Gamma(\alpha-1)}{\beta\cdot\Gamma(\alpha)}=\frac{1}{\beta\cdot(\alpha-1)} \qquad (\alpha>1)$$

$$E[Y^2] = \int_0^\infty \frac{x^{\alpha-3}}{\Gamma(\alpha)\beta^\alpha} \exp\left(\frac{-x}{\beta}\right) dx \qquad \frac{x}{\beta} = y \quad \frac{dx}{dy} = \beta$$

$$= \int_0^\infty \frac{(\beta y)^{\alpha-3}}{\Gamma(\alpha)\beta^\alpha} \exp(-y)\, \beta dy = \int_0^\infty \frac{y^{\alpha-3}}{\Gamma(\alpha)\beta^2} \exp(-y) dy$$

$$= \frac{\Gamma(\alpha-2)}{\Gamma(\alpha)\beta^2} = \frac{1}{(\alpha-1)(\alpha-2)\beta^2}$$

$$\therefore V[Y] = \frac{1}{(\alpha-1)(\alpha-2)\beta^2} - \frac{1}{\beta^2(\alpha-1)^2}$$

$$= \frac{1}{(\alpha-1)\beta^2}\left(\frac{1}{\alpha-2} - \frac{1}{\alpha-1}\right) = \frac{1}{(\alpha-1)^2(\alpha-2)} \cdot \frac{1}{\beta^2}$$

(e) 期望值、變異數

$$E[Y] = E[\alpha - r\ln X] = \alpha - r\int_0^\infty \ln x \cdot e^{-x} dx$$

$$E[Y^2] = \int (\alpha^2 - 2r\ln x + r^2(\ln x)^2) e^{-x} dx = \alpha^2 - 2r\underbrace{\int \ln x\, e^{-x} dx} + r^2 \underbrace{\int (\ln x)^2 e^{-x} dx}$$

無法計↑ →

$$V[Y] = E[Y^2] - E[Y]^2$$

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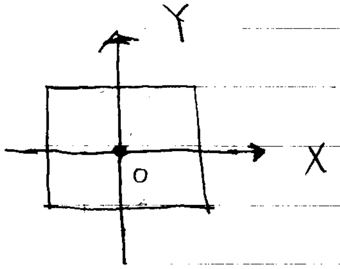
R05246013 森元俊成

作業4. 資料科學之統計基礎 (I)

4.1.



(a) $\iint_{x^2+y^2<1} \frac{1}{4}\,dx\,dy = \iint_{\substack{0\le r\le 1\\ \theta=0\to 2\pi}} \frac{1}{4} r\,dr\,d\theta$

$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

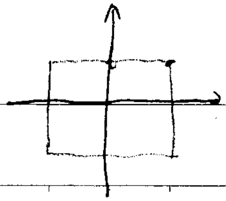
$= \left[\frac{r^2}{8}\right]_{r=0}^{r=1} \left[\theta\right]_0^{2\pi} = \frac{\pi}{4}$

(b) $\iint_{\substack{-1\le x\le 1\\ -1\le y\le 1\\ 2x-y>0}} \frac{1}{4}\,dx\,dy = \int_{\substack{-\frac{1}{2}\le x\le \frac{1}{2}\\ -1\le y\le 2x}} \frac{1}{4}\,dx\,dy$

$+ \int_{\substack{\frac{1}{2}\le x\le 1\\ -1\le y\le 1}} \frac{1}{4}\,dx\,dy$

$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{4}(2x+1)\,dx + \frac{1}{4}\cdot\frac{1}{2}\cdot 2$

$= \left[\frac{1}{4}(x^2+x)\right]_{-\frac{1}{2}}^{\frac{1}{2}} + \frac{1}{4} = \frac{1}{4}+\frac{1}{4} = \frac{1}{2}$



(c) $\displaystyle\int_{\substack{-1<x<1 \\ -1<y<1 \\ |x+y|<2}} \frac{1}{4}\,dx\,dy = \int_{\substack{-1<x<1 \\ -1<y<1}} \frac{1}{4}\,dx\,dy$

$= \frac{1}{4}[x]_{-1}^{1}[y]_{-1}^{1} = 1 \qquad \therefore 1$

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4.6.

考慮等待時間的 T.

令 $X, Y \overset{iid}{\sim} \text{Uniform}(0,1)$ 表示 A 和 B 的抵達時間.

$T = I_{\{X<Y\}} \cdot (Y-X)$

T 為離散與連續混合分佈.

① $\Pr(T=0) = \frac{1}{2}$ ($\because$ X, Y 為連續分佈. $\therefore$ $\Pr(X=Y)=0$
$\therefore \Pr(X>Y) = \Pr(Y<X) = \frac{1}{2}$)

② 接著考慮 $\Pr(T \le t) = \Pr(T \le t, T>0) + \underbrace{\Pr(T \le t, T=0)}_{(=\Pr(T=0))}$

$= \Pr(T \le t \mid T>0) \cdot \underbrace{\Pr(T>0)}_{\frac{1}{2}} + \underbrace{\Pr(T=0)}_{\frac{1}{2}}$

關於 $\Pr(T \le t \mid T>0)$, 這等於 $X_1, X_2 \overset{iid}{\sim} \text{Uniform}(0,1)$

的「樣本範圍」(Sample Range) 的 cdf.
$(X_{(2)} - X_{(1)})$

我們將情況一般化. 考慮 $X_1 \dots X_n \overset{iid}{\sim} \text{Uniform}(0,1)$
的 $R = X_{(n)} - X_{(1)}$ 與 $S = \frac{X_{(n)} + X_{(1)}}{2}$ 的機率分佈. $(n=2)$

$$\Pr(X_{(1)} \le x,\ X_{(n)} \le y) = y^n - (y-x)^n \quad (y \ge x)$$

$$\therefore \frac{\partial^2}{\partial x \partial y} \Pr(X_{(1)} \le x,\ X_{(n)} \le y) = n(n-1)(y-x)^{n-2}.$$

$$\therefore f_{X_{(1)}X_{(n)}}(x,y) = n(n-1)(y-x)^{n-2} \cdot I_{(0 \le x \le y \le 1)}$$

求R,S的联合分布 $dxdy = dRdS$; $0 \le \underset{X_{(1)}}{\underset{\shortparallel}{S-\frac{R}{2}}} \le \underset{X_{(n)}}{\underset{\shortparallel}{S+\frac{R}{2}}} \le 1$

全概率 $1 = \iint_{0 \le x \le y \le 1} n(n-1)(y-x)^{n-2}\,dxdy$

$$= \iint_{0 \le S-\frac{R}{2} \le S+\frac{R}{2} \le 1} n(n-1)R^{n-2}\,dRdS.$$

求R的边际分布. $\int_{S=\frac{R}{2}}^{S=1-\frac{R}{2}} n(n-1)R^{n-2}\,dS = n(n-1)R^{n-2}(1-R)$

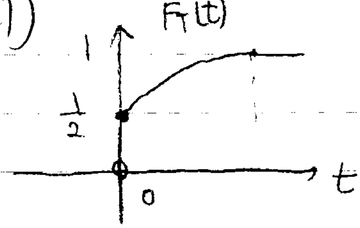
$$\therefore f_R(R) = n(n-1)R^{n-2}(1-R),\quad (0 \le R \le 1)$$

代入 $n=2$ 可知: $X_{(2)} - X_{(1)} = R$ 的pdf: $2(1-R)$ $(0 \le R \le 1)$

$$\therefore \Pr(T \le t \mid T > 0) = \int_0^t 2(1-R)\,dR = \left[2R - R^2\right]_0^t = \underline{2t - t^2}.$$

故 $\Pr(T \le t) = \frac{1}{2} + \frac{1}{2}(2t - t^2)$ $(0 \le t \le 1)$

$$\therefore F_T(t) = \begin{cases} 0 & (t < 0) \\ \frac{1}{2}(1 + 2t - t^2) & (0 \le t \le 1) \\ 1 & (t > 1) \end{cases}$$



No.

Date

4.17

(a) $\Pr(Y=k)=\Pr(k-1\leq X<k)=\int_{k-1}^{k} e^{-x}dx$

$=\left[-e^{-x}\right]_{k-1}^{k} = e^{-k+1}-e^{-k}=e^{-k}(e-1) \quad (k=1,2,3\cdots)$

這是幾何分佈 $Geo(p=e^{-1})$ (Y為嘗試次數)

(b) $X-4 \mid Y\geq 5 = X-4 \mid X\geq 4$

$\therefore \Pr(X-4\leq z \mid X\geq 4)=\dfrac{\Pr(4\leq X\leq z+4)}{\Pr(X\geq 4)}$

$=\dfrac{\int_4^{z+4}\exp(-x)dx}{\int_4^{\infty}\exp(-x)dx}=\dfrac{\exp(-4)-\exp(-(z+4))}{\exp(-4)}$

$=1-\exp(-z) \quad (z\geq 0)$

由此可知, $X-4 \mid Y\geq 5 \sim \exp(1)$

(這件事與指數分佈的「無記憶性」有關)

No.

Date

4.20.

(a) $X_1, X_2 \overset{iid}{\sim} N(0, \sigma^2)$

$$全機率 = 1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi\sigma^2} \exp\left(-\frac{1}{2\sigma^2}(x_1^2+x_2^2)\right) dx_1 dx_2$$

$$\begin{cases} X_1 = \sqrt{Y_1} \cdot Y_2 \\ X_2 = \pm\sqrt{Y_1}\sqrt{1-Y_2^2} \end{cases}$$

$$J = \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{pmatrix} = \begin{pmatrix} \frac{y_2}{2\sqrt{y_1}} & \sqrt{y_1} \\ \frac{\pm\sqrt{1-y_2^2}}{2\sqrt{y_1}} & \frac{\mp\sqrt{y_1}\, y_2}{\sqrt{1-y_2^2}} \end{pmatrix}$$

$$|J| = \frac{\mp 1}{2\sqrt{1-y_2^2}} \qquad \therefore \text{abs}(|J|) = \frac{1}{2\sqrt{1-y_2^2}}$$

$$\therefore 全機率 = \overset{(I)}{\int_{x_1\in\mathbb{R}}\int_{x_2\geq 0} f_{X_1X_2}\, dx_1 dx_2} \quad \text{(根據}x_2\text{的正負分成兩段)}$$

$$+ \overset{(II)}{\int_{x_1\in\mathbb{R}}\int_{x_2<0} f_{X_1X_2}\, dx_1 dx_2} = 1$$

(I) $x_1 \in \mathbb{R} \quad x_2 \geq 0 \quad \Rightarrow \quad y_1 \in [0, \infty) \quad y_2 \in (-1, 1)$

(II) $X_1 \in \mathbb{R} \quad x_2 < 0 \quad \Rightarrow \quad y_1 \in [0, \infty), \quad y_2 \in (-1, 1)$

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全機率 $= 1 = (\mathrm{I}) + (\mathrm{II}) = \overset{(\mathrm{I})}{\int_{y_1 \in [0,\infty),\, y_2 \in (-1,1)}} \frac{1}{2\pi\sigma^2} \exp\left(\frac{-y_1}{2\sigma^2}\right) \cdot \frac{1}{2\sqrt{1-y_2^2}}\, dy_1 dy_2$

$+ \overset{(\mathrm{II})}{\int_{y_1 \in [0,\infty)\, y_2 \in (-1,1)}} \frac{1}{2\pi\sigma^2} \exp\left(\frac{-y_1}{2\sigma^2}\right) \cdot \frac{1}{2\sqrt{1-y_2^2}}\, dy_1 dy_2$

$= \int_{\substack{y_1 \in [0,\infty) \\ y_2 \in [0,1)}} \frac{1}{2\sigma^2} \exp\left(\frac{-1}{2\sigma^2} y_1\right) \cdot \frac{1}{\pi\sqrt{1-y_2^2}}\, dy_1 dy_2$

$\therefore f_{Y_1 Y_2}(y_1, y_2) = \frac{1}{2\sigma^2} \exp\left(\frac{-1}{2\sigma^2} y_1\right) \cdot \frac{1}{\pi\sqrt{1-y_2^2}} \cdot I_{[0,\infty)}(y_1)\, I_{[0,1)}(y_2)$

(b) $f_{Y_1}(y_1) = \int_{y_2=-1}^{y_2=1} \frac{1}{2\sigma^2} \exp\left(\frac{-1}{2\sigma^2} y_1\right) \cdot I_{[0,\infty)}(y_1) \cdot \frac{1}{\pi\sqrt{1-y_2^2}}\, dy_2$

$\left(\frac{1}{2\sigma^2} \exp\left(\frac{-y_1}{2\sigma^2}\right)\right) \cdot I_{[0,\infty)}(y_1) \cdot \left[\frac{1}{\pi} \arcsin y_2\right]_{-1}^{1} = \frac{1}{2\sigma^2} \exp\left(\frac{-y_1}{2\sigma^2}\right) I_{[0,\infty)}(y_1).$

$f_{Y_2}(y_2) = \int_{y_1=0}^{y_1=\infty} \frac{1}{2\sigma^2} \exp\left(\frac{-1}{2\sigma^2} y_1\right) \cdot \frac{1}{\pi\sqrt{1-y_2^2}} \cdot I_{(-1,1)}(y_2)\, dy_1$

$= \frac{1}{\pi\sqrt{1-y_2^2}} I_{(-1,1)}(y_2).$

$\therefore f_{Y_1 Y_2}(y_1, y_2) = f_{Y_1}(y_1) \cdot f_{Y_2}(y_2) \Rightarrow Y_1, Y_2$ 獨立

($\Rightarrow$ 將 $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ 視為 R^2 上的點，距離與角度的分佈是獨立的。)

法二 Y_1 為 σ^2 的完備充份統計量，Y_2 為 σ^2 的輔助統計量，根據 BASU 定理，Y_1, Y_2 獨立。

No.

Date

4.26.

(a) 求 $\Pr(Z\leqq z,\ W=\omega)$

- $\Pr(Z\leqq z,\ W=0)=\Pr(\min\{X,Y\}\leqq z,\ Y\leqq X)$

$=\Pr(Y\leqq X)-\Pr(X\geqq Y>z)$

$$=\int_{y=0}^{y=\infty}\int_{x=y}^{x=\infty}\lambda\exp(-\lambda x)\ \mu\exp(-\mu y)\,dx\,dy$$

$$-\int_{y=z}^{y=\infty}\int_{x=y}^{x=\infty}\lambda\exp(-\lambda x)\ \mu\exp(-\mu y)\,dx\,dy$$

$$=\int_{y=0}^{y=\infty}\mu\exp(-(\lambda+\mu)y)\,dy$$

$$-\int_{y=z}^{y=\infty}\mu\exp(-(\lambda+\mu)y)\,dy$$

$$=\left[\frac{-\mu}{\lambda+\mu}\exp(-(\lambda+\mu)y)\right]_{y=0}^{y=\infty}-\left[\frac{-\mu}{\lambda+\mu}\exp(-(\lambda+\mu)y)\right]_{y=z}^{y=\infty}$$

$$=\frac{\mu}{\lambda+\mu}-\frac{\mu}{\lambda+\mu}\exp(-(\lambda+\mu)z)=\frac{\mu}{\lambda+\mu}\left(1-\exp(-(\lambda+\mu)z)\right)$$

$$\therefore\ \Pr(Z\leqq z,\ W=0)=\frac{\mu}{\lambda+\mu}\left(1-\exp(-(\lambda+\mu)z)\right)$$

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接著求 $Pr(Z\leq z, W=1) = \underset{\sim\sim\sim}{Pr(Z\leq z)} - Pr(Z\leq z, W=0)$

、$Pr(Z\leq z) = 1 - Pr(Z>z) = 1 - \int_z^\infty \int_z^\infty \lambda \exp(-\lambda x)\ \mu \exp(-\mu y)\, dx\, dy$

$= 1 - \exp(-(\lambda+\mu)z)$

$\therefore Pr(Z\leq z, W=1) = 1 - \exp(-(\lambda+\mu)z) - \frac{\mu}{\lambda+\mu}(1-\exp(-(\lambda+\mu)z))$

$= \frac{\lambda}{\lambda+\mu}(1-\exp(-(\lambda+\mu)z))$

綜合 $Pr(Z\leq z, W=0)$ & $Pr(Z\leq z, W=1)$ 得：

$\therefore Pr(Z\leq z, W=w) = \frac{\lambda^w \mu^{1-w}}{\lambda+\mu}(1-\exp(-(\lambda+\mu)z))$ 」

$(w=0 \text{ or } 1)$

(b) $Pr(Z\leq z) = \sum_{w=0,1} \frac{\lambda^w \mu^{1-w}}{\lambda+\mu}(1-\exp(-(\lambda+\mu)z))$

$= 1-\exp(-(\lambda+\mu)z)$

$Pr(W=w) = \lim_{z\to\infty} Pr(Z\leq z, W=w) = \frac{\lambda^w \mu^{1-w}}{\lambda+\mu}$

$\therefore Pr(Z\leq z, W=w) = Pr(Z\leq z)\cdot Pr(W=w)$ 成立，

$\therefore$ 證明完成

No.

Date

4.32

(a) $\Pr(Y=y \mid \Lambda=\lambda) = e^{-\lambda} \cdot \frac{\lambda^y}{y!}$

$\therefore f_{Y,\Lambda}(Y=y, \Lambda=\lambda) = e^{-\lambda} \cdot \frac{\lambda^y}{y!} \cdot \frac{\beta^\alpha \lambda^{\alpha-1}}{\Gamma(\alpha)} \exp(-\beta\lambda)$

求 $\int_{\lambda=0\to\infty} f_{Y\Lambda} \, d\lambda =$ $\quad (Z \overset{\text{def}}{=} (1+\beta)\lambda)$

$\int_{\lambda=0}^{\lambda=\infty} \frac{\beta^\alpha}{y!\,\Gamma(\alpha)} \lambda^{\alpha+y-1} \exp(-(1+\beta)\lambda) \, d\lambda$

$= \int_{Z=0}^{Z=\infty} \frac{\beta^\alpha}{y!\,\Gamma(\alpha)} \cdot \left(\frac{Z}{1+\beta}\right)^{\alpha+y-1} \exp(-Z) \cdot \frac{dZ}{1+\beta}$

$= \frac{\beta^\alpha}{y!\,\Gamma(\alpha)} \cdot \frac{1}{(1+\beta)^{\alpha+y}} \int_{Z=0}^{Z=\infty} Z^{\alpha+y-1} \exp(-Z) \, dZ$

$= \frac{1}{y!\,\Gamma(\alpha)} \cdot \frac{\beta^\alpha}{(1+\beta)^{\alpha+y}} \cdot \Gamma(\alpha+y)$

$= \frac{(\alpha+y-1)!}{y!\,(\alpha-1)!} \left\{\frac{\beta}{(1+\beta)}\right\}^\alpha \cdot \frac{1}{(1+\beta)^y} = \Pr(Y=y)$

(若α為整數)

若α為整數, 則 $Y \sim NB\left(\underset{\text{成功次數}}{\alpha}, \underset{\text{成功機率}}{\frac{\beta}{1+\beta}}\right)$ (Y: 失敗次數)

失 成 { 成

y $\alpha-1$ 1

$E[\Lambda]=\frac{\alpha}{\beta}$ $V[\Lambda]=\frac{\alpha}{\beta^2}$

$E[\Lambda^2]=\frac{\alpha}{\beta^2}+\frac{\alpha^2}{\beta^2}$

$\therefore Pr(Y=y)=\frac{\Gamma(\alpha+y)}{y!\,\Gamma(\alpha)}\cdot\left(\frac{\beta}{1+\beta}\right)^{\alpha}\cdot\left(\frac{1}{1+\beta}\right)^{y}$

$Y\sim NB(\alpha,\frac{\beta}{1+\beta})$ (if α : 整數)

接著求 Y的期望值與變異數

$E[Y|\Lambda]=\Lambda$, $E[Y^2|\Lambda]=\Lambda^2+\Lambda$

$\therefore E[Y]=E[\Lambda]=\frac{\alpha}{\beta}$

$E[Y^2]=E[\Lambda^2+\Lambda]=\frac{\alpha}{\beta^2}+\frac{\alpha^2}{\beta^2}+\frac{\alpha}{\beta}$

$\therefore V[Y]=\frac{\alpha}{\beta^2}+\frac{\alpha}{\beta}$

$\therefore E[Y]=\frac{\alpha}{\beta}$ $V[Y]=\frac{\alpha}{\beta^2}+\frac{\alpha}{\beta}$

(b) ↓下一頁

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Date

4.32

(b) 求 $Y|\Lambda$ 的條件分布：

$$\left\{\begin{array}{l} \Pr(Y=y|N) = {}_nC_y \cdot p^y(1-p)^{n-y} \quad (y \le n) \\ \Pr(N=n|\Lambda) = e^{-\lambda} \cdot \frac{\lambda^n}{n!} \end{array}\right.$$

$$\therefore \Pr(Y=y, N=n|\Lambda) = {}_nC_y \cdot p^y(1-p)^{n-y} \cdot e^{-\lambda} \cdot \frac{\lambda^n}{n!}$$

$$= \frac{\cancel{n!}}{y!(n-y)!} \cdot p^y(1-p)^{n-y} \cdot e^{-\lambda} \cdot \frac{\lambda^n}{\cancel{n!}}$$

$$= \left(\frac{p}{1-p}\right)^y \cdot \frac{e^{-\lambda}}{y!} \frac{(\lambda(1-p))^n}{(n-y)!} \quad (y \le n)$$

$$\therefore \sum_{n \ge y}^{\infty} \Pr(Y=y, N=n|\Lambda) = \left(\frac{p}{1-p}\right)^y \cdot \frac{e^{-\lambda}}{y!} \sum_{n \ge y}^{\infty} \frac{(\lambda(1-p))^n}{(n-y)!}$$

$$\left(\overset{\text{put}}{z=n-y}\right) = \left(\frac{p}{1-p}\right)^y \frac{e^{-\lambda}}{y!} \cdot \sum_{z \ge 0} \frac{(\lambda(1-p))^{z+y}}{z!}$$

$$= \left(\frac{p}{\cancel{1-p}}\right)^y \cdot \frac{e^{-\lambda}}{y!} \cdot \lambda^y \cancel{(1-p)^y} \underbrace{\sum_{z \ge 0} \frac{(\lambda(1-p))^z}{z!}}_{\exp(\lambda-\lambda p)}$$

$$= \exp(-\lambda p) \cdot \frac{(\lambda p)^y}{y!}$$

$$\therefore \Pr(Y=y \mid \Lambda) = \exp(-\lambda p) \cdot \frac{(\lambda p)^y}{y!}$$

$$\therefore Y \mid \Lambda \sim Po(p\Lambda) \qquad \left((a): \ Y \mid \Lambda \sim Po(\Lambda) \right)$$

跟 (a) 同樣：

$$\int_{\lambda=0}^{\lambda=\infty} f_{Y,\Lambda} \, d\lambda = \int_0^\infty \exp(-\lambda p) \cdot \frac{(\lambda p)^y}{y!} \cdot \frac{\beta^\alpha \lambda^{\alpha-1}}{\Gamma(\alpha)} \exp(-\beta\lambda) \, d\lambda$$

$$\int_{\lambda=0}^{\lambda=\infty} \frac{p^y}{y!} \cdot \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha+y-1} \exp(-(\beta+p)\lambda) \, d\lambda \qquad z \stackrel{\text{def}}{=} (\beta+p)\lambda$$

$$= \int_{z=0}^{z=\infty} \frac{p^y}{y!} \frac{\beta^\alpha}{\Gamma(\alpha)} \left(\frac{z}{\beta+p} \right)^{\alpha+y-1} \exp(-z) \cdot \frac{1}{\beta+p} \, dz$$

$$= \int_{z=0}^{z=\infty} \frac{p^y \beta^\alpha}{y! \, \Gamma(\alpha)} \frac{z^{\alpha+y-1}}{(\beta+p)^{\alpha+y}} \exp(-z) \, dz$$

$$= \frac{p^y \beta^\alpha}{y! \, \Gamma(\alpha)} \cdot \frac{\Gamma(\alpha+y)}{(\beta+p)^{\alpha+y}}$$

$$= \frac{\Gamma(\alpha+y)}{y! \, \Gamma(\alpha)} \cdot \left(\frac{\beta}{\beta+p} \right)^\alpha \cdot \left(\frac{p}{\beta+p} \right)^y$$

由此可知，題目是騙人的.

($p=1$ 時才會一致)

No.

Date

4.39

- $X_i \mid X_j = x_j \sim Bin\left(m-x_j, \frac{p_j}{1-p_j}\right)$ 的證明

$$Pr(X_i = x_i \mid X_j = x_j) = \frac{Pr(X_i = x_i, X_j = x_j)}{Pr(X_j = x_j)}$$

$$= \frac{\left\{ p_i^{x_i} \cancel{p_j^{x_j}} (1-p_i-p_j)^{m-(x_i+x_j)} \cdot \frac{\cancel{m!}}{x_i! \cancel{x_j!} (m-x_i-x_j)!} \right\}}{\left\{ \cancel{p_j^{x_j}} (1-p_j)^{m-x_j} \frac{\cancel{m!}}{\cancel{x_j!} (m-x_j)!} \right\}}$$

$$= \frac{(m-x_j)!}{x_i! (m-x_i-x_j)!} \left(\frac{1-p_i-p_j}{1-p_j}\right)^{m-x_j-x_i} \left(\frac{p_i}{1-p_j}\right)^{x_i}$$

$$= {}_{m-x_j}C_{x_i} \cdot \left(\frac{p_j}{1-p_j}\right)^{x_i} \cdot \left(1 - \frac{p_i}{1-p_j}\right)^{(m-x_j)-x_i}$$

由此可知, $X_i \mid X_j = x_j \sim Bin\left(m-x_j, \frac{p_j}{1-p_j}\right)$

- $$Pr(X_j = x_j) = \sum_{\substack{0 \le x_1 \\ \vdots \\ 0 \le x_n \\ x_1+x_2+\cdots+x_j+x_{j+1}+\cdots+x_n = m-x_j}} \frac{m!}{x_1! \cdots x_n!} p_1^{x_1} \cdots p_n^{x_n}$$

$$= \frac{m!}{x_j! (m-x_j)!} \cdot p_j^{x_j} \underbrace{\sum_{\substack{0 \le x_1, x_2 \cdots x_n \\ x_1+\cdots x_{j-1}+x_{j+1}+\cdots+x_n = m-x_j}} \frac{(m-x_j)!}{x_1! x_2! \cdots x_{j-1}! x_{j+1}! \cdots x_n!} p_1^{x_1} p_2^{x_2} \cdots p_{j-1}^{x_{j-1}} p_{j+1}^{x_{j+1}} \cdots p_n^{x_n}}_{\parallel \\ (p_1+\cdots p_{j-1}+p_{j+1}+\cdots p_n)^{m-x_j}}$$

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$$= \frac{m!}{x_j!(m-x_j)!} p_j^{x_j} (p_1+p_2+\cdots p_{j-1}+p_{j+1}+\cdots p_n)^{m-x_j}$$

$$= \frac{m!}{x_j!(m-x_j)!} p_j^{x_j} (1-p_j)^{m-x_j}$$ 由此可知 $X_j \sim Bn(m, p_j)$

$\left(P(X_i=x_i, X_j=x_j) \right.$ 也可以用同樣方法求$\left. \right)$

- $cov[X_i, X_j]$ — 利用 MGF (cumulant)

$$E[e^{t_iX_i+t_jX_j}] = \sum_{\substack{x_1+\cdots x_n=m \\ x_1\geq 0 \\ \vdots \\ x_n \geq 0}} \frac{m!}{x_1!\cdots x_n!} p_1^{x_1} \cdots (p_ie^{t_i})^{x_i} \cdots (p_je^{t_j})^{x_j} \cdots p_n^{x_n}$$

$$= (p_1+p_2+\cdots p_ie^{t_i}+\cdots p_je^{t_j}+\cdots p_n)^m = M(t_i,t_j)$$

$$K(t_i,t_j) = \ln M(t_i,t_j) = m\cdot \ln(p_ie^{t_i}+p_je^{t_j}+(1-p_i-p_j))$$

$$\frac{\partial K}{\partial t_i} = m\cdot \frac{p_ie^{t_i}}{p_ie^{t_i}+p_je^{t_j}+1-p_i-p_j}$$

$$\frac{\partial^2 K}{\partial t_i \partial t_j} = m\cdot \frac{-p_ie^{t_i}p_je^{t_j}}{(p_ie^{t_i}+p_je^{t_j}+1-p_i-p_j)^2}$$

$$\therefore \left.\frac{\partial^2 K}{\partial t_i \partial t_j}\right|_{\substack{t_i=0 \\ t_j=0}} = m\cdot \frac{-p_ip_j}{(p_i+p_j+1-p_i-p_j)^2} = -mp_ip_j$$

$$\therefore cov[X_i, X_j] = -mp_ip_j$$

No.

Date

4.46.

(a) $E[X] = E[a_X Z_1 + b_X Z_2 + c_X] = a_X E[Z_1] + b_X E[Z_2] + c_X$
$= c_X$

$V[X] = V[a_X Z_1 + b_X Z_2 + c_X] = a_X^2 V[Z_1] + b_X^2 V[Z_2]$
$= a_X^2 + b_X^2$

($E[Y]$ $V[Y]$ 將 a_X, b_X, c_X 改為 a_Y, b_Y, c_Y.)

$\mathrm{cov}[a_X Z_1 + b_X Z_2 + c_X,\ a_Y Z_1 + b_Y Z_2 + c_Y]$

$= a_X a_Y \underbrace{\mathrm{cov}[Z_1, Z_1]}_{(= V[Z_1] = 1)} + b_X a_Y \underbrace{\mathrm{cov}[Z_2, Z_1]}_{=0} + \underbrace{\mathrm{cov}[c_X, a_Y Z_1]}_{=0}$

$+ a_X b_Y \underbrace{\mathrm{cov}[Z_1, Z_2]}_{=0} + b_X b_Y \underbrace{\mathrm{cov}[Z_2, Z_2]}_{=1} + \underbrace{c_X b_Y \mathrm{cov}[1, Z_2]}_{=0}$

$+ a_X c_Y \underbrace{\mathrm{cov}[Z_1, 1]}_{=0} + b_X c_Y \underbrace{\mathrm{cov}[Z_2, 1]}_{=0} + \underbrace{c_X c_Y \mathrm{cov}[1, 1]}_{=0}$

$= a_X a_Y + b_X b_Y$

(b) $E[X] = c_X = \mu_X$
$E[Y] = c_Y = \mu_Y$
$V[X] = a_X^2 + b_X^2 = \left(\frac{1+\rho}{2} + \frac{1-\rho}{2}\right)\sigma_X^2 = \sigma_X^2$
$V[Y] = a_Y^2 + b_Y^2 = \left(\frac{1+\rho}{2}\ \frac{1-\rho}{2}\right)\sigma_Y^2 = \sigma_Y^2$
$\mathrm{cov}[X, Y] = \frac{1+\rho}{2}\sigma_X\sigma_Y - \frac{1-\rho}{2}\sigma_X\sigma_Y - \rho\sigma_X\sigma_Y$

(c) 利用積分的變數轉換...

$$\underset{\text{全機率}}{1} = \int_{z_1\in\mathbb{R}}\int_{z_2\in\mathbb{R}} \frac{1}{2\pi} \exp\left(-\frac{1}{2}(z_1^2+z_2^2)\right) dz_1 dz_2$$

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix} + \underbrace{\begin{pmatrix} a_X & b_X \\ a_Y & b_Y \end{pmatrix}}_{A} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

$$J = \begin{pmatrix} \frac{\partial X}{\partial z_1} & \frac{\partial X}{\partial z_2} \\ \frac{\partial Y}{\partial z_1} & \frac{\partial Y}{\partial z_2} \end{pmatrix} \to A \qquad |\det A|\, dz_1 dz_2 = dx\,dy$$

$\because \det A \neq 0 \quad \therefore A: \mathbb{R}^2 \to \mathbb{R}^2$ (onto, one-to-one)

$$\therefore \int_{(x,y)\in\mathbb{R}^2} \frac{1}{2\pi} \exp\left(-\frac{1}{2} (z_1\ z_2) \underbrace{\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}}_{= A^{-1}\begin{pmatrix} X-\mu_X \\ Y-\mu_Y \end{pmatrix}}\right) \cdot |\det A|^{-1}\, dx\,dy$$

$\Sigma \overset{\text{def}}{=} AA^t$ (由此可知 Σ 為正定矩陣. $a^t\Sigma a = (A^t a)^t (A^t a) \geq 0$, $\det A \neq 0$)

$\therefore$

$$f_{XY} = \frac{1}{2\pi} \frac{1}{|\Sigma|^{\frac{1}{2}}} \exp\left(-\frac{1}{2} (X-\mu_X, Y-\mu_Y)\, \Sigma^{-1} \begin{pmatrix} X-\mu_X \\ Y-\mu_Y \end{pmatrix}\right)$$

$\Rightarrow$ 二維常態分佈

$$\Sigma = AA^t = \begin{pmatrix} a_X & b_X \\ a_Y & b_Y \end{pmatrix} \begin{pmatrix} a_X & a_Y \\ b_X & b_Y \end{pmatrix} = \begin{pmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{pmatrix}$$

$$\therefore \begin{pmatrix} X \\ Y \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix}, \begin{pmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{pmatrix}\right)$$

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(d) 利用特徵值分解, 求A使得

$$AA^t = \begin{pmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2 \end{pmatrix} \qquad A = \begin{pmatrix} a_x & b_x \\ a_y & b_y \end{pmatrix}$$

求特徵值 $\begin{pmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2 \end{pmatrix} \vec{v} = \lambda \vec{v}$

$$(\lambda - \sigma_x^2)(\lambda - \sigma_y^2) - \rho^2 \sigma_x^2 \sigma_y^2 = 0$$

$$\therefore \lambda^2 - \lambda(\sigma_x^2 + \sigma_y^2) - \rho^2 \sigma_x^2 \sigma_y^2 = 0$$

設 $\{\lambda_1, \lambda_2\}$ 為其根

$$\begin{cases} U_1 \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} U_1^t = \begin{pmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2 \end{pmatrix} = AA^t \\ U_2 \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_1 \end{pmatrix} U_2^t = \begin{pmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2 \end{pmatrix} = AA^t \end{cases}$$

$$\therefore A = U_1 \begin{pmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{pmatrix} \text{ or } U_2 \begin{pmatrix} \sqrt{\lambda_2} & 0 \\ 0 & \sqrt{\lambda_1} \end{pmatrix}$$

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$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \underbrace{\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}}_{\Sigma}\right)$$

- $\mathrm{Corr}[X, Y] = \dfrac{\mathrm{Cov}[X, Y]}{\sqrt{V[X]V[Y]}} = \dfrac{\rho}{\sqrt{1}\sqrt{1}} = \rho$

- $\mathrm{Corr}[X^2, Y^2] = \dfrac{\mathrm{Cov}[X^2, Y^2]}{\sqrt{V[X^2] \cdot V[Y^2]}}$

① $V[X^2] = E[X^4] - E[X^2]^2 = 2.$

$X \sim N(0,1)$ (周辺分布) $\quad E[X^4] = 3. \quad E[X^2] = 1$

$$\left(\because E[e^{tX}] = \exp\left(\frac{t^2}{2}\right) = 1 + \frac{t^2}{2} + \underbrace{\frac{t^4}{8}}_{= \frac{M^{(4)}(0)}{4!}} + \cdots \quad \therefore M^{(4)}(0) = 3 = E[X^4] \right)$$

同様 $V[Y^2] = 2$

②
$$\begin{aligned} \mathrm{Cov}[X^2, Y^2] &= E[X^2Y^2] - E[X^2]E[Y^2] \\ &= E[X^2Y^2] - 1 \\ &= E[X^2 \cdot E[Y^2 | X]] - 1 \end{aligned}$$

· 接著求 Y|X 的條件分佈.（首先考慮一般分組的情況）

假設 $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}\right)$ $\quad X_1 \in R^{p_1}$, $X_2 \in R^{p_2}$

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \overset{def}{=} \underbrace{\begin{pmatrix} I_{p_1} & 0 \\ -\Sigma_{21}\Sigma_{11}^{-1} & I_{p_2} \end{pmatrix}}_{A} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

$$A\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = \begin{pmatrix} I_{p_1} & 0 \\ -\Sigma_{21}\Sigma_{11}^{-1} & I_{p_2} \end{pmatrix}\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \mu_2 - \Sigma_{21}\Sigma_{11}^{-1}\mu_1 \end{pmatrix}$$

$$A\begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}A^t = \begin{pmatrix} I_{p_1} & 0 \\ -\Sigma_{21}\Sigma_{11}^{-1} & I_{p_2} \end{pmatrix}\begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}A^t$$

$$= \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ 0 & \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12} \end{pmatrix}\begin{pmatrix} I_{p_1} & -\Sigma_{11}^{-1}\Sigma_{12} \\ 0 & I_{p_2} \end{pmatrix}$$

$$= \begin{pmatrix} \Sigma_{11} & 0 \\ 0 & \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12} \end{pmatrix}$$

∴ 由此可知，Y_1, Y_2 獨立（p-維常態分佈：無相關⇔獨立）

∴ X_1 與 $X_2 - \Sigma_{21}\Sigma_{11}^{-1}X_1$ 獨立.

∴ $X_2 - \Sigma_{21}\Sigma_{11}^{-1}X_1 \mid X_1 \sim N(\mu_2 - \Sigma_{21}\Sigma_{11}^{-1}\mu_1,\ \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12})$

∴ $X_2 \mid X_1 \sim N(\mu_2 + \Sigma_{21}\Sigma_{11}^{-1}(x_1 - \mu_1),\ \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12})$

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由此可知, $Y|X \sim N(\rho X, (1-\rho^2))$

$\therefore E[Y^2|X] = (1-\rho^2) + \rho^2 X^2$

$\therefore E[X^2 \cdot E[Y^2|X]] = E[(1-\rho^2)\cdot X^2 + \rho^2 X^4]$

$= (1-\rho^2) + 3\rho^2 = 2\rho^2+1 = E[X^2Y^2]$

$\therefore Cov[X^2, Y^2] = (2\rho^2+1) - 1 = 2\rho^2$

$\therefore Corr[X^2, Y^2] = \dfrac{2\rho^2}{\sqrt{2}\sqrt{2}} = \rho^2$ //

$\therefore$ 證明完成

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4.52

$R^2 = (X_1 - X_2)^2 + (Y_1 - Y_2)^2 \quad X_1, X_2, Y_1, Y_2 \sim N(0,1)$

$X_1 - X_2 \sim N(0,2) \quad Y_1 - Y_2 \sim N(0,2)$

$\frac{X_1 - X_2}{\sqrt{2}} \sim N(0,1) \quad \frac{Y_1 - Y_2}{\sqrt{2}} \sim N(0,1)$

$R^2 = 2\left(\frac{X_1 - X_2}{\sqrt{2}}\right)^2 + 2\left(\frac{Y_1 - Y_2}{\sqrt{2}}\right)^2$

自由度1的卡方分佈

$= \Gamma\left(\frac{1}{2}, 2\right)$

$2\left(\frac{X_1 - X_2}{\sqrt{2}}\right)^2 \sim \Gamma\left(\frac{1}{2}, 4\right)$

$2\left(\frac{X_1 - X_2}{\sqrt{2}}\right)^2 + 2\left(\frac{Y_1 - Y_2}{\sqrt{2}}\right)^2 \sim \Gamma(1, 4) = \exp(4) \quad (\text{Mean } 4)$

$\therefore R^2 \sim \exp(4)$

$T \overset{\text{def}}{=} R^2 \sim \exp(4)$

全機率 $= 1 = \int_{t=0}^{t=\infty} \frac{1}{4} \exp\left(\frac{t}{4}\right) dt =$

$$\left(\frac{dt}{dr} = 2r\right) \quad \therefore \int_{r=0}^{r=\infty} \frac{1}{4} \exp\left(\frac{-r^2}{4}\right) \cdot 2r\,dr$$

$$= \int_{r=0}^{r=\infty} \frac{r}{2} \exp\left(\frac{-r^2}{4}\right) dr$$

$$\therefore f_R(r) = \frac{r}{2} \cdot \exp\left(\frac{-r^2}{4}\right) \cdot I_{(r \geq 0)}$$

(Weibull 分布)

No. 資料科學之統計基礎

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作業5. R05246013 林元俊成

$\boxed{5.2}$ $N=\min\{n\geq 2 \mid X_n > X_1\}$

$Pr(N\geq k+1) = Pr(X_2,\dots X_k < X_1) = \int_{-\infty}^{\infty}\int_{x_2,\dots x_k<x_1} f(x_1)\dots f(x_k)\, dx_1\dots dx_k$

$= \int_{-\infty}^{\infty} f(x_1)\, F(x_1)^{k-1} dx_1$

$= \left[\frac{F(x_1)^k}{k}\right]_{-\infty}^{\infty} = \frac{1}{k}$

$\therefore Pr(N\geq k) - Pr(N\geq k+1) = \frac{1}{k-1} - \frac{1}{k}$ $(k=2,3\dots)$

$\therefore Pr(N=k) = \left(\frac{1}{k-1}-\frac{1}{k}\right)$ $(k\geq 2)$

$E[N] = \sum_{k\geq 2}^{\infty} k\left(\frac{1}{k-1}-\frac{1}{k}\right) = \sum_{k=2}^{\infty}\frac{1}{k-1} = \sum_{k=1}^{\infty}\frac{1}{k} = +\infty$

$\therefore$ 證明完成

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$\boxed{53}$

$Y_i \sim \text{bernoulli}(1 - F_X(\mu))$

$\therefore \sum_{i=1}^{n} Y_i \sim \text{Bin}(n, 1 - F_X(\mu))$

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$\boxed{5.6}$

(a) $\begin{pmatrix} Z \\ W \end{pmatrix} \overset{def}{=} \begin{pmatrix} X-Y \\ Y \end{pmatrix} \Rightarrow \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} Z+W \\ W \end{pmatrix}$

$J = \begin{pmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial w} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \therefore |J| = 1 \quad \therefore$

全機率 $1 = \iint f_X(x) f_Y(y)\, dx\, dy = \iint f_X(w+z) f_Y(w)\, dw\, dz$

$f_Z(z) = \int_{w \in R} f_X(w+z) f_Y(w)\, dw$

(b) $\begin{pmatrix} Z \\ W \end{pmatrix} \overset{def}{=} \begin{pmatrix} XY \\ X \end{pmatrix} \Rightarrow \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} W \\ \frac{Z}{W} \end{pmatrix}$

$J = \begin{pmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial w} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{1}{w} & \frac{-z}{w^2} \end{pmatrix} \quad |J| = \frac{-1}{w}$

全機率 $1 = \iint f_X(x) f_Y(y)\, dx\, dy = \iint f_X(w) f_Y\left(\frac{z}{w}\right) \cdot \left|\frac{1}{w}\right| \cdot dw\, dz$

$f_Z(z) = \int_{w \in R} f_X(w) f_Y\left(\frac{z}{w}\right) \frac{1}{|w|}\, dw$

(c) $\begin{pmatrix} Z \\ W \end{pmatrix} \overset{def}{=} \begin{pmatrix} \frac{X}{Y} \\ Y \end{pmatrix} \Rightarrow \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} WZ \\ W \end{pmatrix}$

$J = \begin{pmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial w} \end{pmatrix} = \begin{pmatrix} w & z \\ 0 & 1 \end{pmatrix} \quad |J| = w$

全機率 $1 = \iint f_X(x) f_Y(y)\, dx\, dy = \iint f_X(wz) f_Y(w) |w|\, dw\, dz$

$f_Z(z) = \int_{w \in R} f_X(wz) f_Y(w) |w|\, dw$

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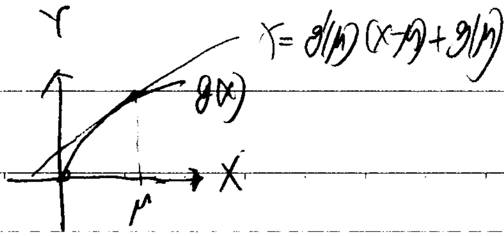

$\boxed{5.11}$

利用Jensen's inequality：$g(x)=\sqrt{x}$

$g'(\mu)(x-\mu)+g(\mu)\geq g(x)$ （∵圖↑）

∴ $E[g'(\mu)(X-\mu)+g(\mu)]\geq E[g(X)]$

∴ $\mu^{\frac{1}{2}}\geq E[X^{\frac{1}{2}}]$

$E[S^2]=\sigma^2$

∴ 將 S^2,σ^2 分別代入 X,μ

得：$\sigma\geq E[S]$，

另外，$g'(\mu)(X-\mu)+g(\mu)-g(X)\geq 0$ （∵↑）

故 $g'(\mu)(X-\mu)+g(\mu)-g(X)$ 為非負的可測函數

$$\left.\begin{aligned} &E[g'(\mu)(X-\mu)+g(\mu)-g(X)]=\int\{\underbrace{g'(\mu)(X-\mu)+g(\mu)-g(X)}_{\geq 0}\}\,dP \\ &=0 \Leftrightarrow g'(\mu)(X-\mu)+g(\mu)-g(X)=0 \quad (a.s) \end{aligned}\right\}$$ ※（證明寫在右頁）

如圖所示，$g'(\mu)(X-\mu)+g(\mu)-g(X)=0$ (a.s)

$\Leftrightarrow X=\mu$ (a.s)

注 a.s = almost surely

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由此可知,「Jensen's inequality 等號成立

$\Leftrightarrow X=\mu$ (a.s)」

∴ 若 $\sigma=E[S]$ 成立 $\Leftrightarrow S^2=\sigma^2$ (a.s)

$S^2=\sigma^2$ (a.s) 只有 $X_j=\mu$ $(j=1\sim n)$ (a.s) 才會成立

$\Leftrightarrow \sigma^2=0$

∴ $\sigma=E[S] \Leftrightarrow \sigma^2=0$

∴ $\sigma>E[S] \Leftrightarrow \sigma^2>0$

■ 證明完成

※ 的證明: $f\in M((S,\mathcal{A})\to[0,\infty])$, $(S,\mathcal{A},\mu)$: 測度空間

$\int f d\mu \geq \int_{\{f>\frac{1}{n}\}} f d\mu \geq \frac{1}{n}\mu(\{f>\frac{1}{n}\})$ (for all $n=1,2,\ldots$)

∴ $\int f d\mu=0 \Rightarrow \frac{1}{n}\mu(\{f>\frac{1}{n}\})=0 \Rightarrow \mu(\{f>\frac{1}{n}\})=0$

$$\mu\left(\bigcup_{n=1}^{\infty}\{f>\tfrac{1}{n}\}\right)\leq\sum_{n=1}^{\infty}\mu(\{f>\tfrac{1}{n}\})=0$$

$\parallel$

$\mu(\{f>0\})$

∴ $f=0$ (μ-a.e)

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$\boxed{5.13}$ $T = \sum_{k=1}^{n} \frac{(X_k - \bar{X})^2}{\sigma^2}$ $\quad T \sim \chi^2_{n-1} = \Gamma\left(\frac{n-1}{2}, \frac{1}{2}\right)$

$$E\left[T^{\frac{1}{2}}\right] = \int_0^\infty t^{\frac{1}{2}} \cdot \frac{t^{\frac{n-1}{2}-1}}{\Gamma\left(\frac{n-1}{2}\right) \cdot 2^{\frac{n-1}{2}}} \exp\left(-\frac{t}{2}\right) dt$$

$$= \int_0^\infty \frac{t^{\frac{n}{2}-1}}{\Gamma\left(\frac{n-1}{2}\right) \cdot 2^{\frac{n-1}{2}}} \exp\left(-\frac{t}{2}\right) dt$$

$u \overset{\text{put}}{=} \frac{t}{2} \quad \frac{du}{dt} = \frac{1}{2}$

$$E\left[T^{\frac{1}{2}}\right] = \int_0^\infty \frac{(2u)^{\frac{n}{2}-1}}{\Gamma\left(\frac{n-1}{2}\right) 2^{\frac{n-1}{2}}} \exp(-u) \cdot 2du$$

$$= \int_0^\infty \frac{u^{\frac{n}{2}-1}}{\Gamma\left(\frac{n-1}{2}\right)} \frac{2^{\frac{n}{2}}}{2^{\frac{n-1}{2}}} \cdot \exp(-u) du$$

$$= \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} \cdot 2^{\frac{1}{2}}$$

$\frac{\sigma^2 T}{n-1} = S^2$

$$\therefore E\left[\left\{\frac{S^2}{\sigma^2}(n-1)\right\}^{\frac{1}{2}}\right] = E\left[T^{\frac{1}{2}}\right] = \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} \cdot 2^{\frac{1}{2}}$$

$$\therefore E\left[\sqrt{n-1} \cdot \frac{S}{\sigma}\right] = \frac{\Gamma\left(\frac{n}{2}\right)\sqrt{2}}{\Gamma\left(\frac{n-1}{2}\right)}$$

$$\therefore E\left[\frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \frac{\sqrt{n-1}}{\sqrt{2}} S\right] = \sigma$$

$$\therefore \frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \cdot \frac{\sqrt{n-1}}{\sqrt{2}} S$$

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$\boxed{5.25}$

全機率 $= 1 = \int_{0 \le X_{(1)} \le X_{(2)} \le \cdots X_{(n)} \le \theta} n! \frac{a X_{(1)}^{a-1}}{\theta^a} \cdots \frac{a X_{(n)}^{a-1}}{\theta^a} \, dx_{(1)} \cdots dx_{(n)}$

$Y_1 \overset{put}{=} \frac{X_{(1)}}{X_{(2)}}$ $Y_2 \overset{put}{=} \frac{X_{(2)}}{X_{(3)}}$, ..., $Y_{n-1} = \frac{X_{(n-1)}}{X_{(n)}}$, $Y_n = X_{(n)}$

$$\begin{cases} X_{(1)} = Y_1 Y_2 \cdots Y_n \\ X_{(2)} = Y_2 \cdots Y_n \\ \vdots \\ X_{(n)} = Y_{(n)} \end{cases} \qquad J = \begin{pmatrix} Y_2 \cdots Y_n & * & * \\ 0 & Y_3 \cdots Y_n & * \\ \vdots & 0 & \ddots \\ 0 & 0 & 1 \end{pmatrix}$$

J 為上三角矩陣 $\det J = Y_2^1 Y_3^2 \cdots Y_n^{n-1}$
（看對角元素即可）

全機率 $= 1 = \int_{\substack{0 \le Y_1 \cdots Y_n \le Y_2 \cdots Y_n \le \\ \cdots \le Y_n}} \frac{n! a^n}{\theta^{na}} (Y_1 Y_2^2 \cdots Y_n^n)^{a-1} \cdot Y_2 Y_3^2 \cdots Y_n^{n-1} \, dy_1 \cdots dy_n$

$= \int_{\substack{0 \le Y_1 \le 1 \\ 0 \le Y_2 \le 1 \\ \vdots \\ 0 \le Y_{n-1} \le 1 \\ 0 \le Y_n \le \theta}} \frac{n! a^n}{\theta^{na}} y_1^{a-1} y_2^{2a-1} \cdots y_n^{na-1} \, dy_1 \cdots dy_n$

$= \int_{\substack{0 \le Y_j \le 1 \ (j=1 \sim n-1) \\ 0 \le Y_n \le \theta}} \underbrace{(a y_1^{a-1}) \cdot (2a y_2^{2a-1}) \cdots ((n-1) a y_{n-1}^{(n-1)a-1}) \cdot \frac{na y_n^{na-1}}{\theta^{na}}}_{= f_{Y_1 \cdots Y_n}(y_1, \ldots, y_n)} \, dy_1 \cdots dy_n$

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求$Y_1 \sim Y_n$之邊際分佈…

$$f_{Y_1}(y_1) = \int_{\substack{y_2 \in [0,1] \\ \vdots \\ y_{n-1} \in [0,1] \\ y_n \in [0,\theta]}} (a y_1^{a-1})(2a y_2^{2a-1}) \cdots \left(\frac{na\, y_n^{na-1}}{\theta^{na}}\right) dy_2 \cdots dy_n$$

$$= a y_1^{a-1} \quad (0 \leq y_1 \leq 1)$$

同理，$f_{Y_j}(y_j) = ja\, y_j^{ja-1} \quad (0 \leq y_j \leq 1) \quad (j = 1 \sim n-1)$

$$f_{Y_n}(y_n) = \frac{na\, y_n^{na-1}}{\theta^{na}} \quad (0 \leq y_n \leq \theta)$$

$f_{Y_1 \cdots Y_n}(y_1 \cdots y_n) = \prod_{j=1}^{n} f_{Y_j}(y_j)$ 成立，

故$Y_1 \sim Y_n$為獨立的隨機變數.

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[5.27]. (a) 求 $X_{(k)}$ 與 $X_{(m)}$ 的聯合分佈. ($k<m$ 的 case)

全機率 $=1=\int_{X_{(1)}\le\cdots\le X_{(n)}} n!\, f(x_{(1)})\cdots f(x_{(n)})\, dx_{(1)}\cdots dx_{(n)}$

STEP 1 $\int dx_{(1)} \sim dx_{(k-1)}$ 的計算.

$$\int\cdots dx_{(1)} = n!\left[F(x_{(1)})\right]_{x_{(1)}=-\infty}^{x_{(1)}=x_{(2)}} f(x_{(2)})\cdots f(x_{(n)})$$

$$= n!\, f(x_{(2)})\, F(x_{(2)})\cdot f(x_{(3)})\cdots f(x_{(n)})$$

$$\int\cdots dx_{(2)} = n!\left[\frac{1}{2}F(x_{(2)})^2\right]_{-\infty}^{x_{(3)}} f(x_{(3)})f(x_{(4)})\cdots f(x_{(n)})$$

$$= \frac{n!}{2!} f(x_{(3)})F(x_{(3)})^2 f(x_{(4)})\cdots f(x_{(n)})$$

$$\int\cdots dx_{(j)} = \frac{n!}{j!} f(x_{(j+1)})F(x_{(j+1)})^j f(x_{(j+2)})\cdots f(x_{(n)})$$

$j=k-1$ (代入) $\dfrac{n!}{(k-1)!} f(x_{(k)})F(x_{(k)})^{k-1} f(x_{(k+1)})\cdots f(x_{(n)})$

STEP 2 $\int dx_{(k+1)}\cdots dx_{(m-1)}$ 的計算

$$\int_{x_{(k)}}^{x_{(k+2)}} f(x_{(k+1)})\,dx_{(k+1)} = \left[F(x_{(k+1)})\right]_{x_{(k)}}^{x_{(k+2)}} = F(x_{(k+2)})-F(x_{(k)})$$

$$\int_{x_{(k)}}^{x_{(k+3)}} f(x_{(k+2)})\left(F(x_{(k+2)})-F(x_{(k)})\right)dx_{(k+2)} = \left[\frac{1}{2!}\left(F(x_{(k+2)})-F(x_{(k)})\right)^2\right]_{x_{(k)}}^{x_{(k+3)}}$$

$$= \frac{1}{2!}\left(F(x_{(k+3)})-F(x_{(k)})\right)^2$$

$$\int_{x_{(k)}}^{x_{(k+j)}} f(x_{(k+j-1)})\left(F(x_{(k+j-1)})-F(x_{(k)})\right)^{j-1}/(j-1)!\; dx_{(k+j-1)}$$

$$= \frac{1}{(j-1)!}\left(F(x_{(k+j)})-F(x_{(k)})\right)^{j-1}$$

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$j = m-k$ (代入)

$$\frac{1}{(m-k-1)!}\left(F(x_{(m)})-F(x_{(k)})\right)^{m-k-1}$$

综合 STEP1, STEP2 的结論：

$$\int \cdots dx_{(1)} \cdots dx_{(k-1)}\, dx_{(k+1)} \cdots dx_{(m-1)} = \frac{n!}{(k-1)!} f(x_{(k)}) \cdot F(x_{(k)})^{k-1} \cdot \frac{1}{(m-k-1)!}$$
$$\left(F(x_{(m)})-F(x_{(k)})\right)^{m-k-1} f(x_{(m)}) \cdot f(x_{(m+1)}) \cdots f(x_{(n)})$$

$\boxed{\text{STEP3}}$ $\int dx_{(n)} \cdots dx_{(m+1)}$ 的計算

$$\int_{x_{(n-1)}}^{\infty} f(x_{(n)})\,dx_{(n)} = 1-F(x_{(n-1)})$$

$$\int_{x_{(n-2)}}^{\infty} f(x_{(n-1)}) \cdot \left(1-F(x_{(n-1)})\right)\underset{dx_{(n-1)}}{\wedge} = \left[\frac{-1}{2!}\left(1-F(x_{(n-1)})^2\right)\right]_{x_{(n-2)}}^{\infty} = \frac{1}{2!}\left(1-F(x_{(n-2)})\right)^2$$

$$\int_{x_{(n-j)}}^{\infty} \cdots dx_{(n-j+1)} = \frac{1}{j!}\left(1-F(x_{(n-j)})\right)^j$$

$j = n-m$ (代入) $\cdots$ $\dfrac{1}{(n-m)!}\left(1-F(x_{(m)})\right)^{n-m}$

故 $\int \sim dx_{(1)} \cdots dx_{(k-1)}\, dx_{(k+1)} \cdots dx_{(m-1)}\, dx_{(m+1)} \cdots dx_{(n)}$

$$= n! \cdot \frac{f(x_{(k)})\, F(x_{(k)})^{k-1}}{(k-1)!} \cdot \frac{\left(F(x_{(m)})-F(x_{(k)})\right)^{m-k-1} f(x_{(m)})}{(m-k-1)!} \cdot \frac{\left(1-F(x_{(m)})\right)^{n-m}}{(n-m)!}$$

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$X_{(k)}$ 與 $X_{(m)}$ 的聯合 pdf：$(k<m)$

$$f_{X_{(k)}X_{(m)}}(x_{(k)},X_{(m)}) = n!\,\frac{f(X_{(k)})\,F(X_{(k)})^{k-1}}{(k-1)!}\,\frac{f(X_{(m)})\,(F(X_{(m)})-F(X_{(k)}))^{m-k-1}}{(m-k-1)!}$$

$$\frac{(1-F(m))^{n-m}}{(n-m)!} \quad \left(\overset{\text{when}}{X_{(k)} \le X_{(m)}}\right)$$

接著求 $X_{(m)}$ 的邊際分佈.

$$\int_{-\infty}^{X_{(m)}} f_{X_{(k)}X_{(m)}}(X_{(k)},X_{(m)})\,dx_{(k)} \quad \downarrow \circledast\ Z_k \overset{\text{def}}{=} F(X_{(k)}) \quad \frac{dZ_k}{dX_{(k)}} = f(X_{(k)})$$

$$\int_0^{F(X_{(m)})} n!\,\frac{\cancel{f(X_{(k)})}}{(k-1)!}\,Z_k^{k-1}\,\frac{f(X_{(m)})\cdot(F(X_{(m)})-Z_k)^{m-k-1}}{(m-k-1)!}\cdot\frac{(1-F(m))^{n-m}}{(n-m)!}\cdot\frac{dZ_k}{\cancel{f(X_{(k)})}}$$

$$= \int_0^{F(X_{(m)})} n!\cdot\frac{Z_k^{k-1}}{(k-1)!}\cdot\frac{(F(X_{(m)})-Z_k)^{m-k-1}}{(m-k-1)!}\cdot\frac{(1-F(m))^{n-m}}{(n-m)!}\,f(X_{(m)})\,dZ_k$$

$$\downarrow \circledast\ U_k \overset{\text{def}}{=} \frac{Z_k}{F(X_{(m)})} \qquad \frac{dU_k}{dZ_k} = \frac{1}{F(X_{(m)})}$$

$$= \int_0^1 n!\,\frac{F(X_{(m)})^{k-1}\,U_k^{k-1}}{(k-1)!}\cdot\frac{F(X_{(m)})^{m-k-1}\,(1-U_k)^{m-k-1}}{(m-k-1)!}\cdot\frac{(1-F(m))^{n-m}}{(n-m)!}\cdot F(X_{(m)})\,f(X_{(m)})\,dU_k$$

$$= \int_0^1 n!\,F(X_{(m)})^{m-1}\,(1-F(X_{(m)}))^{n-m}\,U_k^{k-1}\,(1-U_k)^{m-k-1}\cdot\frac{f(X_{(m)})\,dU_k}{(k-1)!\,(m-k-1)!\,(n-m)!}$$

$$= n!\cdot B_e(k,m-k)\cdot\frac{1}{(k-1)!\,(m-k-1)!\,(n-m)!}\cdot F(X_{(m)})^{m-1}\,(1-F(X_{(m)}))^{n-m}\cdot f(X_{(m)})$$

$$= n!\cdot\frac{\cancel{(k-1)!}\,\cancel{(m-k-1)!}}{(m-1)!}\,\frac{1}{\cancel{(k-1)!}\,\cancel{(m-k-1)!}\,(n-m)!}\,F(X_{(m)})^{m-1}\,(1-F(X_{(m)}))^{n-m}\,f(X_{(m)})$$

$$= \frac{n!}{(m-1)!(n-m)!} F(x_{(m)})^{m-1} (1-F(x_{(m)}))^{n-m} f(x_{(m)})$$

故 $f_{X_{(k)}|X_{(m)}}(x_{(k)}|x_{(m)}) =$

$$\frac{\left\{ \frac{\cancel{n!}}{(k-1)!(m-k-1)!} f(x_{(k)}) F(x_{(k)})^{k-1} \cancel{f(x_{(m)})} (F(x_{(m)}) - F(x_{(k)}))^{m-k-1} \cdot \cancel{\frac{(1-F(m))^{n-m}}{(n-m)!}} \right\}}{\frac{\cancel{n!}}{(m-1)!\cancel{(n-m)!}} F(x_{(m)})^{m-1} \cancel{(1-F(x_{(m)}))^{n-m}} \cancel{f(x_{(m)})}}$$

$$= \frac{(m-1)!}{(k-1)!(m-k-1)!} \cdot \frac{f(x_{(k)})}{F(x_{(m)})} \cdot \left(\frac{F(x_{(k)})}{F(x_{(m)})} \right)^{k-1} \left(1 - \frac{F(x_{(k)})}{F(x_{(m)})} \right)^{m-k-1}$$

$$\therefore \frac{1}{Be(k,m-k)} \cdot \frac{f(x_{(k)})}{F(x_{(m)})} \cdot \left(\frac{F(x_{(k)})}{F(x_{(m)})} \right)^{k-1} \left(1 - \frac{F(x_{(k)})}{F(x_{(m)})} \right)^{m-k-1} \quad (0 \le x_{(k)} \le x_{(m)})$$

$(0 \because \text{elsewhere})$ 為所求的條件 pdf.

注)

若 $k>m$: $\frac{f_{X_{(k)},X_{(m)}}(x_{(k)},x_{(m)})}{f_{X_{(m)}}(x_{(m)})}$ 的分子改為以下 $f_{X_{(m)},X_{(k)}}(x_{(m)},x_{(k)})$

$$f_{X_{(m)},X_{(k)}}(x_{(m)},x_{(k)}) \overset{def}{=} n! \cdot \frac{f(x_{(m)}) F(x_{(m)})^{m-1}}{(m-1)!} \cdot \frac{f(x_{(k)}) (F(x_{(k)}) - F(x_{(m)}))^{k-m-1}}{(k-m-1)!} \cdot \frac{(1-F(k))^{n-k}}{(n-k)!} \quad (x_{(m)} \le x_{(k)})$$

(替換 k.m 而已)

$\therefore f_{X_{(m)},X_{(k)}}(x_{(m)},x_{(k)}) / f_{X_{(m)}}(x_{(m)})$ 為所求的條件 pdf

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(b) V 改為 S，求 S|R 的分佈.

先求 $X_{(1)}, X_{(n)}$ 的聯合 pdf.

$$\Pr(X_{(1)} \le x, X_{(n)} \le y) = \Pr(X_{(n)} \le y) - \Pr(x < X_{(1)}, X_{(n)} \le y)$$

$$= F(y)^n - (F(y) - F(x))^n$$

$$f_{X_{(1)}X_{(n)}}(x,y) = \frac{\partial^2}{\partial x \partial y}\left(F(y)^n - (F(y)-F(x))^n\right)$$

$$= n(n-1) f(x) f(y) (F(y)-F(x))^{n-2}$$

$$\begin{cases} r = y - x \\ s = \frac{x+y}{2} \end{cases} \quad |J| = 1 \quad \therefore dxdy = drds$$

$$全機率 = 1 = \int_{-\infty<x<y<\infty} n(n-1) f(x) f(y) (F(y)-F(x))^{n-2} dxdy$$

$$= \int_{-\infty<s-\frac{r}{2}<s+\frac{r}{2}<\infty} n(n-1) f(s-\tfrac{r}{2}) f(s+\tfrac{r}{2}) (F(s+\tfrac{r}{2}) - F(s-\tfrac{r}{2}))^{n-2} drds$$

$$\therefore f_{SR}(s,r) = n(n-1) f(s-\tfrac{r}{2}) f(s+\tfrac{r}{2}) (F(s+\tfrac{r}{2}) - F(s-\tfrac{r}{2}))^{n-2}$$

$$f_R(r) = \int_{r>0} n(n-1) f(s-\tfrac{r}{2}) f(s+\tfrac{r}{2}) (F(s+\tfrac{r}{2}) - F(s-\tfrac{r}{2}))^{n-2} dr$$

$$\therefore f_{S|R}(s|r) = \frac{n(n-1) f(s-\frac{r}{2}) f(s+\frac{r}{2}) (F(s+\frac{r}{2}) - F(s-\frac{r}{2}))^{n-2}}{\int_{r>0} n(n-1) f(s-\frac{r}{2}) f(s+\frac{r}{2}) (F(s+\frac{r}{2}) - F(s-\frac{r}{2}))^{n-2} dr}$$

※這題有點不清楚，pdf 是 f 還是 $\frac{1}{a} I_{(0,a)}(x)$

若 $X_1 \cdots X_n \sim \text{Uniform}(0,a)$：

$f_{S,R}(s,r) = n(n-1)\frac{1}{a^2}\cdot\left(\frac{r}{a}\right)^{n-2} = n(n-1)\frac{r^{n-2}}{a^n}$

$0 \leq s-\frac{r}{2} < s+\frac{r}{2} \leq a$

$\left(\Rightarrow \frac{r}{2} \leq s \leq a-\frac{r}{2}\right)$

$\therefore f_R(r) = n(n-1)\frac{r^{n-2}}{a^n}(a-r)$

$\therefore \frac{f_{S,R}(s,r)}{f_R(r)} = \frac{1}{a-r}$ $\therefore S|R \sim \text{Uni}\left(\frac{r}{2}, a-\frac{r}{2}\right)$

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$\boxed{5.28}$ (a) 由於變數很多，故 cdf 需要依照 $\{x_{(i_1)}, \dots x_{(i_k)}\}$ 的大小考慮很多 case. 我們先討論 pdf.

基本的做法與 $\boxed{5.27}$ (a) 相同.（反覆積分）

$$f_{X_{(1)} \dots X_{(n)}}(x_{(1)}, \dots x_{(n)}) = n!\, f(x_{(1)}) \cdots f(x_{(n)}) \cdot I_{(x_{(1)} \le \dots \le x_{(n)})}$$

① $\int \dots dx_{(1)} dx_{(2)} \dots dx_{(i_1-1)}$ 的積分

$$\int dx_{(1)} = \int_{-\infty}^{x_{(2)}} n!\, f(x_{(1)}) \cdots f(x_{(n)})\, dx_{(1)} = n!\, F(x_{(2)}) f(x_{(2)}) \cdots f(x_{(n)}) \cdot$$

$$\int dx_{(2)} = \int_{-\infty}^{x_{(3)}} n!\, F(x_{(2)}) f(x_{(2)}) \cdots f(x_{(n)})\, dx_{(2)} = \frac{n!}{2!} F(x_{(3)})^2 \cdot f(x_{(3)}) \cdots f(x_{(n)})$$

$\vdots$

$$\int dx_{(k-1)} = \frac{n!}{(k-1)!} F(x_{(k)})^{k-1} f(x_{(k)}) \cdots f(x_{(n)})$$

$\overset{\text{代入}}{k = i_1}$ $$\underbrace{\frac{n!}{(i_1-1)!} F(x_{(i_1)})^{i_1-1} f(x_{(i_1)}) f(x_{(i_1+1)}) \cdots f(x_{(n)})}_{①}$$

② $\int \dots dx_{(i_1+1)} \dots dx_{(i_2-1)}$ 的積分（先忽略 ① 的部分）

$$\int dx_{(i_1+1)} = \int_{x_{(i_1)}}^{x_{(i_1+2)}} f(x_{(i_1+1)}) \cdots f(x_{(n)})\, dx_{(i_1+1)} = \left(F(x_{(i_1+2)}) - F(x_{(i_1)})\right) f(x_{(i_1+1)}) \cdots$$

$$\int dx_{(i_1+2)} = \frac{1}{2!} \left(F(x_{(i_1+3)}) - F(x_{(i_1)})\right)^2 f(x_{(i_1+3)}) f(x_{(i_1+4)}) \cdots f(x_{(n)})$$

$\vdots$

$$\int dx_{(i_1+k-1)} = \frac{1}{(k-1)!} \left(F(x_{(i_1+k)}) - F(x_{(i_1)})\right)^{k-1} f(x_{(i_1+k)}) \cdots f(x_{(n)})$$

$\overset{\text{代入}}{k = i_2 - i_1}$ $$\frac{1}{(i_2-i_1)!} \left(F(x_{(i_2)}) - F(x_{(i_1)})\right)^{i_2-i_1-1} f(x_{(i_2)}) \cdots f(x_{(n)})$$

① & ②: $$\int dx_{(1)} \dots dx_{(i_2-1)} = \frac{n!}{(i_1-1)!} F(x_{(i_1)})^{i_1-1} f(x_{(i_1)}) \frac{1}{(i_2-i_1)!} \left(F(x_{(i_2)}) - F(x_{(i_1)})\right)^{i_2-i_1-1} f(x_{(i_2)}) \cdots f(x_{(n)})$$

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② ': $\int dx_{(i_2)} \cdots dx_{(i_3-1)}, \cdots, \int dx_{(i_{l-1})} \cdots dx_{(i_l-1)}$ (反覆②的操作)

得: $n! \frac{1}{(i_1-1)!} F(x_{(i_1)})^{i_1-1} f(x_{(i_1)}) \cdot \frac{1}{(i_2-i_1)!} \left(F(x_{(i_2)})-F(x_{(i_1)})\right)^{i_2-i_1-1} f(x_{(i_2)}) \cdot$

$\frac{1}{(i_3-i_2)!} \left(F(x_{(i_3)})-F(x_{(i_2)})\right)^{i_3-i_2-1} f(x_{(i_3)}) \cdots$

$\frac{1}{(i_l-i_{l-1})!} \left(F(x_{(i_l)})-F(x_{(i_{l-1})})\right)^{i_l-i_{l-1}-1} f(x_{(i_l)}) \underbrace{f(x_{(i_l+1)}) \cdots f(x_{(n)})}_{②'}$

③ 最後考慮 ②' 的積分

$\int dx_{(n)} = \int_{x_{(n-1)}}^{\infty} \cdots dx_{(n)} = f(x_{(i_l+1)}) \cdots f(x_{(n-1)}) \left(1-F(x_{(n-1)})\right)$

$\int dx_{(n-1)} = \int_{x_{(n-2)}}^{\infty} \cdots dx_{(n-1)} = f(x_{(i_l+1)}) \cdots f(x_{(n-2)}) \cdot \frac{\left(1-F(x_{(n-2)})\right)^2}{2!}$

$\vdots$

$\int dx_{(n-k)} = f(x_{(i_l+1)}) \cdots f(x_{(n-k-1)}) \cdot \frac{\left(1-F(x_{(n-k-1)})\right)^{(k+1)}}{(k+1)!}$

代入 $k = n-i_l-1 \cdots \frac{\left(1-F(x_{(i_l)})\right)^{n-i_l}}{(n-i_l)!}$

故 結合 ②' 與 ③: $f^*(x_{(i_1)}, x_{(i_2)} \cdots x_{(i_l)}) =$

$n! \cdot \frac{1}{(i_1-1)!} F(x_{(i_1)})^{i_1-1} f(x_{(i_1)}) \cdot \frac{1}{(i_2-i_1)!} \left(F(x_{(i_2)})-F(x_{(i_1)})\right)^{i_2-i_1-1} f(x_{(i_2)}) \cdot$

$\frac{1}{(i_3-i_2)!} \left(F(x_{(i_3)})-F(x_{(i_2)})\right)^{i_3-i_2-1} f(x_{(i_3)}) \cdots \frac{1}{(i_l-i_{l-1})!} \left(F(x_{(i_l)})-F(x_{(i_{l-1})})\right)^{i_l-i_{l-1}-1} \cdot f(x_{(i_l)})$

$\frac{\left(1-F(x_{(i_l)})\right)^{n-i_l}}{(n-i_l)!} \cdot \mathbb{I}(x_{(i_1)} \leq x_{(i_2)} \leq \cdots \leq x_{(i_l)})$ ⊥

(指示函數)

另外 cdf: $F(t_{(i_1)}, \cdots t_{(i_l)}) = \int_{-\infty}^{t_{(i_1)}} \cdots \int_{-\infty}^{t_{(i_l)}} f^*(x_{(i_1)}, \cdots x_{(i_l)})\, dx_{(i_1)} \cdots dx_{(i_l)}$

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(b) 為了方便起見，令 $\{X_{(k_1)},\dots X_{(k_{\ell+m})}\} \overset{\text{def}}{=} \{X_{(i_1)},\dots X_{(i_\ell)}\} \cup \{X_{(j_1)},\dots X_{(j_m)}\}$ where $X_{(k_1)} < \dots < X_{(k_{\ell+m})}$.

我們該求 $\dfrac{(X_{(k_1)},\dots X_{(k_{\ell+m})})\text{ 的聯合 pdf}}{(X_{(j_1)}\dots X_{(j_m)})\text{ 的聯合 pdf}} \cdots \dfrac{\boxed{\text{分子}}}{\boxed{\text{分母}}}$

我們於(a)已經求 $(X_{(i_1)},\dots X_{(i_\ell)})$ 的聯合 pdf.

所以可以直接將 $(i_1)\dots(i_\ell)$ 改為 $(k_1)\dots(k_{\ell+m})$ 以及 $(j_1)\dots(j_m)$.

$\boxed{\text{分子}}$ $n!\dfrac{F(X_{(k_1)})^{k_1-1}f(x_{(k_1)})\,(F(X_{(k_2)})-F(X_{(k_1)}))^{k_2-k_1-1}f(X_{(k_2)})\cdots(F(X_{(k_{\ell+m})})-F(X_{(k_{\ell+m-1})}))^{k_{\ell+m}-k_{\ell+m-1}}}{(k_1-1)!\,(k_2-k_1)!\cdots(k_{\ell+m}-k_{\ell+m-1})!\,(n-k_{\ell+m})!}$

$\cdot f(x_{(k_{\ell+m})})\cdot(1-F(x_{(k_{\ell+m})}))^{n-k_{\ell+m}}\cdot I(X_{(k_1)}\le\dots\le X_{(k_{\ell+m})})$

$\boxed{\text{分母}}$ $n!\dfrac{F(x_{(j_1)})^{j_1-1}f(X_{(j_1)})\,(F(X_{(j_2)})-F(X_{(j_1)}))^{j_2-j_1-1}f(X_{(j_2)})\cdots(F(X_{(j_m)})-F(X_{(j_{m-1})}))^{j_m-j_{m-1}}}{(j_1-1)!\,(j_2-j_1)!\cdots(j_m-j_{m-1})!\,(n-j_m)!}$

$\cdot f(X_{(j_m)})\cdot(1-F(x_{(j_m)}))^{n-j_m}\,I(X_{(j_1)}\le\dots\le X_{(j_m)})$

$\therefore \dfrac{\boxed{\text{分子}}}{\boxed{\text{分母}}}$ 為所求的 pdf.

$$\text{cdf} = \int_{-\infty}^{t_{(i_1)}}\cdots\int_{-\infty}^{t_{(i_\ell)}} \frac{\boxed{\text{分子}}}{\boxed{\text{分母}}}\,dx_{(i_1)}\cdots dx_{(i_\ell)}$$

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$\boxed{530}$ 利用 chebyshev 不等式:

$$E[\bar{X}_1-\bar{X}_2]=0,\quad V[\bar{X}_1-\bar{X}_2]=\frac{2\sigma^2}{n}$$

$$Pr\left(|\bar{X}_1-\bar{X}_2|\geq k\left(\frac{2\sigma^2}{n}\right)^{\frac{1}{2}}\right)\leq\frac{1}{k^2}$$

$$k=\frac{1}{\sqrt{2}}\frac{\sqrt{n}}{5}\ \text{(代入)}$$

$$Pr\left(|\bar{X}_1-\bar{X}_2|\geq\frac{\sigma}{5}\right)\leq\frac{50}{n}$$

$$\therefore\ Pr\left(|\bar{X}_1-\bar{X}_2|<\frac{\sigma}{5}\right)\geq 1-\frac{50}{n}\geq 0.99$$

$$0.01\geq\frac{50}{n}$$

$$n\geq 5000$$

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若利用中央極限定理 ($\because \sigma^2 < \infty$)

$Z_j \overset{\text{def}}{=} X_{1j} - X_{2j}$ $\quad E[Z_j]=0 \quad V[Z_j] = 2\sigma^2 < \infty$

$\{Z_j\}_{j=1\sim n}$ 為 iid 的隨機變數

$$\sqrt{n}\left(\frac{Z_1+\dots+Z_n}{n}-0\right) \xrightarrow{d} N(0, 2\sigma^2) \quad (\because \text{中央極限定理})$$

$$\parallel$$

$$\sqrt{n}(\overline{X}_1-\overline{X}_2) \xrightarrow{d} N(0, 2\sigma^2)$$

$$\frac{\sqrt{n}}{\sqrt{2}\sigma}(\overline{X}_1-\overline{X}_2) \xrightarrow{d} N(0,1)$$

$$\therefore P\left(\left|\frac{\sqrt{n}}{\sqrt{2}\sigma}(\overline{X}_1-\overline{X}_2)\right| \geq \frac{\sqrt{n}}{\sqrt{2}\cdot 5}\right) = 2\cdot\left(1-\Phi\left(\frac{\sqrt{n}}{5\sqrt{2}}\right)\right) \leq 0.01$$

$$\therefore \Phi\left(\frac{\sqrt{n}}{5\sqrt{2}}\right) \geq 0.995 \qquad \frac{\sqrt{n}}{5\sqrt{2}} \geq \Phi^{-1}(0.995) \doteqdot 2.57$$

$\therefore n \geq 330.245$ 　　約 331

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$\boxed{5.50}$ $\begin{cases} X_1 = \cos(2\pi U_1)\sqrt{-2\ln U_2} \\ X_2 = \sin(2\pi U_1)\sqrt{-2\ln U_2} \end{cases}$

$\tan(2\pi U_1) = \dfrac{X_2}{X_1}$

$U_1 \sim \mathrm{Uniform}(0,1)$，故 $2\pi U_1 = \arctan\left(\dfrac{X_2}{X_1}\right) + \dfrac{\pi}{2}$ or $\left(0 \leq U_1 < \dfrac{1}{2}\right)$

$\arctan\left(\dfrac{X_2}{X_1}\right) + \dfrac{3\pi}{2}$ $\left(\dfrac{1}{2} \leq U_1 < 1\right)$

・$U_1 = \dfrac{1}{2\pi}\arctan\left(\dfrac{X_2}{X_1}\right) + \dfrac{1}{4}$ or $\dfrac{1}{2\pi}\arctan\left(\dfrac{X_2}{X_1}\right) + \dfrac{3}{4}$

$\left(0 \leq U_1 < \dfrac{1}{2}\right)$ $\left(\dfrac{1}{2} \leq U_1 < 1\right)$

・$U_2 = \exp\left(\dfrac{-1}{2}(X_1^2 + X_2^2)\right)$

$$J = \begin{pmatrix} \dfrac{\partial U_1}{\partial X_1} & \dfrac{\partial U_1}{\partial X_2} \\ \dfrac{\partial U_2}{\partial X_1} & \dfrac{\partial U_2}{\partial X_2} \end{pmatrix} = \begin{pmatrix} \dfrac{\frac{-X_2}{X_1^2}}{1+\frac{X_2^2}{X_1^2}} \cdot \dfrac{1}{2\pi} & \dfrac{\frac{1}{X_1}}{1+\frac{X_2^2}{X_1^2}} \cdot \dfrac{1}{2\pi} \\ -X_1 \cdot \exp\left(\frac{-1}{2}(X_1^2+X_2^2)\right) & -X_2 \exp\left(\frac{-1}{2}(X_1^2+X_2^2)\right) \end{pmatrix}$$

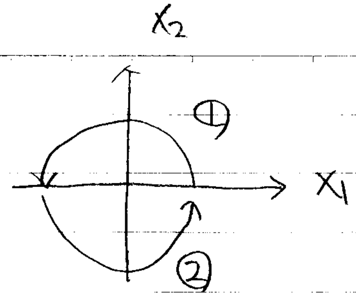
$$\det J = \left\{ \frac{X_2^2}{X_1^2+X_2^2}\exp\left(\frac{-1}{2}(X_1^2+X_2^2)\right) + \frac{X_1^2}{X_1^2+X_2^2}\exp\left(\frac{-1}{2}(X_1^2+X_2^2)\right) \right\} \cdot \frac{1}{2\pi}$$

$$= \frac{1}{2\pi}\exp\left(\frac{-1}{2}(X_1^2+X_2^2)\right)$$

全機率 $= 1 = \int_{U_1 \in (0,1)} \int_{U_2 \in (0,1)} du_1 du_2$

$= \int_{U_1 \in [0,\frac{1}{2})} \int_{U_2 \in [0,1)} du_1 du_2 + \int_{U_1 \in [\frac{1}{2},1)} \int_{U_2 \in [0,1)} du_1 du_2$

① $\downarrow$ $\dfrac{1}{2\pi}\exp\left(\dfrac{-1}{2}(X_1^2+X_2^2)\right)$ ② $\downarrow$ $\dfrac{1}{2\pi}\exp\left(\dfrac{-1}{2}(X_1^2+X_2^2)\right)$



$$= \int_{\substack{x_1 \in R \\ x_2 \in R^+}} \frac{1}{2\pi} \exp\left(\frac{-1}{2}(x_1^2+x_2^2)\right) dx_1 dx_2 + \int_{\substack{x_1 \in R \\ x_2 \in R^-}} \frac{1}{2\pi} \exp\left(\frac{-1}{2}(x_1^2+x_2^2)\right) dx_1 dx_2$$

$$= \int_{(x_1,x_2) \in R^2} \frac{1}{2\pi} \exp\left(\frac{-1}{2}(x_1^2+x_2^2)\right) dx_1 dx_2$$

由此可知 $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$

No. 資料科學與統計基礎(Ⅰ)

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作業6. R05246013 森元俊成

$\boxed{1}$ $X \sim Po(\lambda)$, $Y|X=x \sim Bin(x, p_1)$

$Z|X=x, Y=y \sim Bin(y, p_2)$

(a) $Pr(Z=z \mid X=x, Y=y) = {}_yC_z\, p_2^{z}(1-p_2)^{y-z} \cdots ①$ $(0 \leq z \leq y)$

$$Pr(X=x, Y=y) = Pr(Y=y \mid X=x)\cdot Pr(X=x)$$
$$= {}_xC_y\, p_1^{y}(1-p_1)^{x-y}\cdot e^{-\lambda}\cdot \frac{\lambda^x}{x!} \cdots ②$$
$(0 \leq y \leq x)$

$$\therefore Pr(X=x, Y=y, Z=z) = Pr(Z=z \mid X=x, Y=y)\, Pr(X=x, Y=y)$$
$$= ①\cdot②$$
$$= {}_yC_z\, p_2^{z}(1-p_2)^{y-z}\cdot {}_xC_y\, p_1^{y}(1-p_1)^{x-y}\cdot e^{-\lambda}\cdot\frac{\lambda^x}{x!}$$
$(0 \leq z \leq y \leq x)$

(b) $Pr(X=x, Y=y) =$

$$\frac{\cancel{x!}}{y!(x-y)!} p_1^{y}(1-p_1)^{x-y}\cdot e^{-\lambda}\cdot\frac{\lambda^x}{\cancel{x!}}$$

$$= \frac{p_1^{y}\cdot e^{-\lambda}}{y!}\cdot\frac{(1-p_1)^{x-y}}{(x-y)!}\cdot\lambda^{(x-y)}\cdot\lambda^{y}$$

$$= \frac{(\lambda p_1)^{y}}{y!}e^{-\lambda}\cdot\frac{(\lambda(1-p_1))^{x-y}}{(x-y)!} \quad (z = x-y)$$

$$\sum_{z=0}^{\infty}\frac{(\lambda p_1)^{y}}{y!}e^{-\lambda}\frac{(\lambda(1-p_1))^{z}}{z!} = \frac{(\lambda p_1)^{y}}{y!}e^{-\lambda}\cdot e^{\lambda-\lambda p_1} = \frac{e^{-\lambda p_1}(\lambda p_1)^{y}}{y!}$$

$= Pr(Y=y)$ $\therefore Y \sim Po(\lambda p_1)$

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(c) $\Pr(Z=z \mid X=x, Y=y) = {}_yC_z \cdot p_2^z (1-p_2)^{y-z}$ $\quad (\mathrm{Bn}(y, p_2))$

$E[e^{tZ} \mid X=x, Y=y] = (p_2 e^t + q_2)^y$

$E[e^{tZ}] = E[(p_2 e^t + q_2)^Y] = \sum_{y\ge 0} (p_2 e^t + q_2)^y \cdot \frac{e^{-\lambda p_1}(\lambda p_1)^y}{y!}$

$\downarrow$ $\quad (p_2 + q_2 = 1)$

Z的 mgf.

$= \sum_{y\ge 0} \frac{1}{y!} e^{-\lambda p_1} (\lambda p_1 p_2 e^t + \lambda p_1 q_2)^y$

$= e^{-\lambda p_1} \exp(\lambda p_1 p_2 e^t + \lambda p_1 q_2)$

$= \exp(\lambda p_1 p_2 e^t + \lambda p_1 (q_2 - 1))$

$= \exp(\lambda p_1 p_2 e^t - \lambda p_1 p_2)$

$= \exp(\lambda p_1 p_2 (e^t - 1))$

由此可知 $Z \sim \mathrm{Po}(\lambda p_1 p_2)$

(d) $W \overset{def}{=} X - Y$

$\Pr(X=x, Y=y) = \frac{(\lambda p_1)^y}{y!} \cdot e^{-\lambda} \cdot \frac{(\lambda(1-p_1))^{x-y}}{(x-y)!} \cdot I(x \ge y)\, I(y \ge 0)$

$\therefore \quad = \frac{(\lambda p_1)^y}{y!} \cdot e^{-\lambda} \frac{(\lambda(1-p_1))^w}{w!} I(w \ge 0)\, I(y \ge 0)$

$= \frac{e^{-\lambda p_1}(\lambda p_1)^y}{y!} \cdot e^{-\lambda(1-p_1)} \cdot \frac{(\lambda(1-p_1))^w}{w!} \cdot I(y \ge 0) \cdot I(w \ge 0)$

$\therefore \Pr(Y=y, W=w) = \frac{e^{-\lambda p_1}(\lambda p_1)^y}{y!} I(y \ge 0) \cdot \frac{e^{-\lambda(1-p_1)}(\lambda(1-p_1))^w}{w!} \cdot I(w \ge 0)$

(由此可知，Y, W 獨立)

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(e) $V \overset{def}{=} Y - Z \quad W \overset{def}{=} X - Y$

$$\Pr(X=x, Y=y, Z=z) = {}_yC_z \cdot p_2^{z} (1-p_2)^{y-z} \cdot {}_xC_y \cdot p_1^{y} (1-p_1)^{x-y} e^{-\lambda} \frac{\lambda^x}{x!} \cdot \mathbb{I}(0 \le z \le y \le x)$$

用 Z, V, W 來表達: $\begin{cases} y = v+z \\ x = v+z+w \end{cases}$

$$\Pr(X=x, Y=y, Z=z) = {}_{v+z}C_z \cdot p_2^{z} (1-p_2)^{v} \cdot {}_{v+z+w}C_{v+z}\, p_1^{v+z} (1-p_1)^{w} e^{-\lambda} \cdot \frac{\lambda^{v+z+w}}{(v+z+w)!} \cdot \mathbb{I}(z\ge 0)\mathbb{I}(v\ge 0)\mathbb{I}(w\ge 0)$$

$$= \frac{\cancel{(v+z)!}}{z!\,v!} \cdot \frac{\cancel{(v+z+w)!}}{\cancel{(v+z)!}\,w!} \cdot (p_1p_2)^{z} (p_1(1-p_2))^{v} (1-p_1)^{w} \cdot e^{-\lambda} \frac{\lambda^{v+z+w}}{\cancel{(v+z+w)!}} \mathbb{I}(z\ge 0)\mathbb{I}(v\ge 0)\mathbb{I}(w\ge 0)$$

$$= \frac{e^{-\lambda p_1 p_2} (\lambda p_1 p_2)^{z}}{z!} \cdot \frac{e^{-\lambda p_1(1-p_2)} (\lambda p_1 (1-p_2))^{v}}{v!} \cdot \frac{e^{-\lambda(1-p_1)} (\lambda(1-p_1))^{w}}{w!}$$

$$\mathbb{I}(z\ge 0)\mathbb{I}(v\ge 0)\mathbb{I}(w\ge 0)$$

$$\therefore \Pr(Z=z, \overset{Y-Z}{V}=v, \overset{X-Y}{W}=w) = \frac{e^{-\lambda p_1 p_2} (\lambda p_1 p_2)^{z}}{z!} \cdot \frac{e^{-\lambda p_1(1-p_2)} (\lambda p_1(1-p_2))^{v}}{v!} \cdot \frac{e^{-\lambda(1-p_1)} (\lambda(1-p_1))^{w}}{w!}$$

(when $z\ge 0$, $v \ge 0$, $w \ge 0$) (0 : elsewhere)

由此可知 Z, V, W 獨立 $\sim Po(\lambda p_1 p_2)$, $Po(\lambda p_1(1-p_2))$, $Po(\lambda(1-p_1))$

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(f) $X = Z + V + W$

我們已知 Z, V, W 為獨立且分別服從 $Po(\lambda p_1 p_2)$、$Po(\lambda p_1(1-p_2))$ $Po(\lambda(1-p_1))$

$Z + V + W \sim Po(\lambda)$.

$$\therefore \Pr(Z+V+W=n) = e^{-\lambda} \cdot \frac{\lambda^n}{n!}$$

$$\Pr(Z=z,\ V=v,\ W=w \mid Z+V+W=n)$$

$$= \frac{\Pr(Z=z,\ V=v,\ W=w,\ Z+V+W=n)}{\Pr(Z+V+W=n)}$$

$$= \frac{\frac{\cancel{e^{-\lambda p_1 p_2}}(\lambda p_1 p_2)^z}{z!} \cdot \frac{\cancel{e^{-\lambda p_1(1-p_2)}}(\lambda p_1(1-p_2))^v}{v!} \cdot \frac{\cancel{e^{-\lambda(1-p_1)}}(\lambda(1-p_1))^w}{w!}}{\left(\cancel{e^{-\lambda}} \cdot \frac{\lambda^n}{n!}\right)} \quad (z+v+w=n)$$

$$= \frac{n!}{z!\,v!\,w!} \cdot \frac{(\lambda p_1 p_2)^z (\lambda p_1(1-p_2))^v (\lambda(1-p_1))^w}{\lambda^n} \quad (z+v+w=n)$$

$$= \frac{n!}{z!\,v!\,w!} (p_1 p_2)^z (p_1(1-p_2))^v (1-p_1)^w \quad (z+v+w=n)$$

由此可知, $Z, V, W \mid Z+V+W=n \sim \text{Multinomial}(n;\ p_1 p_2,\ p_1(1-p_2),\ (1-p_1))$

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$\boxed{2}$ $U_j \overset{def}{=} X_j - (\theta - \frac{1}{2}) \sim \text{Uniform}(0,1) \quad E[U_j] = \frac{1}{2} \quad V[U_j] =$

(a) $E[\bar{X}] = E\left[\frac{X_1 + X_2 + \cdots X_5}{5}\right] = E\left[\frac{U_1 + U_2 + \cdots U_5}{5} + (\theta - \frac{1}{2})\right]$

$= \frac{1}{2} \times 5 \div 5 + \theta - \frac{1}{2} = \theta$

• $V = X_{(3)} = U_{(3)} + \theta - \frac{1}{2} \qquad E[V] = E[U_{(3)}] + \theta - \frac{1}{2}$

故證明 $E[U_{(3)}] = \frac{1}{2}$ 即可.

$U_{(3)} \sim \text{Beta}(3,3)$ （後面證明）

$E[U_{(3)}] = \int_0^1 \frac{u \cdot u^2(1-u)^2}{Be(3,3)} du = \frac{Be(4,3)}{Be(3,3)} = \frac{\Gamma(4)\Gamma(3)}{\Gamma(7)} \frac{\Gamma(6)}{\Gamma(3)\Gamma(3)} = \frac{3! \cdot 5!}{6! \cdot 2!}$

$= \frac{3}{6} = \frac{1}{2} \qquad \therefore \; E[V] = \frac{1}{2} + \theta - \frac{1}{2} = \theta$

* 在此證明 $U_{(3)} \sim Be(3,3)$

一般而言, $U_1, \dots, U_n \overset{iid}{\sim} \text{Uniform}(0,1) \Rightarrow U_{(k)} \sim \text{Beta}(k, n-k+1)$

全機率 $= 1 = \int_{u_{(1)} \le \cdots \le u_{(n)}} n! \, du_{(1)} du_{(2)} \cdots du_{(n)}$

step1 : $du_{(1)}$: $\int_{0 \le u_{(1)} \le u_{(2)}} n! \, du_{(1)} = n! \left[u_{(1)}\right]_0^{u_{(2)}} = n! \, u_{(2)}$

step2 $du_{(2)}$: $\int_{0 \le u_{(2)} \le u_{(3)}} n! \, u_{(2)} du_{(2)} = n! \cdot \frac{u_{(3)}^2}{2!}$

⋮

step k-1 $du_{(k-1)}$ $\int_{0 \le u_{(k-1)} \le u_{(k)}} \sim du_{(k-1)} = n! \cdot \frac{u_{(k)}^{k-1}}{(k-1)!}$

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同樣處理, $du_{(k+1)} \sim du_{(n)}$: 得 $n! \cdot \dfrac{u_{(k)}^{k-1}(1-u_{(k)})^{n-k}}{(k-1)!\,(n-k)!}$ $(0 \leq u_{(k)} \leq 1)$

$$= \frac{u_{(k)}^{k-1}(1-u_{(k)})^{(n-k+1)-1}}{Be(k, n-k+1)}$$

由此可知 $U_{(k)} \sim Be(k, n-k+1)$ ($n=5$ $k=3$ 代入即可)

- $W = \frac{1}{2}\left(X_{(1)}+X_{(5)}\right) = \frac{1}{2}\left(U_{(1)}+U_{(5)}\right) + \left(\theta - \frac{1}{2}\right)$

同樣證明 $E[W] = \theta \Leftrightarrow E\left[\frac{1}{2}\left(U_{(1)}+U_{(5)}\right)\right] = \frac{1}{2}$

(雖然可以利用 $U_{(1)} \sim Be(1,5)$, $U_{(5)} \sim Be(5,1)$ 這個事實,
但為了方便起見, 我們還是求 $\frac{1}{2}\left(U_{(1)}+U_{(5)}\right)$ 的分布)

我們考慮一般化的情況, 設 $U_1, U_2, \ldots U_n \overset{iid}{\sim} \text{Uniform}(0,1)$.
$R \overset{def}{=} U_{(n)} - U_{(1)}$, $S \overset{def}{=} \frac{1}{2}\left(U_{(n)}+U_{(1)}\right)$
先求 R, S 的聯合分布, 再求 S 的邊際分佈. (最後 $n \overset{代入}{=} 5$)

、$U_{(1)}, U_{(n)}$ 的聯合分布. $Pr\left(U_{(1)} \leq u_{(1)}, U_{(n)} \leq u_{(n)}\right)$ $(0 \leq u_{(1)} \leq u_{(n)} \leq 1)$
$= u_{(n)}^n - \left(u_{(n)} - u_{(1)}\right)^n$.

$\therefore \dfrac{\partial^2}{\partial u_{(1)} \partial u_{(n)}} Pr\left(U_{(1)} \leq u_{(1)}, U_{(n)} \leq u_{(n)}\right) = n(n-1)\left(u_{(n)} - u_{(1)}\right)^{n-2}$ $(0 \leq u_{(1)} \leq u_{(n)} \leq 1)$

$\therefore f_{U_{(1)}U_{(n)}} = n(n-1)\left(u_{(n)}-u_{(1)}\right)^{n-2} \cdot \mathbb{I}_{(0 \leq u_{(1)} \leq u_{(n)} \leq 1)}$

全機率 $= 1 = \iint_{0 \leq u_{(1)} \leq u_{(n)} \leq 1} n(n-1)\left(u_{(n)}-u_{(1)}\right)^{n-2} du_{(n)}\, du_{(1)}$

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$$U_{(n)} = S + \frac{r}{2} \qquad J = \begin{pmatrix} \frac{\partial U_{(n)}}{\partial S} & \frac{\partial U_{(n)}}{\partial r} \\ \frac{\partial U_{(1)}}{\partial S} & \frac{\partial U_{(1)}}{\partial r} \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2} \\ 1 & -\frac{1}{2} \end{pmatrix}$$

$$U_{(1)} = S - \frac{r}{2}$$

$\therefore |J| = -1 \quad \therefore abs|J| = 1 \quad \therefore dU_{(1)}\,dU_{(n)} = dr\,ds$

$$1 = \iint_{0 \leq S - \frac{r}{2} \leq S + \frac{r}{2} \leq 1} n(n-1) \cdot r^{n-2} \cdot dr\,ds$$

$\downarrow$

$\therefore$ $0 \leq r \leq 2S$

$0 \leq r \leq 2(1-S)$

$\therefore \quad 0 \leq r \leq \min\{2S, 2(1-S)\}$

$$f_S(S) = \int_0^{\min\{2S, 2(1-S)\}} n(n-1) r^{n-2}\,dr$$

$$= \left[n r^{n-1} \right]_0^{\min\{2S, 2(1-S)\}}$$

$$= n \cdot \min\{2S, 2(1-S)\}^{n-1} \quad (0 \leq S \leq 1)$$

$$\therefore f_S(S) = \begin{cases} n(2S)^{n-1} & (0 \leq S \leq \frac{1}{2}) \\ n(2(1-S))^{n-1} & (\frac{1}{2} < S \leq 1) \end{cases}$$

$$\therefore E[S] = \int_0^{\frac{1}{2}} 2^{n-1} \cdot n S^n\,ds + \int_{\frac{1}{2}}^1 2^{n-1} n S (1-S)^{n-1}\,ds$$

$$= \int_0^{\frac{1}{2}} 2^{n-1} \cancel{n} S^n\,ds - \int_{\frac{1}{2}}^1 2^{n-1} \cdot \cancel{n (1-S)^n}\,ds + \int_{\frac{1}{2}}^1 2^{n-1} n (1-S)^{n-1}\,ds$$

$$= \int_0^{\frac{1}{2}} 2^{n-1} n S^{n-1}\,ds = \left[2^{n-1} \cdot S^n \right]_0^{\frac{1}{2}} = \frac{2^{n-1}}{2^n} = \frac{1}{2}$$

由此可知 $E[S] = E\left[\frac{1}{2}(U_{(1)}+U_{(n)})\right] = \frac{1}{2}$ (for all n) $\quad \therefore E[W] = \theta$

(b) $Var[U] = Var\left[\frac{1}{5}(X_1+X_2+\cdots X_5)\right] = Var\left[\frac{1}{5}(U_1+\cdots U_5) + \theta - \frac{1}{2}\right]$

$= \frac{1}{5} V[U_1] = \frac{1}{60}$

$Var[V] = Var\left[U_{(3)} + \theta - \frac{1}{2}\right] = Var[U_{(3)}]$

$\because$ Beta(3,3) 的 變異數 $\quad E[U_{(3)}^2] = \int_0^1 \frac{u^4(1-u)^2}{Be(3,3)}\,du$

$= \frac{Be(5,3)}{Be(3,3)} = \frac{\Gamma(5)\cancel{\Gamma(3)}}{\Gamma(8)} \frac{\Gamma(6)}{\Gamma(3)\cancel{\Gamma(3)}} = \frac{5!\,4!}{7!\,2!} = \frac{4\cdot 3}{7\cdot 6} = \frac{2}{7}$

$\therefore V[U_{(3)}] = \frac{2}{7} - \left(\frac{1}{2}\right)^2 = \frac{2}{7} - \frac{1}{4} = \frac{8-7}{28} = \frac{1}{28}$

$Var[W] = Var\left[\frac{1}{2}(X_{(1)}+X_{(5)})\right] = Var\left[\frac{1}{2}(U_{(1)}+U_{(5)}) + \theta - \frac{1}{2}\right]$

$= Var\left[\frac{1}{2}(U_{(1)}+U_{(5)})\right] = Var[S] \quad \left(E[S] = \frac{1}{2}\right)$

$(n=5)$

$E[S^2] = \int_0^{\frac{1}{2}} 2^{n-1} n\, s^{n+1}\,ds + \int_{\frac{1}{2}}^1 2^{n-1} n\, s^2 (1-s)^{n-1}\,ds$

$s^2 = (1-s)^2 - 2(1-s) + 1$

一樣

$= \int_0^{\frac{1}{2}} 2^{n-1} n\, s^{n+1}\,ds + \int_{\frac{1}{2}}^1 2^{n-1} n (1-s)^{n+1} - 2\int_{\frac{1}{2}}^1 2^{n-1} n (1-s)^n\,ds + \int_{\frac{1}{2}}^1 2^{n-1} n (1-s)^{n-1}\,ds$

$= 2^n \cdot n \int_0^{\frac{1}{2}} s^{n+1}\,ds - 2^n \cdot n \int_0^{\frac{1}{2}} s^n\,ds + 2^{n-1} n \int_0^{\frac{1}{2}} s^{n-1}\,ds$

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$$2^n\cdot n\left[\frac{1}{n+2}\cdot s^{n+2}\right]_0^{\frac{1}{2}} - 2^n\cdot n\left[\frac{1}{n+1}s^{n+1}\right]_0^{\frac{1}{2}} + 2^{n-1}\cdot n\left[\frac{1}{n}\cdot s^{n}\right]_0^{\frac{1}{2}}$$

$$= \frac{n}{4}\cdot\frac{1}{n+2} - \frac{n}{2}\cdot\frac{1}{n+1} + \frac{1}{2}$$

$$\therefore V[S] = \frac{n}{4}\cdot\frac{1}{n+2} - \frac{n}{2}\cdot\frac{1}{n+1} + \frac{1}{2} - \left(\frac{1}{2}\right)^2$$

$$= \frac{n}{4}\cdot\frac{1}{n+2} - \frac{n}{2}\cdot\frac{1}{n+1} + \frac{1}{4}$$

代入

$\therefore n=5$

$$V[S] = \frac{5}{4}\cdot\frac{1}{7} - \frac{5}{2}\cdot\frac{1}{6} + \frac{1}{4}$$

$$= \frac{5}{28} - \frac{5}{12} + \frac{3}{12} = \frac{5}{28} - \frac{1}{6} = \frac{15-14}{84} = \frac{1}{84}$$

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註 $\mathbb{1}(\cdot)$ 為 indicator function

$\mathbb{1}(\text{true})=1,\ \mathbb{1}(\text{false})=0$

3

(a) 由於 $(X_{(1)},\cdots,X_{(n)})$ 與 S 為獨立，故 $(X_{(1)}\cdots X_{(n)},S)$ 的聯合 pdf 為 $f_{X_{(1)}\cdots X_{(n)},S}(x_{(1)}\cdots x_{(n)},s)=f_{X_{(1)}\cdots X_{(n)}}\cdot f_S$

$\therefore\ n!\cdot\mathbb{1}_{(0\le x_{(1)}\le\cdots\le x_{(n)}\le 1)}\cdot\dfrac{s^{n-1}}{\Gamma(n+1)}\exp(-s)\cdot\mathbb{1}_{(s\ge 0)}$

(b) 利用積分的變數轉換：

$$\begin{cases}Y_1=SX_{(1)}\\ Y_2=S(X_{(2)}-X_{(1)})\\ Y_3=S(1-X_{(2)})\end{cases}$$

$\Downarrow$

$$\begin{cases}SX_{(1)}=Y_1\\ SX_{(2)}=Y_1+Y_2\\ S=Y_1+Y_2+Y_3\end{cases}\Rightarrow\begin{cases}X_{(1)}=\dfrac{Y_1}{Y_1+Y_2+Y_3}\\ X_{(2)}=\dfrac{Y_1+Y_2}{Y_1+Y_2+Y_3}\\ S=Y_1+Y_2+Y_3\end{cases}$$

$0\le X_{(1)}\le X_{(2)}\le 1\ \&\ S\ge 0$

「$0\le\dfrac{Y_1}{Y_1+Y_2+Y_3}\le\dfrac{Y_1+Y_2}{Y_1+Y_2+Y_3}\le 1,\ Y_1+Y_2+Y_3\ge 0$」

$\Downarrow$

$\therefore$「$0\le Y_1\le Y_1+Y_2\le Y_1+Y_2+Y_3,\ Y_1+Y_2+Y_3\ge 0$」

得

$\therefore\ Y_1\ge 0\ \ Y_2\ge 0,\ Y_3\ge 0$

$Y_1 \geq 0,\ Y_2 \geq 0,\ Y_3 \geq 0$

(n=2)

全機率 $= 1 = \int_{\substack{0 \leq X_{(1)} \leq X_{(2)} \leq 1 \\ S \geq 0}} \cancel{2!} \frac{S^2}{\cancel{\Gamma(3)}} \exp(-S)\, dX_{(1)} dX_{(2)} dS \cdots ⊛$

$$J = \begin{pmatrix} \frac{\partial y_1}{\partial S} & \frac{\partial y_1}{\partial X_{(1)}} & \frac{\partial y_1}{\partial X_{(2)}} \\ \frac{\partial y_2}{\partial S} & \frac{\partial y_2}{\partial X_{(1)}} & \frac{\partial y_2}{\partial X_{(2)}} \\ \frac{\partial y_3}{\partial S} & \frac{\partial y_3}{\partial X_{(1)}} & \frac{\partial y_3}{\partial X_{(2)}} \end{pmatrix} = \begin{pmatrix} X_{(1)} & S & 0 \\ X_{(2)} - X_{(1)} & -S & S \\ 1 - X_{(2)} & 0 & -S \end{pmatrix}$$

$$|J| = \det \begin{pmatrix} X_{(1)} & S & 0 \\ X_{(2)} & 0 & S \\ 1 - X_{(2)} & 0 & -S \end{pmatrix} = \det \begin{pmatrix} X_{(1)} & S & 0 \\ X_{(2)} & 0 & S \\ 1 & 0 & 0 \end{pmatrix} = -\det \begin{pmatrix} 1 & 0 & 0 \\ X_{(2)} & 0 & S \\ X_{(1)} & S & 0 \end{pmatrix}$$

$$= -\det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & S \\ 0 & S & 0 \end{pmatrix} = S^2 \qquad \therefore\ dy_1 dy_2 dy_3 = S^2 dS dX_{(1)} dX_{(2)}$$

$$\therefore\ ⊛ = \int_{Y_1 \geq 0,\ Y_2 \geq 0,\ Y_3 \geq 0} \exp(-Y_1 - Y_2 - Y_3)\, dy_1 dy_2 dy_3$$

$$\therefore\ f_{Y_1 Y_2 Y_3} = \exp(-y_1) \exp(-y_2) \exp(-y_3) \quad (y_1 \geq 0,\ y_2 \geq 0,\ y_3 \geq 0)$$

由此可知 $Y_1, Y_2, Y_3 \overset{iid}{\sim} \exp(1)$

(c)&(d) $Y_1, \dots, Y_{n+1} \overset{iid}{\sim} \exp(1)$ ⋯ 以下證明這件事

$$\begin{cases} Y_1 = S X_{(1)} \\ Y_2 = S(X_{(2)} - X_{(1)}) \\ Y_3 = S(X_{(3)} - X_{(2)}) \\ \vdots \\ Y_n = S(X_{(n)} - X_{(n+1)}) \\ Y_{n+1} = S(1 - X_{(n)}) \end{cases} \Rightarrow \begin{cases} S X_{(1)} = Y_1 \\ S X_{(2)} = Y_1 + Y_2 \\ S X_{(3)} = Y_1 + Y_2 + Y_3 \\ \vdots \\ S X_{(n)} = Y_1 + Y_2 + \cdots Y_n \\ S = Y_1 + Y_2 + \cdots Y_n + Y_{n+1} \end{cases}$$

$$\Downarrow$$

$\therefore$

$$\begin{cases} X_{(1)} = \dfrac{Y_1}{Y_1 + Y_2 + \cdots Y_{n+1}} \\ X_{(2)} = \dfrac{Y_1 + Y_2}{Y_1 + Y_2 + \cdots Y_{n+1}} \\ \vdots \\ X_{(n)} = \dfrac{Y_1 + Y_2 + \cdots + Y_n}{Y_1 + Y_2 + \cdots + Y_{n+1}} \\ S = Y_1 + Y_2 + \cdots + Y_{n+1} \end{cases}$$

$$0 \le X_{(1)} \le X_{(2)} \le \cdots \le X_{(n)} \le 1 \quad \& \quad S \ge 0$$

$$\Downarrow$$

$$\begin{cases} 0 \le \dfrac{Y_1}{Y_1 + \cdots + Y_{n+1}} \le \dfrac{Y_1 + Y_2}{Y_1 + \cdots + Y_{n+1}} \le \cdots \le \dfrac{Y_1 + \cdots + Y_n}{Y_1 + \cdots + Y_{n+1}} \le 1 \\ Y_1 + \cdots + Y_{n+1} \ge 0 \end{cases}$$

$$\Downarrow$$

$$0 \le Y_1 \le Y_1 + Y_2 \le Y_1 + Y_2 + Y_3 \le \cdots \le Y_1 + Y_2 + \cdots + Y_n \le Y_1 + Y_2 + \cdots + Y_{n+1}$$

由此可知, $Y_1 \ge 0, Y_2 \ge 0, \dots, Y_{n+1} \ge 0$

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接著計算 Jacobi 矩陣

$$J=\begin{pmatrix} \frac{\partial Y_1}{\partial X_{(1)}} & \cdots & \frac{\partial Y_1}{\partial X_{(n)}} & \frac{\partial Y_1}{\partial S} \\ \vdots & & \vdots & \vdots \\ \frac{\partial Y_{n+1}}{\partial X_{(1)}} & & \frac{\partial Y_{n+1}}{\partial X_{(n)}} & \frac{\partial Y_{n+1}}{\partial S} \end{pmatrix} = \begin{pmatrix} S & 0 & & 0 & X_{(1)} \\ -S & S & & \vdots & X_{(2)}-X_{(1)} \\ 0 & -S & \ddots & 0 & \vdots \\ \vdots & \vdots & & S & \\ 0 & 0 & & -S & 1-X_{(n)} \end{pmatrix}$$

(第1行加到第2行)

$$|J| = \det\begin{pmatrix} S & 0 & \cdots & 0 & X_{(1)} \\ 0 & S & \cdots & 0 & X_{(2)} \\ 0 & -S & \cdots & 0 & X_{(3)}-X_{(2)} \\ \vdots & \vdots & & S & X_{(n)}-X_{(n-1)} \\ 0 & 0 & & -S & 1-X_{(n)} \end{pmatrix}$$

$$= \det\begin{pmatrix} S & 0 & & 0 & X_{(1)} \\ 0 & S & & \vdots & X_{(2)} \\ \vdots & 0 & \ddots & & \vdots \\ & \vdots & & S & X_{(n)} \\ 0 & 0 & & 0 & 1 \end{pmatrix} = S^n \det\begin{pmatrix} 1 & 0 & & 0 & X_{(1)} \\ 0 & 1 & & \vdots & X_{(2)} \\ \vdots & 0 & \ddots & 1 & X_{(n)} \\ 0 & 0 & & 0 & 1 \end{pmatrix}$$

n個

$$= S^n \cdot \det\begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}_{(n+1)\times(n+1)} = S^n \det I_{n+1} = S^n$$

由此可知，$dy_1 dy_2 \cdots dy_{n+1} = S^n \cdot dx_{(1)} dx_{(2)} \cdots dx_{(n)} ds$

全機率 $= 1 = \int_{\substack{0\le X_{(1)}\le X_{(2)}\le\cdots\le X_{(n)}\le 1 \\ S\ge 0}} \cancel{n!}\, \frac{S^n}{\cancel{\Gamma(n+1)}} \exp(-S)\, dx_{(1)} dx_{(2)} \cdots dx_{(n)} ds$

$$= \int_{y_1\ge 0, y_2\ge 0, \ldots, y_{n+1}\ge 0} \exp(-y_1-y_2-\cdots-y_{n+1})\, dy_1 dy_2 \cdots dy_n$$

$\therefore f_{Y_1 Y_2 \cdots Y_{n+1}}(y_1, \ldots, y_{n+1}) = \exp(-y_1-y_2-\cdots-y_{n+1}) \cdot \mathbb{I}(y_1\ge 0) \cdots \mathbb{I}(y_{n+1}\ge 0)$

故 $Y_1, Y_2, \ldots, Y_{n+1} \overset{iid}{\sim} \exp(1)$

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$\boxed{4}$ (a) $\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \phi & 1 & 0 \\ \phi^2 & \phi & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \sigma^2 I_{3\times 3} \right)$$

$$\therefore \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} \sim N\left(A\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \underbrace{\sigma^2 A A^t}_{= \Sigma} \right) \quad \text{where } A = \begin{pmatrix} 1 & 0 & 0 \\ \phi & 1 & 0 \\ \phi^2 & \phi & 1 \end{pmatrix}$$

$$\sigma^2 A A^t = \sigma^2 \begin{pmatrix} 1 & 0 & 0 \\ \phi & 1 & 0 \\ \phi^2 & \phi & 1 \end{pmatrix} \begin{pmatrix} 1 & \phi & \phi^2 \\ 0 & 1 & \phi \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \sigma^2 \begin{pmatrix} 1 & \phi & \phi^2 \\ \phi & \phi^2+1 & \phi^3+\phi \\ \phi^2 & \phi^3+\phi & \phi^4+\phi^2+1 \end{pmatrix}$$

$$\therefore \Sigma = \sigma^2 \begin{pmatrix} 1 & \phi & \phi^2 \\ \phi & \phi^2+1 & \phi^3+\phi \\ \phi^2 & \phi^3+\phi & \phi^4+\phi^2+1 \end{pmatrix} \quad \cdots \text{(a1)}$$

$$f_{Y_1Y_2Y_3} = \left(\frac{1}{2\pi}\right)^{\frac{3}{2}} \cdot \frac{1}{|\Sigma|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}(y_1\ y_2\ y_3)\, \Sigma^{-1} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \right)$$

$$\Sigma^{-1} = \frac{1}{\sigma^2} \begin{pmatrix} 1+\phi^2 & -\phi & 0 \\ -\phi & 1+\phi^2 & -\phi \\ 0 & -\phi & 1 \end{pmatrix} \qquad |\Sigma| = (\sigma^2)^3$$

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$$\therefore f_{Y_1Y_2Y_3}(y_1,y_2,y_3)=\left(\frac{1}{2\pi\sigma^2}\right)^{\frac{3}{2}}\exp\left(-\frac{1}{2\sigma^2}(y_1\ y_2\ y_3)\begin{pmatrix}1+\phi^2 & -\phi & 0\\ -\phi & 1+\phi^2 & -\phi\\ 0 & -\phi & 1\end{pmatrix}\begin{pmatrix}y_1\\ y_2\\ y_3\end{pmatrix}\right)$$

$\cdots$ (a2)

· 接著求條件分佈的公式

設 $\begin{pmatrix}X_1\\ X_2\end{pmatrix}\sim N\left(\begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix},\begin{pmatrix}\Sigma_{11} & \Sigma_{12}\\ \Sigma_{21} & \Sigma_{22}\end{pmatrix}\right)$

$$\begin{pmatrix}I & 0\\ -\Sigma_{21}\Sigma_{11}^{-1} & I\end{pmatrix}\begin{pmatrix}X_1\\ X_2\end{pmatrix}\sim N\left(\begin{pmatrix}\mu_1\\ \mu_2-\Sigma_{21}\Sigma_{11}^{-1}\mu_1\end{pmatrix},\begin{pmatrix}\Sigma_{11} & 0\\ 0 & \Sigma_{22}-\Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}\end{pmatrix}\right)$$

$(\because)$

平均 $\begin{pmatrix}I & 0\\ -\Sigma_{21}\Sigma_{11}^{-1} & I\end{pmatrix}\begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix}=\begin{pmatrix}\mu_1\\ \mu_2-\Sigma_{21}\Sigma_{11}^{-1}\mu_1\end{pmatrix}$

變異 $\begin{pmatrix}I & 0\\ -\Sigma_{21}\Sigma_{11}^{-1} & I\end{pmatrix}\begin{pmatrix}\Sigma_{11} & \Sigma_{12}\\ \Sigma_{21} & \Sigma_{22}\end{pmatrix}\begin{pmatrix}I & -\Sigma_{11}^{-1}\Sigma_{12}\\ 0 & I\end{pmatrix}$

$=\begin{pmatrix}\Sigma_{11} & \Sigma_{12}\\ 0 & \Sigma_{22}-\Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}\end{pmatrix}\begin{pmatrix}I & -\Sigma_{11}^{-1}\Sigma_{12}\\ 0 & I\end{pmatrix}$

$=\begin{pmatrix}\Sigma_{11} & 0\\ 0 & \Sigma_{22}-\Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}\end{pmatrix}$

由此可知, X_1 與 $X_2-\Sigma_{21}\Sigma_{11}^{-1}X_1$ 為獨立

$$\therefore X_2-\Sigma_{21}\Sigma_{11}^{-1}X_1\mid X_1\sim N(\mu_2-\Sigma_{21}\Sigma_{11}^{-1}\mu_1,\ \Sigma_{22}-\Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12})$$

$$\therefore X_2\mid X_1\sim N(\mu_2+\Sigma_{21}\Sigma_{11}^{-1}(X_1-\mu_1),\ \Sigma_{22}-\Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12})$$

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求Y_2與Y_3的聯合分佈

$$\begin{pmatrix} Y_2 \\ Y_3 \end{pmatrix} = \underbrace{\begin{pmatrix} \phi & 1 & 0 \\ \phi^2 & \phi & 1 \end{pmatrix}}_{B} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$$

$$\begin{pmatrix} Y_2 \\ Y_3 \end{pmatrix} \sim N\left(B\begin{pmatrix} 0 \\ 0 \end{pmatrix}, B(\sigma^2 I)B^t \right)$$

$$= N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \sigma^2 BB^t \right)$$

$$= N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \sigma^2 \begin{pmatrix} \phi & 1 & 0 \\ \phi^2 & \phi & 1 \end{pmatrix} \begin{pmatrix} \phi & \phi^2 \\ 1 & \phi \\ 0 & 1 \end{pmatrix} \right)$$

$$= N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \sigma^2 \begin{pmatrix} \phi^2+1 & \phi^3+\phi \\ \phi^3+\phi & \phi^4+\phi^2+1 \end{pmatrix} \right)$$

$\Sigma_{11} = \phi^2+1$ $\Sigma_{22} = \phi^4+\phi^2+1$

$\Sigma_{12} = \phi^3+\phi$

$\Sigma_{21} = \phi^3+\phi$

利用剛才求的公式

$$Y_3 \mid Y_2 = y_2 \sim N\left(0 + \Sigma_{21}\Sigma_{11}^{-1}(y_2 - 0), \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12} \right)$$

$$= N\left(\frac{\phi^3+\phi}{\phi^2+1} y_2, (\phi^4+\phi^2+1) - \frac{(\phi^3+\phi)^2}{\phi^2+1} \right)$$

$$= N\left(\phi y_2, \phi^4+\phi^2+1 - \phi^2(\phi^2+1) \right)$$

$$= N(\phi y_2, 1)$$

$\therefore Y_3 \mid Y_2 = y_2 \sim N(\phi y_2, 1)$ ... (a3)

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同樣道理 $\begin{pmatrix} Y_3 \\ Y_2 \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \phi^4+\phi^2+1 & \phi^3+\phi \\ \phi^3+\phi & \phi^2+1 \end{pmatrix}\right)$

$\Sigma_{11} = \phi^4+\phi^2+1$

$\Sigma_{12} = \Sigma_{21} = \phi^3+\phi$

$\Sigma_{22} = \phi^2+1$

$$Y_2 \mid Y_3 = y_3 \sim N\left(0+\Sigma_{21}\Sigma_{11}^{-1}(y_3-0),\ \Sigma_{22}-\Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}\right)$$

$$= N\left(\frac{\phi^3+\phi}{\phi^4+\phi^2+1}y_3,\ \phi^2+1-\frac{\phi^2(\phi^2+1)^2}{\phi^4+\phi^2+1}\right)$$

$$\downarrow$$

$$\frac{(\phi^6+\phi^4+\phi^2+\phi^4+\phi^2+1)-\phi^2(\phi^4+2\phi^2+1)}{\phi^4+\phi^2+1}$$

$$= \frac{(\phi^6+2\phi^4+2\phi^2+1)-(\phi^6+2\phi^4+\phi^2)}{\phi^4+\phi^2+1}$$

$$= \frac{\phi^2+1}{\phi^4+\phi^2+1}$$

$$\therefore Y_2 \mid Y_3 = y_3 \sim N\left(\frac{(\phi^2+1)(\phi y_3)}{\phi^4+\phi^2+1},\ \frac{(\phi^2+1)}{\phi^4+\phi^2+1}\right) \quad \cdots (\#4)$$

最後考慮 a, b 使得 $aY_1+Y_2+bY_3$ 與 (Y_1, Y_3) 獨立

$$\underbrace{\begin{pmatrix} a & 1 & b \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{=C}\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} = \begin{pmatrix} aY_1+Y_2+bY_3 \\ Y_1 \\ Y_3 \end{pmatrix}$$

其變異矩陣為 $C\Sigma C^t = C\begin{pmatrix} 1 & \phi & \phi^2 \\ \phi & \phi^2+1 & \phi^3+\phi \\ \phi^2 & \phi^3+\phi & \phi^4+\phi^2+1 \end{pmatrix}C^t$

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$$= \begin{pmatrix} a & 1 & b \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \phi & \phi^2 \\ \phi & \phi^2+1 & \phi^3+\phi \\ \phi^2 & \phi^3+\phi^2 & \phi^4+\phi^2+1 \end{pmatrix} \begin{pmatrix} a & 1 & 0 \\ 1 & 0 & 0 \\ b & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} a+\phi+b\phi^2 & 1+a\phi+(b+1)\phi^2+b\phi^3 & b+\phi+(a+b)\phi^2+\phi^3+b\phi^4 \\ 1 & \phi & \phi^2 \\ \phi^2 & \phi^3+\phi^2 & \phi^4+\phi^2+1 \end{pmatrix} \begin{pmatrix} a & 1 & 0 \\ 1 & 0 & 0 \\ b & 0 & 1 \end{pmatrix}$$

$cov[aY_1+Y_2+bY_3, Y_1] = a+\phi+b\phi^2 = 0$ ⋯ ①

$cov[aY_1+Y_2+bY_3, Y_3] = b+\phi+(a+b)\phi^2+\phi^3+b\phi^4 = 0$ ⋯ ②

$\therefore$ ① $\Rightarrow a = -\phi - b\phi^2$

② $= b+\phi+(\cancel{b}-\phi-b\phi^2)\phi^2+\cancel{\phi^3}+\cancel{b\phi^4}$

$= b(\phi^2+1)+\phi = 0$

$\therefore b = \dfrac{-\phi}{\phi^2+1} \quad a = \dfrac{\phi^3}{\phi^2+1} - \phi = \dfrac{-\phi}{\phi^2+1}$

$\therefore a = b = \dfrac{-\phi}{\phi^2+1}$ ⋯ (a5)

(*)

(b) & (d) 寫在一起。

(b).(c)(d) ⇒

利用矩陣表達 pdf，反而不方便求 MLE，故按照提示來做。

$$f_{Y_1,Y_2,\dots,Y_n} = f_{Y_n|Y_{n-1}} \cdot f_{Y_{n-1}|Y_{n-2}} \cdots f_{Y_2|Y_1} \cdot f_{Y_1}$$

$$Y_j | Y_{j-1} = y_{j-1} \sim N(\phi y_{j-1}, \sigma^2) \quad (j = 1 \sim n) \quad (但\ y_0 = 0)$$

$$\therefore f_{Y_j|Y_{j-1}} = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-1}{2\sigma^2}(y_j - \phi y_{j-1})^2\right)$$

$$\therefore f_{Y_1Y_2\cdots Y_n} = \prod_{j=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-1}{2\sigma^2}(y_j - \phi y_{j-1})^2\right) \quad (\text{where } y_0 = 0)$$

$\cdots$ (c)

$$\therefore 概似函數\ L(\phi,\sigma^2) = \prod_{j=1}^{n} \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right) \exp\left(\frac{-1}{2\sigma^2}(y_j - \phi y_{j-1})^2\right)$$

$$\ell(\phi,\sigma^2) = \ln L(\phi,\sigma^2) = \frac{-n}{2}\ln(2\pi\sigma^2) - \sum_{j=1}^{n} \frac{1}{2\sigma^2}(y_j - \phi y_{j-1})^2$$

若 σ^2 為已知參數…

$$\frac{\partial \ell}{\partial \phi} = -\sum_{j=1}^{n} \frac{1}{\sigma^2}(y_j - \phi y_{j-1}) \cdot (-y_{j-1}) = 0$$

$$\Leftrightarrow \sum_{j=1}^{n} y_{j-1}(y_j - \phi y_{j-1}) = 0$$

$$\Leftrightarrow \sum_{j=1}^{n} y_{j-1} y_j = \phi \sum_{j=1}^{n} y_{j-1}^2$$

$$\therefore \phi = \frac{\sum_{j=1}^{n-1} y_j y_{j+1}}{\sum_{j=1}^{n-1} y_j^2} \qquad \therefore \hat{\phi}_{MLE} = \frac{\sum_{j=1}^{n-1} y_j y_{j+1}}{\sum_{j=1}^{n-1} y_j^2}$$

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接著求 $I(\phi)$ (Fisher 資訊量)

$$\frac{\partial l}{\partial \phi}=\frac{1}{\sigma^2}\sum_{j=1}^{n} y_{j-1}(y_j-\phi y_{j-1})$$

$$\frac{\partial^2 l}{\partial \phi^2}=-\frac{1}{\sigma^2}\sum_{j=1}^{n} y_{j-1}^2$$

$$\therefore E\left[-\frac{\partial^2 l}{\partial \phi^2}\right]=\frac{1}{\sigma^2}\sum_{j=1}^{n} E[Y_{j-1}^2]=\frac{1}{\sigma^2}\sum_{j=1}^{n-1} E[Y_j^2]$$

利用雙重期望值原理 (得關係遞迴式)

$$E[Y_j^2]=E\left[E[Y_j^2 \mid Y_{j-1}]\right]=E[\sigma^2+\phi^2 Y_{j-1}^2]$$

由此可知 $\underbrace{E[Y_j^2]}_{a_j}=\sigma^2+\phi^2\underbrace{E[Y_{j-1}^2]}_{a_{j-1}}$

$$a_1=E[Y_1^2]=\sigma^2$$

$$\left(a_j-\frac{\sigma^2}{1-\phi^2}\right)=\phi^2\left(a_{j-1}-\frac{\sigma^2}{1-\phi^2}\right)$$

$$\therefore \left(a_j-\frac{\sigma^2}{1-\phi^2}\right)=\phi^{2j-2}\cdot\frac{-\phi^2\sigma^2}{1-\phi^2}$$

$$\therefore a_j=\frac{\sigma^2}{1-\phi^2}-\frac{\sigma^2\phi^{2j}}{1-\phi^2}=\frac{\sigma^2}{1-\phi^2}(1-\phi^{2j})$$

$$I(\phi)=\frac{1}{\sigma^2}\sum_{j=1}^{n-1}\left(\frac{\sigma^2}{1-\phi^2}(1-\phi^{2j})\right)=\frac{1}{1-\phi^2}\sum_{j=1}^{n-1}(1-\phi^{2j})$$

總而言之，若σ^2為已知參數，則 $\hat{\phi}_{MLE} = \dfrac{\sum_{j=1}^{n-1} z_j z_{j+1}}{\sum_{j=1}^{n-1} z_j^2}$ ⋯ (d1)

(& (b1))

$$I(\phi) = \frac{1}{1-\phi^2} \sum_{j=1}^{n-1} \left(1-\phi^{2j}\right)$$

最後σ^2與ϕ皆為未知參數

$$\ell(\phi,\sigma^2) = -\frac{n}{2}\ln(2\pi\sigma^2) - \sum_{j=1}^{n} \frac{1}{2\sigma^2}(z_j - \phi z_{j-1})^2$$

$$\frac{\partial \ell}{\partial \phi} = 0 \Rightarrow \quad \hat{\phi}_{MLE} = \frac{\sum_{j=1}^{n-1} z_j z_{j+1}}{\sum_{j=1}^{n-1} z_j^2}$$

$$\frac{\partial \ell}{\partial \sigma^2} = 0 \Rightarrow \quad -\frac{n}{2\sigma^2} + \sum_{j=1}^{n} \frac{1}{2\sigma^4}(z_j - \phi z_{j-1})^2 = 0$$

$$\therefore \hat{\sigma}^2_{MLE} = \frac{1}{n}\sum_{j=1}^{n}\left(z_j - \hat{\phi}_{MLE} z_{j-1}\right)^2$$

$$\therefore \begin{cases} \hat{\phi}_{MLE} = \dfrac{\sum_{j=1}^{n-1} z_j z_{j+1}}{\sum_{j=1}^{n-1} z_j^2} \\ \hat{\sigma}^2_{MLE} = \dfrac{1}{n}\sum_{j=1}^{n}\left(z_j - \hat{\phi}_{MLE} z_{j-1}\right)^2 \end{cases} \quad \cdots \text{(d2)}$$

No. 資料科學之統計基礎 (I)　7.1, 7.2, 7.6, 7.4

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作業7.　森元俊成

☐1 Casella 7.1

$\hat{\theta}_{MLE}$ 為觀測到 x 的情況下, 使 $f(x|\theta)$ $(=L(\theta|x))$ 為最大的 θ

若 $x=0$ 時 $\theta=1$ 使得 $f(x|\theta)=L(x|\theta)$ 為最大

同樣, $x=1$ … $\theta=1$

$x=2$ … $\theta=2$ or 3

$x=3$ … $\theta=3$

$x=4$ … $\theta=3$

故 $\hat{\theta}_{MLE}=\begin{cases} 1 & (x=0,1) \\ 2 \text{ or } 3 & (x=2) \\ 3 & (x=3,4) \end{cases}$

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$\boxed{2}$ (Casella 7.17)

$$f(\vec{x}|\theta) = \prod_{j=1}^{n} \left(\frac{1}{2\sqrt{x_j}}\right)^{\theta} I_{(0,1)}(x_j)$$

θ只會取$\{0,1\}$.

考慮 $\dfrac{f(\vec{x}|1)}{f(\vec{x}|0)} = \prod_{j=1}^{n}\left(\frac{1}{2\sqrt{x_j}}\right) \geq 1$

$\Rightarrow \prod_{j=1}^{n}\left(2\sqrt{x_j}\right) \leq 1$

$\Rightarrow 2^n (x_1 x_2 \cdots x_n)^{\frac{1}{2}} \leq 1$

$\Rightarrow x_1 x_2 \cdots x_n \leq \frac{1}{2^{2n}}$

由此可知 $x_1 x_2 \cdots x_n \leq \frac{1}{2^{2n}} \Rightarrow \theta = 1$ 使得 $L(\theta|x)$ 最大

$$\therefore \hat{\theta}_{MLE} = \begin{cases} 1 & \left(x_1 x_2 \cdots x_n \leq \frac{1}{2^{2n}}\right) \\ 0 & \left(x_1 x_2 \cdots x_n > \frac{1}{2^{2n}}\right) \end{cases}$$

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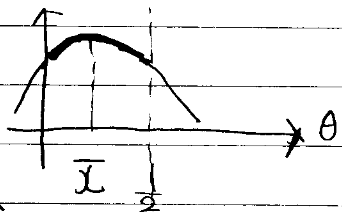
$\boxed{3}$ (casella 7.12) 我們注意 $\theta \in \Theta = [0, \frac{1}{2}]$

(a) $Pr(X_1 = x_1, X_2 = x_2, \dots, X_n \mid \theta) = \theta^{n\bar{x}}(1-\theta)^{n-n\bar{x}}$

$\ell(\theta \mid \vec{x}) = \ln Pr(X_1 = x_1, \dots, X_n = x_n \mid \theta) = n\bar{x}\ln\theta + (n - n\bar{x})\ln(1-\theta)$

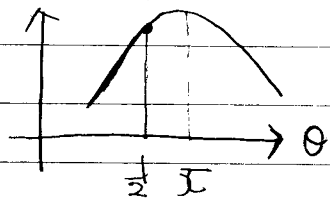
$$\frac{\partial \ell}{\partial \theta} = \frac{n\bar{x}}{\theta} - \frac{n(1-\bar{x})}{1-\theta} = \frac{n\bar{x}(1-\theta) - n(1-\bar{x})\theta}{\theta(1-\theta)} = \frac{n(\bar{x}-\theta)}{\theta(1-\theta)}$$

case I … $0 \le \bar{x} \le \frac{1}{2}$



$\theta = \bar{x}$ 使得 ℓ 最大

case II … $\frac{1}{2} < \bar{x} \le 1$



$\theta = \frac{1}{2}$ 使得 ℓ 最大.

故 $\hat{\theta}_{MLE} = \min\{\frac{1}{2}, \bar{x}\}$

接著求 $\hat{\theta}_{MME}$ (動差法估計量)

$E[X_j] = \theta \quad (j = 1 \sim n)$

故 $\hat{\theta}_{MME} = \bar{X}$

$\therefore$ (a) $\begin{cases} \hat{\theta}_{MME} = \bar{X} \\ \hat{\theta}_{MLE} = \min\{\frac{1}{2}, \bar{x}\} \end{cases}$

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(b) $\hat{\theta}_{MLE}$的MSE $= E[(\hat{\theta}_{MLE}-\theta)^2]$ $\quad Y=\sum_{i=1}^{n} X_i \sim Bin(n,\theta)$

$= E[(\min\{\frac{1}{2},\bar{X}\}-\theta)^2] = E[(\frac{1}{n}\min\{\frac{n}{2},\sum X_i\}-\theta)^2]$

$= \sum_{y=0}^{n}(\frac{1}{n}\min\{\frac{n}{2},y\}-\theta)^2 \cdot {}_nC_y \cdot \theta^y(1-\theta)^{n-y}$

$= \sum_{y=0}^{[\frac{n}{2}]}(\frac{1}{n}y-\theta)^2 \cdot {}_nC_y\, \theta^y(1-\theta)^{n-y} + \sum_{y=[\frac{n}{2}]+1}^{n}(\frac{1}{2}-\theta)^2 \cdot {}_nC_y\, \theta^y(1-\theta)^{n-y}$

$\hat{\theta}_{MME}$的MSE $= E[(\hat{\theta}_{MME}-\theta)^2] = V[\hat{\theta}_{MME}] = V[\bar{X}] = \frac{\theta(1-\theta)}{n}$

(c) 以MSE作為評估估計量的判斷標準

$MSE[\hat{\theta}_{MLE},\theta] - MSE[\hat{\theta}_{MME},\theta]$

$= \sum_{y=0}^{[\frac{n}{2}]}(\frac{y}{n}-\theta)^2 \cdot {}_nC_y \cdot \theta^y(1-\theta)^{n-y} + \sum_{y=[\frac{n}{2}]+1}^{n}(\frac{1}{2}-\theta)^2 \cdot {}_nC_y\, \theta^y(1-\theta)^{n-y}$

$-\sum_{y=0}^{[\frac{n}{2}]}(\frac{y}{n}-\theta)^2 \cdot {}_nC_y\, \theta^y(1-\theta)^{n-y} - \sum_{y=[\frac{n}{2}]+1}^{n}(\frac{y}{n}-\theta)^2 \cdot {}_nC_y\, \theta^y(1-\theta)^{n-y}$

$= \sum_{y=[\frac{n}{2}]+1}^{n} \underbrace{(\frac{1}{2}-\frac{y}{n})}_{負}\underbrace{(\frac{1}{2}+\frac{y}{n}-2\theta)}_{正} \cdot {}_nC_y\, \theta^y(1-\theta)^{n-y} < 0$ $\quad (\because 0 \le \theta \le \frac{1}{2})$

故 $MSE[\hat{\theta}_{MLE},\theta] < MSE[\hat{\theta}_{MME},\theta]$

$\therefore \hat{\theta}_{MLE}$ (最大概似估計量) 較佳!

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$\boxed{4}$ (Casella 7.14)

重新求 (Z,W) 的 聯合機率密度函數

$$\Pr(Z>z, W=0) = \Pr(X\geq Y>z) = \int_{x\geq y>z} \frac{1}{\lambda}\exp\left(\frac{x}{\lambda}\right)\frac{1}{\mu}\exp\left(\frac{y}{\mu}\right)dx\,dy$$

$$= \int_{y\geq z} \frac{1}{\mu}\exp\left(-\left(\frac{1}{\lambda}+\frac{1}{\mu}\right)y\right)dy = \left[\frac{-\frac{1}{\mu}}{\frac{1}{\lambda}+\frac{1}{\mu}}\exp\left(-\left(\frac{1}{\lambda}+\frac{1}{\mu}\right)y\right)\right]_{y=z}^{y\to\infty}$$

$$= \frac{\frac{1}{\mu}}{\frac{1}{\lambda}+\frac{1}{\mu}}\exp\left(-\left(\frac{1}{\lambda}+\frac{1}{\mu}\right)z\right)$$

$$\Pr(W=0) = \Pr(Z\geq 0, W=0) = \frac{\frac{1}{\mu}}{\frac{1}{\lambda}+\frac{1}{\mu}}$$

$$\therefore \Pr(Z\leq z, W=0) = \frac{\frac{1}{\mu}}{\frac{1}{\lambda}+\frac{1}{\mu}}\left(1-\exp\left(-\left(\frac{1}{\lambda}+\frac{1}{\mu}\right)z\right)\right)$$

$$\therefore \frac{d}{dz}\Pr(Z\leq z, W=0) = \frac{1}{\mu}\exp\left(-\left(\frac{1}{\lambda}+\frac{1}{\mu}\right)z\right) \quad (z\geq 0)$$

同理 $\frac{d}{dz}\Pr(Z\leq z, W=1) = \frac{1}{\lambda}\exp\left(-\left(\frac{1}{\lambda}+\frac{1}{\mu}\right)z\right)$

$$\therefore f_{ZW}(z,w) = \left(\frac{1}{\mu}\right)^{1-w}\cdot\left(\frac{1}{\lambda}\right)^{w}\exp\left(-\left(\frac{1}{\lambda}+\frac{1}{\mu}\right)z\right)$$

故 $\prod_{j=1}^{n} f_{ZW}(z_j,w_j) = \left(\frac{1}{\mu}\right)^{n-\sum w_j}\left(\frac{1}{\lambda}\right)^{\sum w_j}\exp\left(-\sum_{j=1}^{n}\left(\frac{1}{\lambda}+\frac{1}{\mu}\right)z_j\right)$

$$L(\mu,\lambda) = \prod_{j=1}^{n} f_{ZW}(z_j,w_j)$$

$$\ell(\mu,\lambda) = \ln(L(\mu,\lambda))$$

$$\ell(\mu,\lambda) = -\left(n-\sum w_j\right)\cdot\ln\mu - \sum w_j \ln\lambda - \left(\frac{1}{\lambda}+\frac{1}{\mu}\right)\sum_{j=1}^{n} z_j$$

$\frac{\partial \ell}{\partial \mu}=0$; $\frac{\partial \ell}{\partial \lambda}=0$

- $$-\frac{1}{\mu}\left(n-\sum_{j=1}^{n} w_j\right) + \frac{1}{\mu^2}\sum_{j=1}^{n} z_j = 0$$

$$\hat{\mu} = \frac{\sum_{j=1}^{n} z_j}{n-\sum_{j=1}^{n} w_j}$$

- $$-\frac{1}{\lambda}\sum_{j=1}^{n} w_j + \frac{1}{\lambda^2}\sum_{j=1}^{n} z_j$$

$$\hat{\lambda} = \frac{\sum_{j=1}^{n} z_j}{\sum_{j=1}^{n} w_j}$$

考慮 ℓ 的 Hessian：

$$H = \begin{pmatrix} \frac{1}{\mu^2}\left(n-\sum_{j=1}^{n} w_j\right) - \frac{2}{\mu^3}\sum_{j=1}^{n} z_j & 0 \\ 0 & \frac{1}{\lambda^2}\sum_{j=1}^{n} w_j - \frac{2}{\lambda^3}\sum_{j=1}^{n} z_j \end{pmatrix}$$

這是負定矩陣 $\because$ $$\frac{1}{\hat{\mu}^2}\underbrace{\left(n-\sum_{j=1}^{n} w_j\right)} - \frac{2}{\hat{\mu}^3}\sum_{j=1}^{n} z_j$$

$$= \frac{1}{\hat{\mu}^2}\cdot\frac{\sum_{j=1}^{n} z_j}{\hat{\mu}} - \frac{2}{\hat{\mu}^3}\sum_{j=1}^{n} z_j = -\frac{1}{\hat{\mu}^3}\sum_{j=1}^{n} z_j < 0$$

(λ 也同樣)

(H 的特徵值皆為負)

$$\therefore \hat{\mu}_{MLE} = \frac{\sum_{j=1}^{n} z_j}{n-\sum_{j=1}^{n} w_j} \qquad \hat{\lambda}_{MLE} = \frac{\sum_{j=1}^{n} z_j}{\sum_{j=1}^{n} w_j}$$

(不考慮 $\{w_j\}=\{0\}$ 或 $\{w_j\}=\{1\}$ 的狀況)

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$\boxed{5}$ (Casella 7.18)

(a) 求 $\sigma_X^2, \sigma_Y^2, \mu_X, \mu_Y, \rho$ 之動差法估計量

$$\begin{pmatrix} X_j \\ Y_j \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix}, \begin{pmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{pmatrix} \right)$$

$$E[X_j] = \mu_X \quad \therefore \frac{1}{n}(X_1+\dots+X_n) \xrightarrow{P} \mu_X \quad \therefore \hat{\mu}_X = \bar{X}$$

(同理 $\hat{\mu}_Y = \bar{Y}$)

$$E[X_j^2] = \mu_X^2 + \sigma_X^2 \quad \therefore \frac{1}{n}(X_1^2+\dots+X_n^2) \xrightarrow{P} \mu_X^2 + \sigma_X^2$$

$$\therefore \frac{1}{n}(X_1^2+\dots+X_n^2) - \left(\frac{1}{n}(X_1+\dots+X_n)\right)^2 \xrightarrow{P} \sigma_X^2$$

$$\therefore \frac{1}{n}\sum_{j=1}^{n}(X_j-\bar{X})^2 \xrightarrow{P} \sigma_X^2$$

(同理 $\frac{1}{n}\sum_{j=1}^{n}(Y_j-\bar{Y})^2 \xrightarrow{P} \sigma_Y^2$)

$$E[X_jY_j] = \rho\sigma_X\sigma_Y + \mu_X\mu_Y \quad \therefore \frac{1}{n}(X_1Y_1+\dots+X_nY_n) \xrightarrow{P} \rho\sigma_X\sigma_Y + \mu_X\mu_Y$$

$$\therefore \frac{1}{n}(X_1Y_1+\dots+X_nY_n) - \bar{X}\bar{Y} \xrightarrow{P} \rho\sigma_X\sigma_Y$$

$$\therefore \frac{\frac{1}{n}(X_1Y_1+\dots+X_nY_n) - \bar{X}\bar{Y}}{\sqrt{\frac{1}{n}\sum_{j=1}^{n}(X_j-\bar{X})^2}\sqrt{\frac{1}{n}\sum_{j=1}^{n}(Y_j-\bar{Y})^2}} \xrightarrow{P} \rho$$

故 $$\frac{\sum_{j=1}^{n}(X_j-\bar{X})(Y_j-\bar{Y})}{\sqrt{\sum_{j=1}^{n}(X_j-\bar{X})^2}\sqrt{\sum_{j=1}^{n}(Y_j-\bar{Y})^2}} \xrightarrow{P} \rho$$

(b) 接著求 $\sigma_X^2, \sigma_Y^2, \mu_X, \mu_Y, \rho$ 的 MLE.

我們直接考慮一般的情形.

$\vec{X}_j=\begin{pmatrix} x_j \\ y_j \end{pmatrix}$ $\quad X_j \overset{iid}{\sim} N_2(\vec{\mu}, \Sigma_{2\times 2})$ $\quad$ 求 μ, Σ 的 MLE.

$$\prod_{j=1}^{n} f(x_j) = \prod_{j=1}^{n} \left(\frac{1}{2\pi}\right) \frac{1}{|\Sigma|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}(\vec{x}_j-\vec{\mu})^t \Sigma^{-1}(x_j-\vec{\mu})\right)$$

$$L(\mu,\Sigma) = \prod_{j=1}^{n} f(x_j) = \prod_{j=1}^{n} \left(\frac{1}{2\pi}\right) \frac{1}{|\Sigma|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}(x_j-\vec{\mu})^t \Sigma^{-1}(x_j-\vec{\mu})\right)$$

$$\ell(\mu,\Sigma) = \ln L(\mu,\Sigma) = -n\ln(2\pi) - \frac{n}{2}\ln|\Sigma| - \frac{1}{2}\sum_{j=1}^{n}(x_j-\vec{\mu})^t \Sigma^{-1}(x_j-\vec{\mu})$$

$$= -n\ln(2\pi) - \frac{n}{2}\ln|\Sigma| - \frac{1}{2}\sum_{j=1}^{n}(x_j-\bar{x}+\bar{x}-\vec{\mu})^t \Sigma^{-1}(x_j-\bar{x}+\bar{x}-\vec{\mu})$$

$$= -n\ln(2\pi) - \frac{n}{2}\ln|\Sigma| - \frac{n}{2}(\bar{x}-\vec{\mu})^t \Sigma^{-1}(\bar{x}-\vec{\mu}) - \frac{1}{2}\sum_{j=1}^{n}(x_j-\bar{x})^t \Sigma^{-1}(x_j-\bar{x})$$

我們注意 Σ^{-1} 為 (半) 正定矩陣, 故 $(\bar{x}-\vec{\mu})\Sigma^{-1}(\bar{x}-\vec{\mu}) \geq 0$

故 $\vec{\mu}=\bar{x}$ 使得 ℓ 最大. $\therefore \hat{\mu}_{MLE} = \bar{x} = \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}$

$\vec{\mu}=\bar{x}$ 時 $\ell = -n\ln(2\pi) - \frac{n}{2}\ln|\Sigma| - \frac{1}{2}\sum_{j=1}^{n}(x_j-\bar{x})^t \Sigma^{-1}(x_j-\bar{x})$

$$= -n\ln(2\pi) - \frac{n}{2}\ln|\Sigma| - \frac{1}{2}\operatorname{tr}\left(\sum_{j=1}^{n}(x_j-\bar{x})^t \Sigma^{-1}(x_j-\bar{x})\right)$$

$$= -n\ln(2\pi) - \frac{n}{2}\ln|\Sigma| - \frac{1}{2}\sum_{j=1}^{n}\operatorname{tr}\,(x_j-\bar{x})^t \Sigma^{-1}(x_j-\bar{x})$$

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$$= -n\ln(2\pi) - \frac{n}{2}\ln|\Sigma| - \frac{1}{2}\sum_{j=1}^{n} tr\, \Sigma^{-1}(x_j-\bar{x})(x_j-\bar{x})^t$$

$$= -n\ln(2\pi) - \frac{n}{2}\ln|\Sigma| - \frac{1}{2}\, tr \sum_{j=1}^{n}\left(\Sigma^{-1}(x_j-\bar{x})(x_j-\bar{x})^t\right)$$

$$= -n\ln(2\pi) - \frac{n}{2}\ln|\Sigma| - \frac{1}{2}\, tr\, \Sigma^{-1} \underbrace{\sum_{j=1}^{n}(x_j-\bar{x})(x_j-\bar{x})^t}\,)$$

$$S \overset{def}{=} \frac{1}{n}\sum_{j=1}^{n}(x_j-\bar{x})(x_j-\bar{x})^t$$

$$= -n\ln(2\pi) - \frac{n}{2}\ln|\Sigma| - \frac{n}{2}\, tr\, \Sigma^{-1}S$$

我們注意 S 為對稱且(半)正定矩陣, $\exists\, S^{\frac{1}{2}}$ s.t. $S = (S^{\frac{1}{2}})^2$

$$l|_{\mu=\hat{\mu}} = -n\ln(2\pi) - \frac{n}{2}\ln|S^{-\frac{1}{2}}\Sigma S^{-\frac{1}{2}}||S| - \frac{n}{2}\, tr\, S^{\frac{1}{2}}\Sigma^{-1}S^{\frac{1}{2}}$$

$$= -n\ln(2\pi) - \frac{n}{2}\ln|S^{-\frac{1}{2}}\Sigma S^{-\frac{1}{2}}| - \frac{n}{2}\ln|S| - \frac{n}{2}\, tr\, S^{\frac{1}{2}}\Sigma^{-1}S^{\frac{1}{2}}$$

($T \overset{def}{=} S^{\frac{1}{2}}\Sigma^{-1}S^{\frac{1}{2}}$ 為(半)正定矩陣) (almost surely 正定矩陣)

$$= -n\ln(2\pi) + \frac{n}{2}\ln|T| - \frac{n}{2}\ln|S| - \frac{n}{2}\, tr\, T.$$

令 $\{\lambda_1, \cdots \lambda_n\}$ 為 T 的特徵值 (注意 $\lambda_j \geq 0$ (>0))

$$l = -n\ln(2\pi) - \frac{n}{2}\ln|S| + \frac{n}{2}\left(\sum_{j=1}^{n}(\ln\lambda_j - \lambda_j)\right)$$

$g(x) \overset{def}{=} \ln x - x$ $\quad g'(x) = \frac{1}{x} - 1 = 0 \Rightarrow x = 1$ (最大)

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故 $\lambda_1=\lambda_2=\cdots=\lambda_n=1$ 時最大

$T=S^{\frac{1}{2}}\Sigma^{-1}S^{\frac{1}{2}}=I_{n\times n}$ $\therefore \Leftrightarrow \Sigma=S$

故 $\hat{\Sigma}_{MLE}=S=\frac{1}{n}\sum_{j=1}^{n}(x_j-\bar{x})(x_j-\bar{x})^t$ 由 $x_j=\begin{pmatrix}x_j\\y_j\end{pmatrix}$ 之

$$=\frac{1}{n}\sum_{j=1}^{n}\begin{pmatrix}x_j-\bar{x}\\y_j-\bar{y}\end{pmatrix}\begin{pmatrix}x_j-\bar{x} & y_j-\bar{y}\end{pmatrix}$$

$$=\begin{pmatrix}\frac{1}{n}\sum_{j=1}^{n}(x_j-\bar{x})^2 & \frac{1}{n}\sum_{j=1}^{n}(x_j-\bar{x})(y_j-\bar{y})\\ \frac{1}{n}\sum_{j=1}^{n}(x_j-\bar{x})(y_j-\bar{y}) & \frac{1}{n}\sum_{j=1}^{n}(y_j-\bar{y})^2\end{pmatrix}$$

$$=\begin{pmatrix}\hat{\sigma_x^2} & \widehat{\rho\sigma_x\sigma_y}\\ \widehat{\rho\sigma_x\sigma_y} & \hat{\sigma_y^2}\end{pmatrix}$$

根據 MLE 的不變性 $\widehat{\rho\sigma_x\sigma_y}=\hat{\rho}\hat{\sigma_x}\hat{\sigma_y}=\hat{\rho}(\hat{\sigma_x^2})^{\frac{1}{2}}(\hat{\sigma_y^2})^{\frac{1}{2}}$

由此可知
$$\hat{\sigma_x^2}=\frac{1}{n}\sum_{j=1}^{n}(x_j-\bar{x})^2$$
$$\hat{\sigma_y^2}=\frac{1}{n}\sum_{j=1}^{n}(y_j-\bar{y})^2$$
$$\hat{\rho}=\frac{\sum(x_j-\bar{x})(y_j-\bar{y})}{\sqrt{\sum(x_j-\bar{x})^2\sum(y_j-\bar{y})^2}}$$
$$\hat{\mu}=\begin{pmatrix}\hat{\mu_x}\\ \hat{\mu_y}\end{pmatrix}=\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}$$

故 MLE 與 MME 有相同

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$\boxed{6}$ (Casella 7.19) $Y_i \sim N(\beta x_i, \sigma^2)$

$$f_{Y_i}(y_i) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-1}{2\sigma^2}(y_i-\beta x_i)^2\right)$$

$$f(y_1,\dots,y_n) = \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{n}{2}} \exp\left(\sum_{j=1}^{n} \frac{-1}{2\sigma^2}(y_j-\beta x_j)^2\right)$$

$$= \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{n}{2}} \exp\left(\frac{-1}{2\sigma^2}\sum_{j=1}^{n} y_j^2 + \frac{\beta}{\sigma^2}\sum_{j=1}^{n} x_j y_j - \frac{\beta^2}{2\sigma^2}\sum_{j=1}^{n} x_j^2\right)$$

$$\left\{\left(\frac{-1}{2\sigma^2}, \frac{\beta}{\sigma^2}\right) \middle| \ \sigma^2>0, \beta\in\mathbb{R}\right\}$$

包含$\mathbb{R}^2$上的內點，故 $\sum_{j=1}^{n} y_j^2$, $\sum_{j=1}^{n} x_j y_j$

為(β, σ)之完備充份統計量。

$$\ell(\beta,\sigma^2) = \ln f = \frac{-n}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{j=1}^{n}(y_j-\beta x_j)^2$$

$$\frac{\partial \ell}{\partial \beta}=0,\ \frac{\partial \ell}{\partial \sigma^2}=0 : \ \frac{-n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{j=1}^{n}(y_j-\beta x_j)^2 = 0$$

$$\therefore \hat{\sigma}^2 = \frac{1}{n}\sum_{j=1}^{n}(y_j-\hat{\beta}x_j)^2$$

$$\frac{1}{\sigma^2}\sum_{j=1}^{n}(y_j-\beta x_j)(-x_j)=0 \quad \therefore \sum_{j=1}^{n}(x_j y_j - \beta x_j^2)=0$$

$$\therefore \hat{\beta} = \frac{\sum x_j y_j}{\sum x_j^2}$$

考慮 $\hat{\beta}_{MLE}$ 的分佈 $\hat{\beta} = \frac{1}{\sum x_j^2}(x_1y_1 + x_2y_2 + \cdots + x_ny_n)$

$$x_jy_j \sim N(\beta x_j^2, x_j^2\sigma^2)$$

$$\sum_{j=1}^{n} x_jy_j \sim N\left(\sum_{j=1}^{n} \beta x_j^2, \sum_{j=1}^{n} \sigma^2 x_j^2\right)$$

$$\therefore \frac{1}{\sum x_j^2}\sum_{j=1}^{n} x_jy_j \sim N\left(\frac{\sum_{j=1}^{n} \beta x_j^2}{\sum x_j^2}, \frac{\sigma^2 \sum_{j=1}^{n} x_j^2}{\left(\sum_{j=1}^{n} x_j^2\right)^2}\right)$$

$$= N\left(\beta, \frac{\sigma^2}{\sum_{j=1}^{n} x_j^2}\right)$$

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$\boxed{7}$ $\vec{\theta}=\begin{pmatrix}\mu\\ \lambda\end{pmatrix} \in (0,\infty)\times(0,\infty)\subseteq\mathbb{R}^2$

我們假設 $f_{ZW}(zw|\vec{\theta})$ 滿足 Regularity Condition.

$\hat{\theta}_{MLE}$ 的漸近分佈為 $\hat{\theta}_{MLE} \xrightarrow{d} N(\theta, I_n^{-1}(\theta))$

故我們先回答 (b) 小題. $f_{ZW}(zw|\theta)=\left(\frac{1}{\mu}\right)^{1-w}\left(\frac{1}{\lambda}\right)^{w} \exp\left(-\left(\frac{1}{\lambda}+\frac{1}{\mu}\right)z\right)$

$$\ell(\theta|z,w)=-(1-w)\ln\mu - w\ln\lambda - \left(\frac{1}{\lambda}+\frac{1}{\mu}\right)z$$

$$\frac{\partial \ell}{\partial\mu}=\frac{-(1-w)}{\mu}+\frac{1}{\mu^2}z \qquad \frac{\partial^2 \ell}{\partial\mu^2}=\frac{1-w}{\mu^2}-\frac{2z}{\mu^3}$$

$$\frac{\partial \ell}{\partial\lambda}=\frac{-w}{\lambda}+\frac{1}{\lambda^2}z \qquad \frac{\partial^2 \ell}{\partial\lambda^2}=\frac{w}{\lambda^2}-\frac{2z}{\lambda^3}$$

$$\frac{\partial^2 \ell}{\partial\mu\partial\lambda}=0$$

$$\begin{bmatrix}\frac{\partial^2 \ell}{\partial\mu^2} & \frac{\partial^2 \ell}{\partial\lambda\partial\mu}\\ \frac{\partial^2 \ell}{\partial\mu\partial\lambda} & \frac{\partial^2 \ell}{\partial\lambda^2}\end{bmatrix}=\begin{bmatrix}\frac{1-w}{\mu^2}-\frac{2z}{\mu^3} & 0\\ 0 & \frac{w}{\lambda^2}-\frac{2z}{\lambda^3}\end{bmatrix}$$

接著考慮 W, Z 的邊際分佈

$Z\sim \exp\left(\frac{1}{\lambda}+\frac{1}{\mu}\right)$ mean: $\left(\frac{1}{\frac{1}{\lambda}+\frac{1}{\mu}}\right)$

$W\sim \text{bernoulli}\left(\frac{\frac{1}{\lambda}}{\frac{1}{\lambda}+\frac{1}{\mu}}\right)$ $\left(=\text{bernoulli}\left(\frac{\mu}{\lambda+\mu}\right)\right)$

$$E\left[\frac{W}{\mu^2}-\frac{2Z}{\mu^3}\right]=\frac{1}{\mu^2}\left(\frac{\frac{1}{\mu}}{\frac{1}{\lambda}+\frac{1}{\mu}}\right)-\frac{2}{\mu^3}\cdot\frac{1}{\frac{1}{\lambda}+\frac{1}{\mu}}=\frac{1}{\mu^3}\cdot\frac{-1}{\frac{1}{\lambda}+\frac{1}{\mu}}=\frac{\lambda}{\lambda+\mu}\cdot\frac{-1}{\mu^2}$$

$$E\left[\frac{W}{\lambda^2}-\frac{2Z}{\lambda^3}\right]=\frac{1}{\lambda^2}\left(\frac{\frac{1}{\lambda}}{\frac{1}{\lambda}+\frac{1}{\mu}}\right)-\frac{2}{\lambda^3}\cdot\frac{1}{\frac{1}{\lambda}+\frac{1}{\mu}}=\frac{-1}{\lambda^3}\cdot\frac{1}{\frac{1}{\lambda}+\frac{1}{\mu}}=\frac{\mu}{\lambda+\mu}\cdot\frac{-1}{\lambda^2}$$

故 $E\left[\begin{pmatrix}\frac{\partial^2 l}{\partial\mu^2} & \frac{\partial^2 l}{\partial\lambda\partial\mu}\\ \frac{\partial^2 l}{\partial\mu\partial\lambda} & \frac{\partial^2 l}{\partial\lambda^2}\end{pmatrix}\right]=\begin{pmatrix}\frac{\lambda}{\lambda+\mu}\cdot\frac{-1}{\mu^2} & 0\\ 0 & \frac{\mu}{\lambda+\mu}\cdot\frac{-1}{\lambda^2}\end{pmatrix}$

$\therefore I_1(\theta)=\begin{pmatrix}\frac{\lambda}{\lambda+\mu}\cdot\frac{1}{\mu^2} & 0\\ 0 & \frac{\mu}{\lambda+\mu}\cdot\frac{1}{\lambda^2}\end{pmatrix}$

故 $\sqrt{n}(\hat{\theta}_{ML}-\theta)\xrightarrow{d}N(0,I_1^{-1}(\theta))=\begin{pmatrix}\frac{(\lambda+\mu)\mu^2}{\lambda} & 0\\ 0 & \frac{(\lambda+\mu)\lambda^2}{\mu}\end{pmatrix}$

$$\left(\begin{array}{l}\because\sqrt{n}(\hat{\mu}-\mu)\xrightarrow{d}N\left(0,\frac{(\lambda+\mu)\mu^2}{\lambda}\right)\\ \sqrt{n}(\hat{\lambda}-\lambda)\xrightarrow{d}N\left(0,\frac{(\lambda+\mu)\lambda^2}{\mu}\right)\end{array}\right.$$

┘

我們再求用不同方法來驗證 $\hat{\theta}_{MLE}$ 的漸近分佈

為 $\begin{pmatrix}\sqrt{n}(\hat{\mu}-\mu)\\ \sqrt{n}(\hat{\lambda}-\lambda)\end{pmatrix}\xrightarrow{d}N\left(\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}\frac{(\lambda+\mu)}{\lambda}\mu^2 & 0\\ 0 & \frac{(\lambda+\mu)\lambda^2}{\mu}\end{pmatrix}\right)$

我們首先利用 Taylor 展開來將 $g(x_1,x_2)=\frac{\lambda_1}{\lambda_2}$

表示為 (x_1,λ_2) 的線性組合。 (下一頁)

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$$g(x_1,x_2)=\frac{x_1}{x_2}\approx\left(\frac{\partial g}{\partial x_1}\ \frac{\partial g}{\partial x_2}\right)\Bigg|_{(x_1,x_2)=(\alpha_1,\alpha_2)}\begin{pmatrix}x_1-\alpha_1\\ x_2-\alpha_2\end{pmatrix}+g(\alpha_1,\alpha_2)$$

$$=\left(\frac{1}{\alpha_2}\ \ -\frac{\alpha_1}{\alpha_2^2}\right)\begin{pmatrix}x_1-\alpha_1\\ x_2-\alpha_2\end{pmatrix}+\frac{\alpha_1}{\alpha_2}$$

故 $\frac{x_1}{x_2}\approx\frac{(x_1-\alpha_1)}{\alpha_2}-\frac{\alpha_1}{\alpha_2^2}(x_2-\alpha_2)+\frac{\alpha_1}{\alpha_2}$ $\perp$

$$\hat{\mu}_{MLE}=\frac{\sum_{j=1}^{n}Z_j}{\sum_{j=1}^{n}(1-W_j)},\quad \hat{\lambda}_{MLE}=\frac{\sum_{j=1}^{n}Z_j}{\sum_{j=1}^{n}W_j}$$

利用上述近似來表達 ...

① $\hat{\mu}_{MLE}=\frac{\bar{Z}}{\bar{V}}$ (where $V_j=1-W_j$ $\bar{V}=\frac{V_1+\cdots+V_n}{n}$)

$$\frac{\bar{Z}}{\bar{V}}\approx\frac{(\bar{Z}-E[Z_1])}{E[V_1]}-\frac{E[Z_1]}{E[V_1]^2}(\bar{V}-E[V_1])+\frac{E[Z_1]}{E[V_1]}$$

- $Z_j\sim\exp\left(\frac{1}{\lambda}+\frac{1}{\mu}\right)$ $\therefore E[Z_1]=\left(\frac{1}{\lambda}+\frac{1}{\mu}\right)^{-1}$
- $V_j\sim \text{bernulli}\left(\frac{\frac{1}{\mu}}{\frac{1}{\lambda}+\frac{1}{\mu}}\right)$ $E[V_1]=\frac{\frac{1}{\mu}}{\frac{1}{\lambda}+\frac{1}{\mu}}$

$$\therefore\frac{E[Z_1]}{E[V_1]}=\mu$$

$$\sqrt{n}(\hat{\mu}_{MLE}-\mu)=\frac{\sqrt{n}(\bar{Z}-E[Z_1])}{E[V_1]}-\frac{E[Z_1]}{E[V_1]^2}\cdot\sqrt{n}(\bar{V}-E[V_1])$$

② $\hat{\lambda}_{MLE} = \dfrac{\bar{Z}}{\bar{W}}$，同理，

$$\sqrt{n}(\hat{\lambda}_{MLE}-\lambda) = \frac{\sqrt{n}(\bar{Z}-E[Z_1])}{E[W_1]} - \frac{E[Z_1]\cdot\sqrt{n}(\bar{W}-E[W_1])}{E[W_1]^2}$$

我們注意 Z_j 與 $W_j(V_j)$ 為獨立

$$\operatorname{cov}[W_1,V_1] = E[W_1V_1] - E[W_1]E[V_1] = E[W_1-W_1^2] - E[W_1]E[1-W_1] = -V[W_1]$$

故：

$$\Sigma \overset{def}{=} V\begin{pmatrix} Z_1 \\ W_1 \\ V_1 \end{pmatrix} = \begin{pmatrix} \left(\frac{1}{\lambda}+\frac{1}{\mu}\right)^{-2} & 0 & 0 \\ 0 & \frac{1}{\mu}\cdot\frac{1}{\lambda}\cdot\left(\frac{1}{\lambda}+\frac{1}{\mu}\right)^{-2} & -\frac{1}{\mu}\cdot\frac{1}{\lambda}\left(\frac{1}{\lambda}+\frac{1}{\mu}\right)^{-2} \\ 0 & -\frac{1}{\mu}\cdot\frac{1}{\lambda}\left(\frac{1}{\lambda}+\frac{1}{\mu}\right)^{-2} & \frac{1}{\mu}\cdot\frac{1}{\lambda}\left(\frac{1}{\lambda}+\frac{1}{\mu}\right)^{-2} \end{pmatrix}$$

根據中央極限定理，

$$\sqrt{n}\begin{pmatrix} \bar{Z}-E[Z_1] \\ \bar{W}-E[W_1] \\ \bar{V}-E[V_1] \end{pmatrix} \xrightarrow{d} N\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \Sigma\right)$$

另外 $X_n \xrightarrow{d} X \Rightarrow AX_n \xrightarrow{d} AX$

$$A \overset{def}{=} \begin{pmatrix} \frac{1}{E[V_1]} & 0 & \frac{-E[Z_1]}{E[V_1]^2} \\ \frac{1}{E[W_1]} & \frac{-E[Z_1]}{E[W_1]^2} & 0 \end{pmatrix}, \quad \sqrt{n}(\hat{\theta}_{MLE}-\theta) = A\cdot\sqrt{n}\begin{pmatrix} \bar{Z}-E[Z_1] \\ \bar{W}-E[W_1] \\ \bar{V}-E[V_1] \end{pmatrix}$$

$$\hat{\theta} = \begin{pmatrix} \hat{\mu} \\ \hat{\lambda} \end{pmatrix} \quad \theta = \begin{pmatrix} \mu \\ \lambda \end{pmatrix}$$

$$\therefore \sqrt{n}(\hat{\theta}-\theta) \xrightarrow{d} N\left(\begin{pmatrix}0\\0\end{pmatrix}, A\Sigma A^t\right), \quad A\Sigma A^t =$$

$$\begin{pmatrix} \frac{1}{E[V_1]} & 0 & \frac{-E[Z_1]}{E[V_1]^2} \\ \frac{1}{E[W_1]} & \frac{-E[Z_1]}{E[W_1]^2} & 0 \end{pmatrix} \begin{pmatrix} V[Z_1] & 0 & 0 \\ 0 & V[W_1] & cov[W_1, Z_1] \\ 0 & cov[W_1, Z_1] & V[V_1] \end{pmatrix} A^t$$

$$= \begin{pmatrix} \frac{V[Z_1]}{E[V_1]} & \frac{-E[Z_1]}{E[V_1]^2} cov[W_1, Z_1] & \frac{-E[Z_1]}{E[V_1]^2} V[V_1] \\ \frac{V[Z_1]}{E[W_1]} & \frac{-E[Z_1]}{E[W_1]^2} V[W_1] & \frac{-E[Z_1]}{E[W_1]^2} cov[W_1, Z_1] \end{pmatrix} \begin{pmatrix} \frac{1}{E[V_1]} & \frac{1}{E[W_1]} \\ 0 & \frac{-E[Z_1]}{E[W_1]^2} \\ \frac{-E[Z_1]}{E[V_1]^2} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{V[Z_1]}{E[V_1]^2} + \frac{E[Z_1]^2 V[V_1]}{E[V_1]^4}, & \frac{V[Z_1]}{E[V_1]E[W_1]} + \frac{E[Z_1]^2 cov[W_1, Z_1]}{E[V_1]^2 E[W_1]^2} \\ \frac{V[Z_1]}{E[W_1]E[V_1]} + \frac{E[Z_1]^2 cov[W_1, Z_1]}{E[W_1]^2 E[V_1]^2} & \frac{V[Z_1]}{E[W_1]^2} + \frac{E[Z_1]^2 V[W_1]}{E[W_1]^4} \end{pmatrix} = A\Sigma A^t$$

(i) $$\frac{V[Z_1]}{E[V_1]^2} + \frac{E[Z_1]^2 V[V_1]}{E[V_1]^4} = \left(\frac{1}{\lambda}+\frac{1}{\mu}\right)^{-2} \cdot \left(\frac{\frac{1}{\mu}}{\frac{1}{\lambda}+\frac{1}{\mu}}\right)^{-2} + \left(\frac{\frac{1}{\mu}}{\frac{1}{\lambda}+\frac{1}{\mu}}\right)^{-4} \cdot \left(\frac{1}{\lambda}+\frac{1}{\mu}\right)^{-2} \cdot \frac{\left(\frac{1}{\lambda}\cdot\frac{1}{\mu}\right)}{\left(\frac{1}{\lambda}+\frac{1}{\mu}\right)^{-2}}$$

$$= \mu^2 + \frac{\mu^3}{\lambda} = \frac{(\lambda+\mu)\mu^2}{\lambda}$$

(ii) $$\frac{V[Z_1]}{E[V_1]E[W_1]} + \frac{E[Z_1]^2 cov[W_1, Z_1]}{E[V_1]^2 E[W_1]^2} =$$

$$= \cancel{\left(\frac{1}{\lambda}+\frac{1}{\mu}\right)^{-2}} \cdot \frac{\cancel{\left(\frac{1}{\lambda}+\frac{1}{\mu}\right)^{2}}}{\left(\frac{1}{\lambda}-\frac{1}{\mu}\right)} + \cancel{\left(\frac{1}{\lambda}+\frac{1}{\mu}\right)^{-2}} \cdot (-1) \cdot \left(\frac{1}{\mu}\frac{1}{\lambda}\right) \cancel{\left(\frac{1}{\lambda}+\frac{1}{\mu}\right)^{-2}} \cdot \frac{\cancel{\left(\frac{1}{\lambda}+\frac{1}{\mu}\right)^{4}}}{\left(\frac{1}{\lambda}\frac{1}{\mu}\right)^2}$$

$$= \cancel{\lambda\mu} - \cancel{\lambda\mu} = 0$$

$(iii) = (ii) = 0$

$$(iv)\ \frac{V[Z_1]}{E[W_1]^2} + \frac{E[Z_1]^2\, V[W_1]}{E[W_1]^4} = \left(\frac{1}{\lambda}+\frac{1}{\mu}\right)\cdot\left(\frac{\frac{1}{\lambda}}{\frac{1}{\lambda}+\frac{1}{\mu}}\right)^{-2} + \left(\frac{1}{\lambda}+\frac{1}{\mu}\right)\cdot\frac{\frac{1}{\lambda}\frac{1}{\mu}}{\left(\frac{1}{\lambda}+\frac{1}{\mu}\right)^2}\left(\frac{\frac{1}{\lambda}}{\frac{1}{\lambda}+\frac{1}{\mu}}\right)^{-4} = \lambda^2 + \frac{\lambda^3}{\mu} = \frac{(\lambda+\mu)}{\mu}\lambda^2$$

$$故\ \sqrt{n}(\hat{\theta}-\theta) \xrightarrow{d} N\left(\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}\frac{\lambda+\mu}{\lambda}\cdot\mu^2 & 0\\ 0 & \frac{\lambda+\mu}{\mu}\lambda^2\end{pmatrix}\right)$$

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$\boxed{8}$

(a) $Pr(X=x|\lambda)=e^{-\lambda}\cdot\frac{\lambda^x}{x!}=p\cdot\frac{(-\ln p)^x}{x!}$ $(x=0,1,2,\ldots)$

$l_p(p)=\ln Pr(X=x|p)=\frac{1}{p}-\frac{1}{p}\cdot\frac{x}{-\ln p}$

$E[l_p(p|x)^2]=E\left[\frac{1}{p}\left(1-\frac{x}{-\ln x}\right)^2\right]=\frac{1}{p^2}\cdot E\left[\left(1-\frac{x}{\lambda}\right)^2\right]$

$=\frac{V[X]}{\lambda^2p^2}=\frac{\lambda}{\lambda^2p^2}=\frac{1}{\lambda p^2}=\frac{1}{(-\ln p)\cdot p^2}$

$I_X(p)=\frac{1}{(-\ln p)\cdot p^2}$

(b) 利用MLE的不變性 ($-\ln X$為 one-to-one 的函數)

$Pr(X_1\ldots X_n|\lambda)=e^{-n\lambda}\cdot\frac{\lambda^{(X_1+\ldots+X_n)}}{x_1!\ldots x_n!}$

$l_\lambda(\lambda)=-n\lambda+(X_1+X_2+\ldots+X_n)\ln\lambda-\ln(x_1!\ldots x_n!)$

$\frac{\partial l_\lambda(\lambda)}{\partial\lambda}=-n+\frac{1}{\lambda}(X_1+X_2+\ldots+X_n)=0 \Rightarrow \lambda=\bar{X}$

$\frac{\partial^2 l_\lambda(\lambda)}{\partial\lambda^2}=\frac{-n\bar{X}}{\lambda^2}<0 \quad \therefore \hat{\lambda}_{MLE}=\bar{X}$

根據MLE的不變性 $\hat{p}_{MLE}=e^{-\hat{\lambda}_{MLE}}=e^{-\bar{X}}$ (p,λ為 one-to-one)

(c) $g(x)\overset{def}{=}\exp(-x)$

根據中央極限定理, $\sqrt{n}(\bar{X}-\lambda)\xrightarrow{d}N(0,\lambda)$ $V[X_1]$

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利用 δ-method 得：$\sqrt{n}\left(g(\bar{X})-g(\lambda)\right) \xrightarrow{d} N\left(0, \lambda\, g'(\lambda)^2\right) = N\left(0, \lambda \exp(-2\lambda)\right)$
$= N\left(0, (-\ln p)\cdot p^2\right) = N\left(0, I_X^{-1}(p)\right)$

(d) Y_i 取 0 or 1. $Pr(Y_i=0) = Pr(X_i=0) = e^{-\lambda}$

$\therefore Y_i \sim bernoulli(1-e^{-\lambda})$ $(= bernoulli(1-p))$

(e) $Pr(Y=y) = (1-e^{-\lambda})^y \cdot (e^{-\lambda})^{1-y}$ $(y=0,1)$

$= (1-p)^y\, p^{1-y}$

$\frac{\partial}{\partial p} \ln Pr(Y=y) = \frac{-y}{1-p} + \frac{1-y}{p} = \frac{-py+(1-p)(1-y)}{p(1-p)} = \frac{(1-p)-y}{p(1-p)}$

$\therefore I_Y(p) = E\left[\left(\frac{(1-p)-y}{p(1-p)}\right)^2\right] = \frac{V[Y]}{p^2(1-p)^2} = \frac{1}{p(1-p)}$

(f) $1-p$ 的 MLE 為 $\bar{Y}$，根据 MLE 的不變性 $\hat{p} = 1-\bar{Y}$

(g) $\hat{p} = \frac{1}{n}\sum_{i=1}^{n}(1-Y_i)$ $1-Y_i \sim bernoulli(p)$ $V[1-Y_i] = p(1-p)$

根據中央極限定理 $\sqrt{n}(\hat{p}-p) \xrightarrow{d} N(0, p(1-p)) = N(0, I_Y^{-1}(p))$

(h) $\frac{I_Y(p)}{I_X(p)} = \frac{p^2 \cdot (-\ln p)}{p(1-p)} = \frac{1}{1-p} \cdot p(-\ln p)$ $f(p) \overset{def}{=} \frac{1}{1-p} p(-\ln p)$

$f'(p) = \frac{(-\ln p - 1)(1-p) + p(-\ln p)}{(1-p)^2} = \frac{-\ln p - 1 + \cancel{p\ln p} + p - \cancel{p\ln p}}{(1-p)^2} = \frac{-\ln p - 1 + p}{(1-p)^2}$

$h(p) = -\ln p - 1 + p$ $h'(p) = 1 - \frac{1}{p} \geq 0$ $h(1) = 0$ $\therefore h(p) \leq 0$

$\therefore f'(p) \leq 0$ $\therefore f(p)$ 遞減函數 $\lim_{p \to 1} f(p) = 1$ $\therefore f(p) > 1$ $(0<p<1)$

故 $I_Y(p) > I_X(p)$ $\Rightarrow$ $\frac{1}{I_X(p)} > \frac{1}{I_Y(p)}$ $\therefore \sigma_X^2 > \sigma_Y^2$

$\boxed{9}$

(a) $f(x,y) = \dfrac{1}{2\pi(1-\rho^2)^{1/2}} \exp\left(\dfrac{-1}{2(1-\rho^2)}(x\ y)\begin{pmatrix}1 & -\rho\\ -\rho & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}\right)$

$$\ell(\rho) = \ln f = -\ln(2\pi) - \frac{1}{2}\ln(1-\rho^2) - \frac{1}{2(1-\rho^2)}(x^2-2\rho xy+y^2)$$

$$\ell'(\rho) = \frac{\rho}{1-\rho^2} + \frac{1}{2}\cdot\frac{(-2\rho)}{(1-\rho^2)^2}(x^2-2\rho xy+y^2) - \frac{1}{2(1-\rho^2)}(-2xy)$$

$$= \frac{\rho}{1-\rho^2} - \frac{\rho}{(1-\rho^2)^2}(x^2-2\rho xy+y^2) + \frac{xy}{(1-\rho^2)}$$

$$= \frac{\rho}{1-\rho^2} + \frac{1}{(1-\rho^2)^2}\left((1+\rho^2)xy - \rho(x^2+y^2)\right)$$

$$\ell''(\rho) = \frac{(1-\rho^2)-\rho(-2\rho)}{(1-\rho^2)^2} + \frac{(-2)\cdot(-2\rho)}{(1-\rho^2)^3}\left((1+\rho^2)xy-\rho(x^2+y^2)\right)$$
$$+ \frac{1}{(1-\rho^2)^2}(2\rho xy - x^2-y^2)$$

$$= \frac{1+\rho^2}{(1-\rho^2)^2} + \frac{4\rho}{(1-\rho^2)^3}\left((1+\rho^2)xy-\rho(x^2+y^2)\right) + \frac{1}{(1-\rho^2)^2}(2\rho xy - x^2-y^2)$$

$$E[\ell''(\rho)] = \frac{1+\rho^2}{(1-\rho^2)^2} + \frac{4\rho}{(1-\rho^2)^3}\left((1+\rho^2)\rho - 2\rho\right) + \frac{1}{(1-\rho^2)^2}(2\rho^2-2)$$

$(\because E[X^2]=E[Y^2]=1,\ E[XY]=\rho)$

$$= \frac{1+\rho^2}{(1-\rho^2)^2} - \frac{4\rho^2(1-\rho^2)}{(1-\rho^2)^3} - \frac{2(1-\rho^2)}{(1-\rho^2)^2}$$

$$= \frac{1+\rho^2}{(1-\rho^2)^2} - \frac{4\rho^2}{(1-\rho^2)^2} - \frac{2(1-\rho^2)}{(1-\rho^2)^2}$$

$$= \frac{1+\rho^2}{(1-\rho^2)^2} - \frac{2(1+\rho^2)}{(1-\rho^2)^2} = \frac{-(1+\rho^2)}{(1-\rho^2)^2}$$

$\therefore I_\rho(\rho) = -E[\ell(\rho)] = \dfrac{1+\rho^2}{(1-\rho^2)^2}$

(b) $W_j \overset{def}{=} X_j Y_j$

求 $E[W_j]$ 與 $V[W_j]$

$E[W_j] = E[X_jY_j] = cov[X_jY_j] + E[X_j]E[Y_j] = \rho$

$E[W_j^2] = E[(X_jY_j)^2] = E\left[E[X_j^2Y_j^2|X_j]\right] = E[X_j^2 E[Y_j^2|X_j]]$

$\left(Y_j|X_j \sim N(\rho X_j, (1-\rho^2))\right.$ 故 $\left.E[Y_j^2|X_j] = (1-\rho^2)+\rho^2X_j^2\right)$

$= E[\rho^2X_j^4 + (1-\rho^2)X_j^2] = 3\rho^2+(1-\rho^2) = 1+2\rho^2$ $\therefore V[W_j]=1+\rho^2$

根據中央極限定理, $\sqrt{n}(\overline{W}-\rho) \xrightarrow{d} N(0,1+\rho^2)$

(c) 根據 (a), $l'(\rho) = \frac{\rho}{1-\rho^2} + \frac{1}{(1-\rho^2)^2}((1+\rho^2)xy - \rho(x^2+y^2))$ (一個樣本)

$\therefore l_n'(\rho|X_1..X_n,Y_1..Y_n) = \sum_{j=1}^n\left(\frac{\rho}{1-\rho^2}+\frac{1}{(1-\rho^2)^2}((1+\rho^2)X_jY_j - \rho(X_j^2+Y_j^2))\right) = 0$

$\Leftrightarrow \frac{\rho}{1-\rho^2} + \frac{1}{(1-\rho^2)^2}\left((1+\rho^2)\frac{1}{n}\sum_{j=1}^n X_jY_j - \rho\left(\frac{1}{n}\sum_{j=1}^n X_j^2+\frac{1}{n}\sum_{j=1}^n Y_j^2\right)\right)=0$

$\Leftrightarrow \rho(1-\rho^2) + (1+\rho^2)\frac{1}{n}\sum_{j=1}^n X_jY_j - \rho\cdot\frac{1}{n}\sum_{j=1}^n X_j^2 - \rho\cdot\frac{1}{n}\sum_{j=1}^n Y_j^2=0$

$\Leftrightarrow -\rho^3+\rho^2\left(\frac{1}{n}\sum_{j=1}^n X_jY_j\right)+\rho\left(1-\frac{1}{n}\sum_{j=1}^n X_j^2-\frac{1}{n}\sum_{j=1}^n Y_j^2\right)+\frac{1}{n}\sum_{j=1}^n X_jY_j=0$

故 $\hat\rho_{MLE}$ 滿足 $\hat\rho^3-\hat\rho^2\left(\frac{1}{n}\sum_{j=1}^n X_jY_j\right)+\hat\rho\left(\frac{1}{n}\sum_{j=1}^n X_j^2+\frac{1}{n}\sum_{j=1}^n Y_j^2-1\right)-\frac{1}{n}\sum_{j=1}^n X_jY_j=0$

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(d) $\Delta = \frac{-1}{1+\rho^2}(\rho U_n + \rho V_n - (1+\rho^2)W_n) + o_p(n^{-\frac{1}{2}})$

$= \left(\frac{-\rho}{1+\rho^2} \ \frac{-\rho}{1+\rho^2} \ 1 \right) \begin{pmatrix} U_n \\ V_n \\ W_n \end{pmatrix} + \underset{(\xrightarrow{P} 0)}{o_p(n^{-\frac{1}{2}})}$

根據(多變量的)中央極限定理，

$\vec{x}_j = \begin{pmatrix} X_j^2 \\ Y_j^2 \\ X_j Y_j \end{pmatrix} \quad V[\vec{X}_j] = \Sigma \quad , E[\vec{X}_j] = \begin{pmatrix} 1 \\ 1 \\ \rho \end{pmatrix}$

$\underbrace{\sqrt{n}\left(\frac{\vec{x}_1 + \cdots + \vec{x}_n}{n} - \begin{pmatrix} 1 \\ 1 \\ \rho \end{pmatrix} \right)}_{= \begin{pmatrix} U_n \\ V_n \\ W_n \end{pmatrix}} \xrightarrow{d} N(0, \Sigma)$

則 $\begin{pmatrix} U_n \\ V_n \\ W_n \end{pmatrix} \xrightarrow{d} N\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \Sigma \right) \Rightarrow \underbrace{A \begin{pmatrix} U_n \\ V_n \\ W_n \end{pmatrix}}_{\Delta} \xrightarrow{d} N(0, A \Sigma A^t)$

where $A = \left(\frac{-\rho}{1+\rho^2} \ \frac{-\rho}{1+\rho^2} \ 1 \right)$

我們求 Σ ...

$E[X_j^2] = 1 \quad E[X_j^4] = 3 \quad \therefore V[X_j^2] = 2$

$(E[Y_j^2] = 1 \quad E[Y_j^4] = 3 \quad V[Y_j^2] = 2)$

$Cov[X_j^2, Y_j^2] = E[X_j^2 Y_j^2] - E[X_j^2]E[Y_j^2] = (1+2\rho^2) - 1 = 2\rho^2$

$V[X_jY_j] = 1+\rho^2$

$\mathrm{cov}[X_j^2, X_jY_j] = E[X_j^3Y_j] - \underbrace{E[X_j^2]\,E[X_jY_j]}_{1\cdot\rho} = 3\rho-\rho = \underline{\underline{2\rho}}$

$$\left(\begin{aligned} &E[E[X_j^3Y_j \mid X_j]] = E[X_j^3 E[Y_j \mid X_j]] \\ &= E[X_j^3\cdot\rho X_j] = \rho E[X_j^4] = 3\rho \end{aligned}\right)$$

$\mathrm{cov}[Y_j^2, X_jY_j] = \underline{\underline{2\rho}}$ (∵ 對稱性)

$$\therefore \Sigma = \begin{pmatrix} 2 & 2\rho^2 & 2\rho \\ 2\rho^2 & 2 & 2\rho \\ 2\rho & 2\rho & 1+\rho^2 \end{pmatrix}$$

$$\therefore A\Sigma A^t = \left(-\frac{\rho}{1+\rho^2}\ \ -\frac{\rho}{1+\rho^2}\ \ 1\right)\begin{pmatrix} 2 & 2\rho^2 & 2\rho \\ 2\rho^2 & 2 & 2\rho \\ 2\rho & 2\rho & 1+\rho^2 \end{pmatrix}\begin{pmatrix} -\frac{\rho}{1+\rho^2} \\ -\frac{\rho}{1+\rho^2} \\ 1 \end{pmatrix}$$

$$= \left(-\frac{\rho}{1+\rho^2}\ \ -\frac{\rho}{1+\rho^2}\ \ 1\right)\begin{pmatrix} 0 \\ 0 \\ \frac{(1-\rho^2)^2}{1+\rho^2} \end{pmatrix} = \frac{(1-\rho^2)^2}{1+\rho^2}$$

故 $\Delta \xrightarrow{d} N(0, A\Sigma A^t) = N\left(0, \frac{(1-\rho^2)^2}{1+\rho^2}\right) = N(0, I_\rho^{-1}(\rho))$

證明完成

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作業8　R05246013 森元俊成

$\boxed{1}$ (Casella 8.1)

$\{X_j\}_{j=1\sim n}$ $(n=1000)$ $\overset{iid}{\sim}$ Bernoulli(p)

考慮檢定 $H_0: p=\frac{1}{2}$ vs $H_1: p\neq\frac{1}{2}$ $(\alpha=0.05)$

利用概似比檢定,

① $\hat{p}_{MLE}\ (p\in(0,1)) = \bar{X}$

② $\hat{p}_{0,MLE}\ (p=\frac{1}{2}) = \frac{1}{2}$

③ $\Lambda = \dfrac{L(\hat{p}_{0,MLE}\mid x_1 \cdots x_n)}{L(\hat{p}_{MLE}\mid x_1,\cdots x_n)} < c$

$$= \frac{(\hat{p}_0)^{x_1+x_2+\cdots+x_n}(1-\hat{p}_0)^{n-(x_1+\cdots+x_n)}}{(\hat{p}_{MLE})^{x_1+\cdots+x_n}(1-\hat{p}_{MLE})^{n-(x_1+x_2+\cdots+x_n)}}$$

$$= \frac{\left(\frac{1}{2}\right)^n}{\bar{x}^{n\bar{x}}(1-\bar{x})^{n(1-\bar{x})}} < c \iff \left\{\bar{x}^{\bar{x}}(1-\bar{x})^{1-\bar{x}}\right\}^n > \left(\frac{1}{2}\right)^n\cdot\frac{1}{c}$$

故

棄卻域 $C=\left\{(x_1,\cdots x_n)\ \middle|\ \bar{x}^{\bar{x}}(1-\bar{x})^{1-\bar{x}} > c'\right\}$

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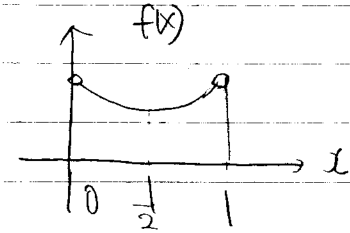
$(0<x<1)$

$f(x) = x^{x}(1-x)^{1-x}$ $\qquad g(x) = \ln f(x) = x\ln x + (1-x)\ln(1-x)$

$g'(x) = \ln x - \ln(1-x) = \ln\left(\frac{x}{1-x}\right) = 0$

$\Rightarrow x = \frac{1}{2}$ 使得 $g(x)$ 最小
$(f(x))$

∴



∴ 棄卻域 $C = \{(x_1,\dots,x_n) \mid \bar{x} < c_1,\ c_2 < \bar{x}\}$ (雙尾檢定)

接下來求 c_1, c_2 使得 $Pr\big((x_1,\dots,x_n) \in C \mid \underset{(p=\frac{1}{2})}{H_0}\big) = \alpha = 0.05$

利用 $\sqrt{n}(\bar{X}-p) \xrightarrow{d} N(0, p(1-p))$

$C:\ \frac{\sqrt{n}(\bar{X}-p)}{\sqrt{p}\sqrt{1-p}} \le -1.96$ or $1.96 \le \frac{\sqrt{n}(\bar{X}-p)}{\sqrt{p}\sqrt{1-p}}$

$C:\ \bar{X} \le \frac{-1.96\sqrt{p(1-p)}}{\sqrt{n}} + p$ or $\frac{1.96\sqrt{p}\sqrt{1-p}}{\sqrt{n}} + p \le \bar{X}$

$p = \frac{1}{2},\ n = 1000$

$C:\ \bar{X} \le -1.96 \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{1000}} + \frac{1}{2}$ or $1.96 \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{1000}} + \frac{1}{2} \le \bar{X}$

reject.

∴ $C:\ \{(x_1,\dots,x_n) \mid \bar{X} \le 0.469,\ 0.531 \le \bar{X}\}$ ∴ $\bar{X} = 0.560 \Rightarrow$ 棄卻

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$\boxed{2}$ (Casella 8.6)

(a) ① $(\theta, \mu) \subseteq (0,\infty)\times(0,\infty)$ 時 $\hat{\theta}_{MLE} = \bar{X}$, $\hat{\mu}_{MLE} = \bar{Y}$

② $(\theta,\mu) \subseteq (0,\infty)\times(0,\infty) \cap \{\theta=\mu\}$ 時

$$\hat{\theta}_{MLE,0} = \hat{\mu}_{MLE,0} = \frac{X_1+X_2+\dots X_n + Y_1+\dots\dots Y_m}{m+n} = \frac{n\bar{X}+m\bar{Y}}{n+m}$$

③
$$\Lambda = \frac{\left(\frac{1}{\hat{\theta}_0}\right)^n \cdot \exp\left(-\frac{1}{\hat{\theta}_0}(X_1+\dots+X_n)\right) \cdot \left(\frac{1}{\hat{\mu}_0}\right)^m \exp\left(-\frac{1}{\hat{\mu}_0}(Y_1+\dots+Y_m)\right)}{\left(\frac{1}{\hat{\theta}}\right)^n \exp\left(-\frac{1}{\hat{\theta}}(X_1+\dots+X_n)\right) \left(\frac{1}{\hat{\mu}}\right)^m \exp\left(-\frac{1}{\hat{\mu}}(Y_1+\dots+Y_m)\right)}$$

$$= \frac{\left(\frac{1}{\hat{\theta}_0}\right)^n \left(\frac{1}{\hat{\mu}_0}\right)^m \cancel{\exp(-(n+m))}}{\left(\frac{1}{\hat{\theta}}\right)^n \left(\frac{1}{\hat{\mu}}\right)^m \cancel{\exp(-(n+m))}}$$

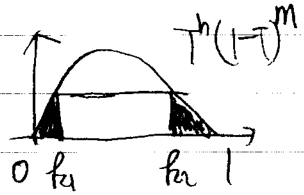
$$= \frac{\hat{\theta}^n \hat{\mu}^m}{\hat{\theta}_0^n \hat{\mu}_0^m} = \frac{\bar{X}^n \bar{Y}^m}{\left(\frac{n\bar{X}+m\bar{Y}}{n+m}\right)^{n+m}} < c' \Leftrightarrow \frac{(n\bar{X})^n \cdot (m\bar{Y})^m}{(n\bar{X}+m\bar{Y})^{n+m}} < c$$

故. $C = \left\{ (x_1,\dots x_n, y_1,\dots y_m) \;\middle|\; \frac{(n\bar{X})^n (m\bar{Y})^m}{(n\bar{X}+m\bar{Y})^{n+m}} < c \right\}$

where $Pr((x_1,\dots x_n, y_1,\dots y_m) \in C \mid H_0) = \alpha$.

(b) $= \left\{\dfrac{n\bar{x}}{(n\bar{x}+m\bar{y})}\right\}^{n}\left\{1-\dfrac{n\bar{x}}{n\bar{x}+m\bar{y}}\right\}^{m}$

$\Leftrightarrow T^{n}(1-T)^{m} < C$



故 $C = \{(X_1,\dots,X_n, Y_1,\dots,Y_m) \mid 0<T<k_1,\ k_2<T<1\}$

∴ 證明完成

(c) 於 H_0 下 $X_i, Y_j \overset{iid}{\sim} \exp(\theta)$

$(i=1\sim n,\ j=1\sim m)$

故 $T = \dfrac{X_1+\dots+X_n}{X_1+\dots X_n+Y_1+\dots+Y_m} \sim \text{Beta}(n,m)$

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3 (Casella 8.14) $p_0 = 0.49$, $p_1 = 0.51$

根據 Neyman-Pearson's lemma

「$\Lambda = \dfrac{f(x_1 \cdots x_n | p_1)}{f(x_1 \cdots x_n | p_0)} > c \Rightarrow$ 棄卻」

$$\Lambda = \frac{p_1^{x_1+x_2+\cdots+x_n}(1-p_1)^{n-(x_1+\cdots+x_n)}}{p_0^{x_1+x_2+\cdots+x_n}(1-p_0)^{n-(x_1+x_2+\cdots+x_n)}}$$

$$= \left(\frac{1-p_1}{1-p_0}\right)^n \cdot \left\{\frac{\left(\frac{p_1}{1-p_1}\right)}{\left(\frac{p_0}{1-p_0}\right)}\right\}^{x_1+x_2+\cdots+x_n} = \left(\frac{1-p_1}{1-p_0}\right)^n \cdot \left(\frac{g(p_1)}{g(p_0)}\right)^{x_1+x_2+\cdots x_n}$$

$g(p) \overset{def}{=} \dfrac{p}{1-p}$ … 遞增函數 $(0<p<1)$

$\because \dfrac{g(p_1)}{g(p_0)} > 1$ $\therefore$ $X_1+X_2+\cdots+X_n$：增加 $\Rightarrow \Lambda$：增加
(p 的充份統計量)

故棄卻域 $C = \{(X_1, X_2, \cdots, X_n) \mid X_1+X_2+\cdots+X_n > k\}$

根據中央極限定理，$\sqrt{n}(\bar{X}-p) \xrightarrow{d} N(0, p(1-p))$

考慮 k 使得 $P_r((X_1,\cdots,X_n) \in C \mid \underbrace{H_0}_{p=0.49}) \leq 0.01$ …①

$P_r((X_1,\cdots,X_n) \notin C \mid \underbrace{H_1}_{p=0.51}) \leq 0.01$ …②

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$$\frac{\sqrt{n}(\bar{X}-p)}{\sqrt{p}\sqrt{1-p}}\Big|_p \sim N(0,1)$$

① $Pr\left(\frac{\sqrt{n}(\bar{X}-p_0)}{\sqrt{p_0}\sqrt{1-p_0}} \geq k' \,\Big|\, H_0\right) \leq 0.01 \qquad k' \geq 2.32$

$\therefore C = \{(X_1, \dots, X_n) \mid X_1+X_2+\dots+X_n \geq \underbrace{np_0 + k'\sqrt{n}\sqrt{p_0}\sqrt{1-p_0}}_{=k}\}$

② 考慮 H_1 為真. $X_1+\dots+X_n \geq np_0 + k'\sqrt{n}\sqrt{p_0}\sqrt{1-p_0}$

$\Leftrightarrow \quad \bar{X} \geq p_0 + \frac{k'}{\sqrt{n}}\sqrt{p_0}\sqrt{1-p_0}$

$\Leftrightarrow \quad \bar{X}-p_1 \geq (p_0-p_1) + \frac{k'}{\sqrt{n}}\sqrt{p_0}\sqrt{1-p_0}$

$\Leftrightarrow \quad \sqrt{n}(\bar{X}-p_1) \geq \sqrt{n}(p_0-p_1) + k'\sqrt{p_0}\sqrt{1-p_0}$

$\Leftrightarrow \quad \frac{\sqrt{n}(\bar{X}-p_1)}{\sqrt{p_1}\sqrt{1-p_1}} \geq \frac{-\sqrt{n}(p_1-p_0)}{\sqrt{p_1}\sqrt{1-p_1}} + k'\frac{\sqrt{p_0}\sqrt{1-p_0}}{\sqrt{p_1}\sqrt{1-p_1}}$

$$Pr\left(\frac{\sqrt{n}(\bar{X}-p_1)}{\sqrt{p_1}\sqrt{1-p_1}} \geq \underbrace{\frac{-\sqrt{n}(p_1-p_0)}{\sqrt{p_1}\sqrt{1-p_1}} + \frac{k'\sqrt{p_0}\sqrt{1-p_0}}{\sqrt{p_1}\sqrt{1-p_1}}}_{\leq -2.32}\right) \geq 0.99$$

$$\therefore \frac{-\sqrt{n}(p_1-p_0)}{\sqrt{p_1}\sqrt{1-p_1}} + \frac{k'\sqrt{p_0}\sqrt{1-p_0}}{\sqrt{p_1}\sqrt{1-p_1}} \leq -2.32$$

$$\therefore \frac{\sqrt{n}(p_1-p_0)}{\sqrt{p_1}\sqrt{1-p_1}} \geq 2.32 + \frac{k'\sqrt{p_0}\sqrt{1-p_0}}{\sqrt{p_1}\sqrt{1-p_1}} \quad (k' \geq 2.32)$$

$$\therefore \sqrt{n} \geq 2.32\left(1 + \cancel{\frac{\sqrt{p_0}\sqrt{1-p_0}}{\sqrt{p_1}\sqrt{1-p_1}}}^{\,1}\right)\cdot\frac{\sqrt{p_1}\sqrt{1-p_1}}{p_1-p_0}$$

$$= 4.64\,\frac{\sqrt{p_1}\sqrt{1-p_1}}{p_1-p_0} = 4.64\cdot 50\cdot\sqrt{p_1}\sqrt{1-p_1}$$

$\therefore n \geq 13451 \qquad k = 6726.$

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$\boxed{4}$ (casella 8.18)

(a) $\beta(\theta) = \Pr\left(\frac{\sqrt{n}}{\sigma}|\bar{X}-\theta_0| \geq c \,\middle|\, \theta\right)$

$= \Pr\left(\bar{X}-\theta_0 \geq \frac{c\sigma}{\sqrt{n}} \text{ or } \bar{X}-\theta_0 \leq \frac{-c\sigma}{\sqrt{n}} \,\middle|\, \theta\right)$

$= \Pr\left(\bar{X}-\theta \geq \theta_0-\theta+\frac{c\sigma}{\sqrt{n}} \text{ or } \bar{X}-\theta \leq \theta_0-\theta+\frac{-c\sigma}{\sqrt{n}} \,\middle|\, \theta\right)$

$= \Pr\left(\frac{\sqrt{n}(\bar{X}-\theta)}{\sigma} \geq \frac{\sqrt{n}(\theta_0-\theta)}{\sigma}+c \text{ or } \frac{\sqrt{n}(\bar{X}-\theta)}{\sigma} \leq \frac{\sqrt{n}(\theta_0-\theta)}{\sigma}-c \,\middle|\, \theta\right)$

$= 1-\Phi\left(\frac{\sqrt{n}(\theta_0-\theta)}{\sigma}+c\right)+\Phi\left(\frac{\sqrt{n}(\theta_0-\theta)}{\sigma}-c\right)$

$\left(\because \frac{\sqrt{n}(\bar{X}-\theta)}{\sigma} \sim N(0,1) \text{ when } X_i \sim N(\theta,\sigma^2)\right)$

(b) $\beta(\theta_0) \leq 0.05$

$\therefore 1-\Phi(c)+\Phi(-c) = 2-2\Phi(c) \leq 0.05$

$\therefore \Phi(c) \geq 0.975 \quad \therefore c \geq 1.96$

$\beta(\theta_1) = \beta(\theta_0+\sigma) \geq 0.75 \quad (\because 1-\beta(\theta_1) \leq 0.25)$

$\therefore 1-\Phi(c-\sqrt{n})+\Phi(-\sqrt{n}-c) \geq 0.75$ → 很小. 所以忽略掉

$\Phi(\sqrt{n}-c) \geq 0.75 = \Phi(0.67)$

$\therefore \sqrt{n}-c \geq 0.67 \quad \therefore \sqrt{n} \geq 2.63 \quad \therefore n \geq 7$

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$\boxed{5}$ (Casella 8.20)

根據 Neyman-Pearson's lemma, $\Lambda = \dfrac{f(x|H_1)}{f(x|H_0)} > c \Rightarrow$ 棄卻.

x	1	2	3	4	5	6	7
Λ	6	5	4	3	2	1	<1

$\because \Lambda(x)$ 為遞減函數

$\therefore C = \{x \mid 1 \le x \le c\}$

$Pr(x \in C \mid H_0) = 0.04$

故 $c=4$ $\because Pr(x \in \{1,2,3,4\} \mid H_0) = 0.01+0.01+0.01+0.01$
$= 0.04$

犯下型II錯誤之機率 $= Pr(x \notin \{1,2,3,4\} \mid H_1)$

$= Pr(x \in \{5,6,7\} \mid H_1) = 0.82$

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$\boxed{6}$ (Casella 8.39)

(a) $\begin{pmatrix} X_i \\ Y_i \end{pmatrix} \overset{iid}{\sim} N\left(\begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix}, \begin{pmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{pmatrix}\right)$

$$W_i = (1\ \ -1)\begin{pmatrix} X_{i} \\ Y_{i} \end{pmatrix} \sim N\left((1\ \ -1)\vec{\mu},\ (1\ \ -1)\Sigma\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right)$$

$$= N\Big(\underbrace{(\mu_X-\mu_Y)}_{\mu_W},\ \underbrace{(\sigma_X^2-2\rho\sigma_X\sigma_Y+\sigma_Y^2)}_{\sigma_W^2}\Big)$$

(b) 根據觀測值 $W_1, \dots W_n$ 來建立檢定

「$H_0: \mu_X=\mu_Y$ vs $H_1: \mu_X\neq\mu_Y$」

可以寫為：「$H_0: \mu_W=0$ vs $H_1: \mu_W\neq 0$」

利用概似比檢定....

① $\hat{\mu}_W=\overline{W}$, $\hat{\sigma}_W^2=\frac{1}{n}\sum_{i=1}^{n}(W_i-\overline{W})^2$

② $\hat{\mu}_{W_0}=0$, $\hat{\sigma}_{W_0}^2=\frac{1}{n}\sum_{i=1}^{n}W_i^2$

③ $$\Lambda=\frac{L(\hat{\mu}_{W_0}, \hat{\sigma}_{W_0}|W_1,\dots,W_n)}{L(\hat{\mu}_W, \hat{\sigma}_W^2|W_1\cdots W_n)} = \frac{\left(\frac{1}{\sqrt{2\pi\hat{\sigma}_0^2}}\right)^n \exp\Big(\underbrace{\frac{-1}{2\hat{\sigma}_{W_0}^2}\sum_{j=1}^{n}(W_j-\hat{\mu}_{W_0})^2}_{\exp(\frac{-n}{2})}\Big)}{\left(\frac{1}{\sqrt{2\pi\hat{\sigma}_W^2}}\right)^n \exp\Big(\underbrace{\frac{-1}{2\hat{\sigma}_W^2}\sum_{j=1}^{n}(W_j-\hat{\mu}_W)^2}_{\exp(\frac{-n}{2})}\Big)}$$

$$= \left(\frac{\hat{\sigma}_W^2}{\hat{\sigma}_{W0}^2}\right)^{\frac{n}{2}} < C \iff \underbrace{\frac{\hat{\sigma}_{W0}^2}{\hat{\sigma}_W^2} > C'}$$

$$\iff \frac{\frac{1}{n}\sum_{i=1}^{n}\left\{(W_i-\bar{W})+\bar{W}\right\}^2}{\frac{1}{n}\sum_{i=1}^{n}(W_i-\bar{W})^2} = \frac{\frac{1}{n}\sum_{i=1}^{n}(W_i-\bar{W})^2+\bar{W}^2}{\frac{1}{n}\sum_{i=1}^{n}(W_i-\bar{W})^2} > C'$$

$$\iff \frac{|\bar{W}|}{\sqrt{\frac{1}{n}\sum_{i=1}^{n}(W_i-\bar{W})^2}} > C'' \iff \frac{|\bar{W}|}{\sqrt{\frac{1}{n}S_W^2}} > C'''$$

$\downarrow$ 檢定統計量

另外，$\dfrac{\bar{W}}{\sqrt{\frac{1}{n}S_W^2}} = \dfrac{\sqrt{n}\,(\bar{W}/\sigma)}{\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}\left(\frac{W_i-\bar{W}}{\sigma}\right)^2}}$

$\sqrt{n}\cdot\frac{\bar{W}}{\sigma} \sim N(0,1)$, $\sum_{i=1}^{n}\left(\frac{W_i-\bar{W}}{\sigma}\right)^2 \sim \chi^2_{n-1}$ (under H_0)

且 $\sqrt{n}\left(\frac{\bar{W}}{\sigma}\right)$ & $\sum_{i=1}^{n}\left(\frac{W_i-\bar{W}}{\sigma}\right)^2$ 為獨立

根據t分佈的定義，$\dfrac{\bar{W}}{\sqrt{\frac{1}{n}S_W^2}} \sim t_{n-1}$ (under H_0)

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7 (Casella 8.40) (a)、(b) 寫在一起

首先考慮線性轉換：$\begin{pmatrix} u_j \\ v_j \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_j \\ y_j \end{pmatrix}$

這是可逆的線性轉換，故 $\left\{ \begin{pmatrix} u_j \\ v_j \end{pmatrix} \right\}_{j=1,2,\ldots,n}$ 亦為參數的充份統計量。根據 Sufficiency Principle，用 $\begin{pmatrix} u_j \\ v_j \end{pmatrix}$ 來推論，得到的結論為相同

$$\begin{pmatrix} U_j \\ V_j \end{pmatrix} \overset{iid}{\sim} N\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & \rho_0\sigma_1\sigma_2 \\ \rho_0\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} \right) \quad (\text{where } \mu_1 = \mu_X - \mu_Y)$$

我們考慮統計檢定：$H_0: \mu_1 = 0$ vs $H_1: \mu_1 \neq 0$

$$L(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho) = \prod_{j=1}^n f(v_j|u_j) \cdot f(u_j) = \underbrace{\prod_{j=1}^n f(v_j|u_j)}_{L_2(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)} \underbrace{\prod_{j=1}^n f(u_j)}_{L_1(\mu_1, \sigma_1^2)}$$

根據 ex 7.18，使得 $L_1(\mu_1, \sigma_1^2)$ 取最大值的 $\hat{\mu}_1, \hat{\sigma}_1^2$ 亦使得 $L_2(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$ 取最大值。

(*) 根據提示，無論 $(\hat{\mu}_1, \hat{\sigma}_1^2)$ or $(\hat{\hat{\mu}}_1, \hat{\hat{\sigma}}_1^2)$，$L_2$ 的最大值為相同。

故概似比 $\Lambda = \dfrac{L_2(\hat{\hat{\mu}}_1, \hat{\hat{\mu}}_2, \hat{\hat{\sigma}}_1^2, \hat{\hat{\sigma}}_2^2, \hat{\hat{\rho}}) L_1(\hat{\hat{\mu}}_1, \hat{\hat{\sigma}}_1^2)}{L_2(\hat{\mu}_1, \hat{\mu}_2, \hat{\sigma}_1^2, \hat{\sigma}_2^2, \hat{\rho}) L_1(\hat{\mu}_1, \hat{\sigma}_1^2)}$

$$= \frac{L_1(\hat{\hat{\mu}}_1, \hat{\hat{\sigma}}_1^2)}{L_1(\hat{\mu}_1, \hat{\sigma}_1^2)} \quad \left(\begin{array}{l} \hat{\theta}: \text{MLE with no restriction} \\ \hat{\hat{\theta}}: \text{MLE under } H_0 \end{array} \right)$$

故其檢定與 ex. 8.39 (b) 相同。

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但其實證明 ⊛ 的部分應該不簡單。
(L2 取相同最大值)

(如果真的可以做到，請給我看詳細的計算過程。)

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$\boxed{8}$ (Casella 8.41)

(a) $f(x_1 \cdots x_n, y_1 \cdots y_m) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x_i-\mu_X)^2\right) \cdot \prod_{j=1}^{m} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(y_j-\mu_Y)^2\right)$

① $l(\mu_X, \mu_Y, \sigma^2) = -\frac{1}{2}(n+m)\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu_X)^2 - \frac{1}{2\sigma^2}\sum_{j=1}^{m}(y_j-\mu_Y)^2$

$\frac{\partial l}{\partial \mu_X} = 0 \Rightarrow \hat{\mu}_X = \bar{X}$, $\frac{\partial l}{\partial \mu_Y} = 0 \Rightarrow \hat{\mu}_Y = \bar{Y}$, $\frac{\partial l}{\partial \sigma^2} = 0 \Rightarrow \hat{\sigma}^2 = \frac{1}{m+n}\left\{\sum_{i=1}^{n}(X_i-\bar{X})^2 + \sum_{j=1}^{m}(Y_j-\bar{Y})^2\right\}$

② $l(\mu, \sigma^2) = -\frac{1}{2}(n+m)\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}\left\{\sum_{i=1}^{n}(X_i-\mu)^2 + \sum_{j=1}^{m}(Y_j-\mu)^2\right\}$

同様 $\hat{\mu}_0 = \frac{n\bar{X}+m\bar{Y}}{n+m}$ $\hat{\sigma}_0^2 = \frac{1}{m+n}\left\{\sum_{i=1}^{n}\left(X_i - \frac{n\bar{X}+m\bar{Y}}{n+m}\right)^2 + \sum_{j=1}^{m}\left(Y_j - \frac{n\bar{X}+m\bar{Y}}{n+m}\right)^2\right\}$

③ $\Lambda = \frac{L(\hat{\mu}_0, \hat{\sigma}_0^2)}{L(\hat{\mu}_X, \hat{\mu}_Y, \hat{\sigma}^2)} = \left(\frac{\hat{\sigma}^2}{\hat{\sigma}_0^2}\right)^{\frac{m+n}{2}} < c \Leftrightarrow \frac{\hat{\sigma}_0^2}{\hat{\sigma}^2} > c'$

整理一下… $\frac{\hat{\sigma}_0^2}{\hat{\sigma}^2} = \frac{\sum_{i=1}^{n}\left(X_i - \bar{X} + \bar{X} - \frac{n\bar{X}+m\bar{Y}}{n+m}\right)^2 + \sum_{j=1}^{m}\left(Y_j - \bar{Y} + \bar{Y} - \frac{n\bar{X}+m\bar{Y}}{n+m}\right)^2}{\sum_{i=1}^{n}(X_i-\bar{X})^2 + \sum_{j=1}^{m}(Y_j-\bar{Y})^2}$

$= 1 + \frac{2\frac{nm(\bar{X}-\bar{Y})^2}{n+m}}{\sum_{i=1}^{n}(X_i-\bar{X})^2 + \sum_{j=1}^{m}(Y_j-\bar{Y})^2} > c'$

$\Leftrightarrow \frac{|\bar{X}-\bar{Y}|}{\sqrt{\frac{1}{n+m-2}\left\{\sum_{i=1}^{n}(X_i-\bar{X})^2 + \sum_{j=1}^{m}(Y_j-\bar{Y})^2\right\}\left(\frac{1}{n}+\frac{1}{m}\right)}} > c''$

故 $C = \{(X_1 \cdots X_n, Y_1 \cdots Y_m) \mid |T| \geq c''\}$

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(b) 注意 $\bar{X}, \bar{Y}$ 與 $\sum_{i=1}^{n}(X_i-\bar{X})^2$, $\sum_{i=1}^{m}(Y_i-\bar{Y})^2$ 獨立

$\mu_X=\mu_Y \Rightarrow \quad \bar{X}-\bar{Y} \sim N\left(0, \left(\frac{1}{n}+\frac{1}{m}\right)\sigma^2\right)$

$$\therefore \frac{(\bar{X}-\bar{Y})}{\sqrt{\left(\frac{1}{n}+\frac{1}{m}\right)\sigma^2}} \sim N(0,1)$$

$$\frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i-\bar{X})^2+\frac{1}{\sigma^2}\sum_{i=1}^{m}(Y_i-\bar{Y})^2 \sim \chi^2_{(n-1)+(m-1)} = \chi^2_{n+m-2}$$

根據 t 分布的定義，
$$\underbrace{\frac{(\bar{X}-\bar{Y})\Big/\sqrt{\left(\frac{1}{n}+\frac{1}{m}\right)\sigma^2}}{\sqrt{\frac{1}{n+m-2}\cdot\frac{1}{\sigma^2}\left\{\sum_{i=1}^{n}(X_i-\bar{X})^2+\sum_{i=1}^{m}(Y_i-\bar{Y})^2\right\}}}}_{= T} \sim t_{n+m-2}$$

$\therefore T \sim t_{n+m-2}$ (under H_0)

(c) 用電腦算：

$t=1.2907 \quad df=21$

$t_{21}(0.975)=2.079$

$\therefore$ not reject H_0

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$\boxed{9}$

$$-2\ln\Lambda_n = -n\ln\left(1-\frac{\Delta^2}{n}\right) - \frac{\Delta^2}{1-\frac{1}{n}\Delta^2}(\bar{X}^2+\bar{Y}^2) + \frac{2\Delta\cdot W}{1-\frac{1}{n}\Delta^2}$$

$$\begin{cases} \frac{1}{1-x} = 1+x+x^2+\cdots & (0<x<1) \\ -\ln(1-x) = x+\frac{x^2}{2}+\frac{x^3}{3}+\cdots & (0<x<1) \end{cases} \quad (\text{Taylor 展開})$$

• $-n\ln\left(1-\frac{\Delta^2}{n}\right) = n\left(\frac{\Delta^2}{n}+\frac{\Delta^4}{2n^2}+\frac{\Delta^6}{3n^3}+\cdots\right)$

$= \Delta^2 + \frac{\Delta^4}{2n} + \frac{\Delta^6}{3n^2} + \cdots$

$n\to\infty$ $-n\ln\left(1-\frac{\Delta^2}{n}\right) \approx \Delta^2$ ($\because$ 其他的項 機率收斂到 0)

• $\frac{\Delta^2}{1-\frac{1}{n}\Delta^2}(\bar{X}^2+\bar{Y}^2) = \Delta^2\cdot\left(1+\frac{\Delta^2}{n}+\frac{\Delta^4}{n^2}+\cdots\right)(\bar{X}^2+\bar{Y}^2)$

$\therefore n\to\infty$ $\frac{\Delta^2(\bar{X}^2+\bar{Y}^2)}{1-\frac{1}{n}\Delta^2} \approx \Delta^2(\bar{X}^2+\bar{Y}^2)$ (其他項 機率收斂到 0)

• $\frac{2\Delta W}{1-\frac{1}{n}\Delta^2} = 2\Delta W\left(1+\frac{\Delta^2}{n}+\frac{\Delta^4}{n^2}+\cdots\right)$

$n\to\infty$ $\frac{2\Delta W}{1-\frac{1}{n}\Delta^2} \approx 2\Delta W$ (其他項 機率收斂到 0)

$\therefore n\to\infty$ $\underline{-2\ln\Lambda_n \approx \Delta^2 - \Delta^2(\bar{X}^2+\bar{Y}^2) + 2\Delta W}$

注 $\bar{X}_2+\bar{Y}_2 \xrightarrow{P} 2$

(註) Slutsky 定理 $\begin{cases} ① & X_n - Y_n \xrightarrow{p} 0 \quad X_n \xrightarrow{d} X \Rightarrow Y_n \xrightarrow{d} Y \\ ② & X_n \xrightarrow{d} X \quad Y_n \xrightarrow{d} a \Rightarrow X_n Y_n \xrightarrow{d} aX \end{cases}$

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注意 $\Delta - W \xrightarrow{p} 0$, $W \xrightarrow{d} N(0,1)$ (∵ 中央極限定理; under $\rho=0$)

- $\{\Delta^2 - \Delta^2(\overline{X^2}+\overline{Y^2}) + 2\Delta W\} - \{W^2 - W^2(\overline{X^2}+\overline{Y^2}) + 2W^2\} \xrightarrow{p} 0$

(∵ $\Delta - W \xrightarrow{p} 0$ & slutsky ②)

- $\{W^2 - W^2(\overline{X^2}+\overline{Y^2}) + 2W^2\} - \{W^2\} \xrightarrow{p} 0$

$$\| $$

$$\underbrace{W^2}_{\to \chi^2(1)} \underbrace{(2 - \overline{X^2}+\overline{Y^2})}_{\to 0 \text{ (in probability)}}$$

(∵ 弱大數法則 $\overline{X^2}+\overline{Y^2} \xrightarrow{p} 2$)

由此可知 $\Delta^2 - \Delta^2(\overline{X^2}+\overline{Y^2}) + 2\Delta W - W^2 \xrightarrow{p} 0$

故考慮 W^2 的漸近分布:

$X_n \xrightarrow{d} X \Rightarrow g(X_n) \xrightarrow{d} g(X)$ (g: 連續函數)

∵ $W \xrightarrow{d} N(0,1) \Rightarrow W^2 \xrightarrow{d} \chi^2(1)$

故 $\Delta^2 - \Delta^2(\overline{X^2}+\overline{Y^2}) + 2\Delta W \xrightarrow{d} \chi^2(1)$ (Slutsky ①)

∴ $-2\ln\Lambda_n \xrightarrow{d} \chi^2(1)$　　∴ 證明完成

No. 資料科學之統計基礎 (I)

Date 9.2., 9.4., 9.34 9.35. 9.52. 11.4

作業9. R05246013 森元俊成

1 (Casella 9.2)

信賴區間 (95%) 為真實的 θ 落在裡面的機率為 0.95 的區間。(而不是新的樣本落在裡面之機率為 0.95 的區間)

$$p = P_r\left(\bar{x} - 1.96/\sqrt{n} \leq X_{n+1} \leq \bar{x} + 1.96/\sqrt{n}\right)$$

$$= P_r\left(\frac{-1.96}{\sqrt{n}} \leq \underbrace{\bar{x} - X_{n+1}}_{} \leq \frac{1.96}{\sqrt{n}}\right)$$

$$\bar{X} \sim N(\theta, \tfrac{1}{n}) \quad X_{n+1} \sim N(\theta, 1)$$

$$\therefore \bar{X} - X_{n+1} \sim N(0, \tfrac{1}{n} + 1)$$

$$= P_r\left(\frac{\frac{-1.96}{\sqrt{n}}}{\sqrt{\frac{n+1}{n}}} \leq \frac{\bar{X} - X_{n+1}}{\sqrt{\frac{n+1}{n}}} \leq \frac{\frac{1.96}{\sqrt{n}}}{\sqrt{\frac{n+1}{n}}}\right)$$

$$= \Phi\left(\frac{1.96}{\sqrt{n+1}}\right) - \Phi\left(\frac{-1.96}{\sqrt{n+1}}\right) < \underbrace{\Phi(1.96)}_{0.975} - \underbrace{\Phi(-1.96)}_{0.025} = 0.95$$

$\therefore$ p 小於 0.95

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$\boxed{2}$ (Casella 9.4)

(a) $L(\sigma_X^2, \sigma_Y^2 \mid X_1,\dots,X_n, Y_1,\dots,Y_m)$

$$\prod_{j=1}^{n} \frac{1}{\sqrt{2\pi\sigma_X^2}} \exp\left(\frac{-X_j^2}{2\sigma_X^2}\right) \cdot \prod_{k=1}^{m} \frac{1}{\sqrt{2\pi\sigma_Y^2}} \exp\left(\frac{-Y_k^2}{2\sigma_Y^2}\right)$$

$\therefore\ \ell(\sigma_X^2, \sigma_Y^2 \mid X_1,\dots,X_n, Y_1,\dots,Y_m)$

$$= \frac{-1}{2}(n+m)\ln(2\pi) - \frac{n}{2}\ln\sigma_X^2 - \frac{m}{2}\ln\sigma_Y^2 - \sum_{j=1}^{n}\frac{X_j^2}{2\sigma_X^2} - \sum_{k=1}^{m}\frac{Y_k^2}{2\sigma_Y^2}$$

$$\frac{\partial \ell}{\partial \sigma_X^2} = \frac{-n}{2\sigma_X^2} + \frac{1}{2\sigma_X^4}\sum_{j=1}^{n} X_j^2 = 0 \quad \therefore\ \hat{\sigma}_X^2 = \frac{1}{n}\sum_{j=1}^{n} X_j^2$$

$$\frac{\partial \ell}{\partial \sigma_Y^2} = \frac{-m}{2\sigma_Y^2} + \frac{1}{2\sigma_Y^4}\sum_{k=1}^{m} Y_k^2 = 0 \qquad \hat{\sigma}_Y^2 = \frac{1}{m}\sum_{k=1}^{m} Y_k^2$$

接下來考慮限制條件下的MLE ($\sigma_Y^2 = \lambda_0 \sigma_X^2$

$L(\sigma_X^2 \mid \lambda_0, X_1,\dots,X_n, Y_1,\dots,Y_m)$

$$\prod_{j=1}^{n} \frac{1}{\sqrt{2\pi\sigma_X^2}} \exp\left(\frac{-X_j^2}{2\sigma_X^2}\right) \cdot \prod_{k=1}^{m} \frac{1}{\sqrt{2\pi\lambda_0\sigma_X^2}} \exp\left(\frac{-Y_k^2}{2\lambda_0\sigma_X^2}\right)$$

$\ell(\sigma_X^2 \mid \lambda_0, X_1,\dots,X_n, Y_1,\dots,Y_m)$

$$= \frac{-1}{2}(n+m)\ln(2\pi) - \frac{m+n}{2}\ln\sigma_X^2 - \frac{m}{2}\ln\lambda_0 - \sum_{j=1}^{n}\frac{X_j^2}{2\sigma_X^2} - \sum_{k=1}^{m}\frac{Y_k^2}{2\lambda_0\sigma_X^2}$$

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$$\frac{\partial \ell}{\partial \sigma_X^2} = \frac{-(m+n)}{2\sigma_X^2} + \frac{1}{2\sigma_X^4}\sum_{j=1}^{n} X_j^2 + \frac{1}{2\lambda_0 \sigma_X^4}\sum_{k=1}^{m} Y_k^2 = 0$$

$$\therefore \tilde{\sigma}_X^2 = \frac{1}{m+n}\left(\sum_{j=1}^{n} X_j^2 + \frac{1}{\lambda_0}\sum_{k=1}^{m} Y_k^2\right)$$

$$\Lambda = \frac{L(\tilde{\sigma}_X^2, \lambda_0 \tilde{\sigma}_X^2 \mid X_1 \ldots X_n, Y_1 \ldots Y_m)}{L(\hat{\sigma}_X^2, \hat{\sigma}_Y^2 \mid X_1 \ldots X_n, Y_1 \ldots Y_m)} =$$

分子: $\left(\frac{1}{\sqrt{2\pi}}\right)^{n+m} \cdot \left(\frac{1}{\tilde{\sigma}_X^2}\right)^{\frac{n+m}{2}} \cdot \left(\frac{1}{\lambda_0}\right)^{\frac{m}{2}} \cdot \exp\left(\frac{-1}{2\tilde{\sigma}_X^2}\underbrace{\left(\sum_{j=1}^{n} X_j^2 + \frac{1}{\lambda_0}\sum_{k=1}^{m} Y_k^2\right)}_{(m+n)\tilde{\sigma}_X^2}\right)$

分母: $\left(\frac{1}{\sqrt{2\pi}}\right)^{n+m} \cdot \left(\frac{1}{\hat{\sigma}_X^2}\right)^{\frac{n}{2}} \cdot \left(\frac{1}{\hat{\sigma}_Y^2}\right)^{\frac{m}{2}} \cdot \exp\left(\frac{-1}{2\hat{\sigma}_X^2}\underbrace{\sum_{j=1}^{n} X_j^2}_{n\hat{\sigma}_X^2} - \frac{1}{2\hat{\sigma}_Y^2}\underbrace{\sum_{k=1}^{m} Y_k^2}_{m\hat{\sigma}_Y^2}\right)$

$$\Lambda = \frac{(\hat{\sigma}_X^2)^{\frac{n}{2}}(\hat{\sigma}_Y^2)^{\frac{m}{2}}}{(\tilde{\sigma}_X^2)^{\frac{m+n}{2}} \cdot \lambda_0^{\frac{m}{2}}} < c$$

為了方便起見，令 $W_j \stackrel{def}{=} X_j^2$, $Z_k = \frac{Y_k^2}{\lambda_0}$.

$$\Lambda = \frac{(\overline{W})^{\frac{n}{2}}(\overline{Z})^{\frac{m}{2}}}{\left(\frac{n\overline{W}+m\overline{Z}}{m+n}\right)^{\frac{m+n}{2}}\lambda_0^{\frac{m}{2}}} < c$$

$$T \stackrel{def}{=} \left(\frac{n\overline{W}}{n\overline{W}+m\overline{Z}}\right)$$

$$\Lambda < c \iff T^{\frac{n}{2}}(1-T)^{\frac{m}{2}} < c'$$

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故 $C=\{(X_1,\dots,X_n,Y_1,\dots,Y_m) \mid T<a,\ b<T,$ where $a^{\frac{n}{2}}(1-a)^{\frac{m}{2}} = b^{\frac{n}{2}}(1-b)^{\frac{m}{2}}\}$

$$T=\frac{X_1^2+\dots X_n^2}{X_1^2+\dots X_n^2+\frac{Y_1^2}{\lambda_0}+\dots\frac{Y_m^2}{\lambda_0}} \quad (a<b)$$

且 $a.b$ 滿足 $Pr((X_1,\dots X_n,Y_1,\dots Y_m)\in C \mid \lambda=\lambda_0) \leq \alpha$

(b) 考慮 $\lambda=\lambda_0$ 為真時狀況:

「$0\leq T<a,\quad b<T\leq 1$」 $\Leftrightarrow$ 「$\frac{1}{a}<\frac{1}{T},\ 1\leq\frac{1}{T}<\frac{1}{b}$」

$$(1<)\ \frac{1}{a}<1+\frac{\frac{Y_1^2}{\lambda_0}+\dots+\frac{Y_m^2}{\lambda_0}}{X_1^2+\dots+X_n^2},\quad 1\leq 1+\frac{\frac{Y_1^2}{\lambda_0}+\dots+\frac{Y_m^2}{\lambda_0}}{X_1^2+\dots+X_n^2}<\frac{1}{b}$$

$$\Leftrightarrow \frac{n}{m}\left(\frac{1}{a}-1\right)<\frac{(Y_1^2+\dots Y_m^2)/\lambda_0\sigma_X^2\cdot m}{(X_1^2+\dots X_n^2)/\sigma_X^2\cdot n},\quad 0\leq\frac{(Y_1^2+\dots+Y_m^2)/\lambda_0\sigma_X^2\cdot m}{(X_1^2+\dots+X_n^2)/\sigma_X^2\cdot n}<\frac{n}{m}\cdot\left(\frac{1}{b}-1\right)$$

$$\therefore C=\left\{(X_1,\dots,X_n,Y_1,\dots Y_m) \;\middle|\; 0\leq\frac{\left(\frac{Y_1^2+\dots Y_m^2}{m\lambda_0}\right)}{\left(\frac{X_1^2+\dots X_n^2}{n}\right)}<\frac{n}{m}\left(\frac{1}{b}-1\right),\ \frac{n}{m}\left(\frac{1}{a}-1\right)<\frac{\frac{Y_1^2+\dots Y_m^2}{m\lambda_0}}{\left(\frac{X_1^2+\dots X_n^2}{n}\right)}\right\}$$

where $\dfrac{\left(\frac{Y_1^2+\dots+Y_m^2}{m\lambda_0}\right)}{\left(\frac{X_1^2+\dots+X_n^2}{n}\right)}\sim F_{mn}$ (under H_0)

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(C) 利用概似比検定

$$Pr\left((X_1, \cdots X_n, Y_1, \cdots, Y_m) \notin C \mid \lambda = \lambda_0 \right) = 1 - \alpha$$

$$Pr\left(\frac{n}{m}\left(\frac{1}{b} - 1\right) \leq \frac{\frac{Y_1^2 + \cdots Y_m^2}{m} \cdot \frac{1}{\lambda_0}}{\frac{X_1^2 + \cdots X_n^2}{n}} \leq \frac{n}{m} \cdot \left(\frac{1}{a} - 1\right) \mid \lambda = \lambda_0 \right) = 1 - \alpha$$

$$\frac{Y_1^2 + \cdots Y_m^2}{X_1^2 + \cdots X_n^2} \cdot \left(\frac{1}{a} - 1\right)^{-1} \leq \lambda_0 \leq \frac{Y_1^2 + \cdots + Y_m^2}{X_1^2 + \cdots + X_n^2} \left(\frac{1}{b} - 1\right)^{-1}$$

$$\therefore \left[\frac{Y_1^2 + \cdots + Y_m^2}{X_1^2 + \cdots + X_n^2} \cdot \frac{a}{1-a}, \frac{Y_1^2 + \cdots + Y_m^2}{X_1^2 + \cdots + X_n^2} \cdot \frac{b}{1-b} \right]$$

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$\boxed{3}$ (casella 9.34) $X_1,\cdots,X_n \sim N(\mu,\sigma^2)$

(a) σ^2：已知 $\cdots$ $\sqrt{n}\left(\frac{\bar{X}-\mu}{\sigma}\right) \sim N(0,1)$

$$Pr\left(-1.96 \leq \frac{\sqrt{n}(\bar{X}-\mu)}{\sigma} \leq 1.96\right) = 0.95$$

$$\Leftrightarrow Pr\left(-1.96 \cdot \frac{\sigma}{\sqrt{n}} \leq \bar{X}-\mu \leq 1.96 \cdot \frac{\sigma}{\sqrt{n}}\right)$$

$$\Leftrightarrow Pr\left(\bar{X}-1.96 \cdot \frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{X}+1.96 \cdot \frac{\sigma}{\sqrt{n}}\right)$$

Length： $3.92\frac{\sigma}{\sqrt{n}} \leq \frac{\sigma}{4}$

$$\therefore n \geq (3.92 \cdot 4)^2 \approx 245.86$$

$$\therefore n \geq 246$$

(b) σ^2 未知 $\cdots$ 利用 t 檢定

$$Pr\left(-t_{n-1}^{-1}(0.975) \leq \frac{\left(\frac{\sqrt{n}(\bar{X}-\mu)}{\sigma}\right)}{\sqrt{\frac{1}{n-1}\frac{1}{\sigma}\sum_{i=1}^{n}(X_i-\bar{X})^2}} \leq t_{n-1}^{-1}(0.975)\right) = 0.95$$

$$Pr\left(-t_{n-1}^{-1}(0.975) \leq \frac{\sqrt{n}(\bar{X}-\mu)}{\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X})^2}} \leq t_{n-1}^{-1}(0.975)\right) = 0.95$$

$$Pr\left(\mu \in \left[\bar{X}-\frac{t_{n-1}^{-1}(0.975)}{\sqrt{n}}\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X})^2},\ \bar{X}+\frac{t_{n-1}^{-1}(0.975)}{\sqrt{n}}\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X})^2}\right]\right.$$

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故 $\text{Length} = \frac{2}{\sqrt{n}} t_{n-1}^{-1} \sqrt{S_X^2} \leq \frac{6}{4}$

$\Leftrightarrow \quad \frac{\sqrt{S_X^2}}{6} \leq \frac{\sqrt{n}}{8} \cdot \frac{1}{t_{n-1}}$

$\Leftrightarrow \quad \underbrace{\frac{1}{6^2} \sum_{i=1}^{n} (X_i - \bar{X})^2}_{\chi^2_{n-1}} \leq \frac{n(n-1)}{64} \cdot \frac{1}{t_{n-1}^2}$

故求最小的 n 使得

$$\Pr\left(\underbrace{\frac{1}{6^2} \sum_{i=1}^{n} (X_i - \bar{X})^2}_{\sim \chi^2_{n-1}} \leq \frac{n(n-1)}{64} \cdot \frac{1}{t_{n-1}^{-1}(0.975)^2} \right) \geq 0.9$$

(需要用電腦算)

$\boxed{4}$ (Casella 9.35)

考慮 $\frac{(b)}{(a)}$：（我們考慮大樣本的情況）

$(a) = 2\cdot\Phi^{-1}(1-\frac{\alpha}{2})\frac{\sigma}{\sqrt{n}}$

$(b) = 2\cdot t_{n-1}^{-1}(1-\frac{\alpha}{2})\cdot\frac{1}{\sqrt{n}}\cdot\sqrt{S_x^2}$

n 很大時，t_n（自由度 n 的 t 分布）$\xrightarrow{d} N(0,1)$

故 $\Phi(1-\frac{\alpha}{2}) \approx t_{n-1}(1-\frac{\alpha}{2}) \Rightarrow \Phi^{-1}(1-\frac{\alpha}{2}) = t_{n-1}^{-1}(1-\frac{\alpha}{2})$

$$\left(\frac{(b)}{(a)}\right)^2 = \frac{1}{n-1}\underbrace{\sum_{i=1}^{n}\left(\frac{X_i-\bar{X}}{\sigma}\right)^2}_{\chi^2_{n-1}} = \frac{1}{n-1}(Z_1^2+\dots+Z_{n-1}^2)$$

(where $Z_1 \sim Z_{n-1} \sim N(0,1)$)

根據弱大數法則，$\left(\frac{(b)}{(a)}\right)^2 \xrightarrow{P} 1$

∴ 在漸近的情況下，$(a) \approx (b)$ ⌟

但題目似乎要求比較 (a) vs (b) 的期望值

$$(b) = 2t_{n-1}^{-1}(1-\frac{\alpha}{2})\frac{\sigma}{\sqrt{n}}\left\{\left(\frac{S_x}{\sigma}\right)^2\right\}^{\frac{1}{2}}$$

$$T \overset{def}{=} \frac{(n-1)S_x^2}{\sigma^2} \sim \chi^2_{n-1}$$

$$(b) = 2t_{n-1}^{-1}(1-\frac{\alpha}{2})\cdot\frac{\sigma}{\sqrt{n}}\cdot\left(\frac{T}{n-1}\right)^{\frac{1}{2}} = 2t_{n-1}^{-1}(1-\frac{\alpha}{2})\cdot\frac{\sigma}{\sqrt{n}}\cdot\frac{1}{\sqrt{n-1}}\cdot T^{\frac{1}{2}}$$

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$T \sim \Gamma\left(\frac{n-1}{2}, \frac{1}{2}\right)$

$$E\left[T^{\frac{1}{2}}\right] = \int_0^\infty t^{\frac{1}{2}} \cdot \frac{t^{\frac{n-1}{2}-1}}{\Gamma\left(\frac{n-1}{2}\right) \cdot 2^{\frac{n-1}{2}}} \exp\left(\frac{-t}{2}\right) dt$$

$$= \int_0^\infty \frac{t^{\frac{n}{2}-1}}{\Gamma\left(\frac{n-1}{2}\right) 2^{\frac{n-1}{2}}} \exp\left(\frac{-t}{2}\right) dt \qquad x = \frac{t}{2} \quad \frac{dx}{dt} = \frac{1}{2}$$

$$= \int_0^\infty \frac{(2x)^{\frac{n}{2}-1}}{\Gamma\left(\frac{n-1}{2}\right) \cdot 2^{\frac{n-1}{2}}} \exp(-x) \cdot 2dx$$

$$= \int_0^\infty \frac{2^{\frac{n}{2}} x^{\frac{n}{2}-1}}{\Gamma\left(\frac{n-1}{2}\right) 2^{\frac{n-1}{2}}} \exp(-x) dx = \frac{\sqrt{2} \cdot \Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}$$

$$\therefore E[(b)] = 2 t_{n-1}^{-1}\left(1-\frac{\alpha}{2}\right) \cdot \frac{\sigma}{\sqrt{n}} \cdot \underbrace{\frac{1}{\sqrt{n-1}} \cdot \frac{\sqrt{2}\, \Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}}$$

$$E[(a)] = 2 \Phi^{-1}\left(1-\frac{\alpha}{2}\right) \cdot \frac{\sigma}{\sqrt{n}}$$

同樣考慮 $n \to \infty$ 的情況. $t_n^{-1}\left(1-\frac{\alpha}{2}\right) \to \Phi^{-1}\left(1-\frac{\alpha}{2}\right)$

$$\frac{\sqrt{2}\, \Gamma\left(\frac{n}{2}\right)}{\sqrt{n-1}\, \Gamma\left(\frac{n-1}{2}\right)}$$: 利用 Stirling's Formula 近似

$\to 1 \quad (\text{as } n \to \infty)$

$\therefore (a) \approx (b)$

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$\boxed{5}$ (Casella 9.52)

(a) 利用概似比検定

① H_0 下的 MLE: $\hat{\mu}_0 = \bar{X}$

$\hat{\sigma}_0^2 = \sigma_0^2$

② MLE: $\hat{\mu} = \bar{X}$

$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X})^2$

③ $$\Lambda = \frac{L(\hat{\mu}_0, \hat{\sigma}_0)}{L(\hat{\mu}, \hat{\sigma})} = \frac{\left(\frac{1}{\sqrt{2\pi\sigma_0^2}}\right)^n \exp\left(-\frac{1}{2\sigma_0^2}\sum_{i=1}^{n}(X_i-\bar{X})^2\right)}{\left(\frac{1}{\sqrt{2\pi\hat{\sigma}^2}}\right)^n \exp\left(-\frac{n}{2}\right)} < C$$

$$= \left(\frac{\hat{\sigma}^2}{\sigma_0^2}\right)^{\frac{n}{2}} \exp\left(\frac{-n\hat{\sigma}^2}{2\sigma_0^2}\right) < C'$$

$$\left(t \overset{def}{=} \frac{n\hat{\sigma}^2}{\sigma_0^2}\right) \Leftrightarrow \frac{t^{\frac{n+2}{2}-1}}{\Gamma\left(\frac{n+2}{2}\right) 2^{\frac{n+2}{2}}} \exp\left(\frac{-t}{2}\right) < C''$$

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$\therefore C = \{(X_1, \dots, X_n) \mid t < a; \quad b < t\}$

$f_{n+2}(a) = f_{n+2}(b)$

$\therefore$ 信賴區間: $a \leq t \leq b \Leftrightarrow a \leq \frac{n\hat{\sigma}^2}{\sigma_0^2} \leq b$

$\Leftrightarrow \frac{n\hat{\sigma}^2}{b} \leq \sigma_0^2 \leq \frac{n\hat{\sigma}^2}{a} \Leftrightarrow \frac{(n-1)S^2}{b} \leq \sigma_0^2 \leq \frac{(n-1)S^2}{b}$

$\therefore$ 證明完成

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(b) 將 $\frac{(n-1)S^2}{a} - \frac{(n-1)}{b}S^2$ 最小化.

且滿足 $\int_a^b f_{n-1}(t)\,dt = 1-\alpha$

利用 Lagrange 未定乘數法:

$$g(a,b) = \frac{(n-1)S^2}{a} - \frac{(n-1)}{b}S^2 - \lambda\left(\int_a^b f_{n-1}(t)\,dt - (1-\alpha)\right) = 0$$

$$\begin{cases} \frac{\partial g}{\partial a} = -\frac{S_{xx}}{a^2} + \lambda f_{n-1}(a) = 0 \\ \frac{\partial g}{\partial b} = \frac{S_{xx}}{b^2} - \lambda f_{n-1}(b) = 0 \end{cases}$$

故 $\lambda = \frac{S_{xx}}{a^2 f_{n-1}(a)} = \frac{S_{xx}}{b^2 f_{n-1}(b)}$

$\therefore\ a^2 f_{n-1}(a) = b^2 f_{n-1}(b)$

$$\frac{a^{\frac{n-1}{2}+1}}{\Gamma\left(\frac{n-1}{2}\right)\cdot 2^{\frac{n-1}{2}}}\exp\left(-\frac{a}{2}\right) = \frac{b^{\frac{n-1}{2}+1}}{\Gamma\left(\frac{n-1}{2}\right) 2^{\frac{n-1}{2}}}\exp\left(\frac{-b}{2}\right)$$

$$\| \qquad\qquad \|$$

$$\frac{a^{\frac{n+3}{2}-1}}{\Gamma\left(\frac{n-1}{2}\right)\cdot 2^{\frac{n-1}{2}}}\exp\left(\frac{-a}{2}\right) = \frac{b^{\frac{n+3}{2}-1}}{\Gamma\left(\frac{n-1}{2}\right)\cdot 2^{\frac{n-1}{2}}}\exp\left(\frac{b}{2}\right)$$

$$\Leftrightarrow \frac{a^{\frac{n+3}{2}-1}}{\Gamma\left(\frac{n+3}{2}\right) 2^{\frac{n+3}{2}}}\exp\left(\frac{a}{2}\right) = \frac{b^{\frac{n+3}{2}-1}}{\Gamma\left(\frac{n+3}{2}\right) 2^{\frac{n+3}{2}}}\exp\left(\frac{b}{2}\right)$$

$\therefore\ f_{n+3}(a) = f_{n+3}(b).$

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(c) 將 $\frac{b}{a}$ 最小化

$\therefore g\left(\frac{b}{a}\right) = \frac{b}{a} - \lambda\left(\int_a^b f_{n-1}(t)\,dt - (1-\alpha)\right)$

$$\frac{\partial g}{\partial a} = \frac{-b}{a^2} + \lambda f_{n-1}(a) = 0$$

$$\frac{\partial g}{\partial b} = \frac{1}{a} - \lambda f_{n-1}(b) = 0$$

$$\therefore \lambda = \frac{b}{a^2 f_{n-1}(a)} = \frac{1}{f_{n-1}(b)\cdot a}$$

$$\therefore a f_{n-1}(a) = b f_{n-1}(b)$$

$$\therefore \frac{a \cdot a^{\frac{n-1}{2}-1} \exp\left(\frac{-a}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right) 2^{\frac{n-1}{2}}} = \frac{b \cdot b^{\frac{n-1}{2}-1} \exp\left(\frac{-b}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right) 2^{\frac{n-1}{2}}}$$

$$\Leftrightarrow \frac{a^{\frac{n+1}{2}-1} \exp\left(\frac{-a}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right) 2^{\frac{n+1}{2}}} = \frac{b^{\frac{n+1}{2}-1} \exp\left(\frac{-b}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right) 2^{\frac{n+1}{2}}}$$

$$\therefore f_{n+1}(a) = f_{n+1}(b)$$

(d) 題目已經給答案

(e) 省略.

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$\boxed{6}$ (Casella 11.4)

(a) $\sqrt{n}(\hat{\rho}-\rho) \xrightarrow{d} N(0, (1-\rho^2)^2)$

$g(x) \stackrel{\text{def}}{=} \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \quad g'(x) = \frac{1}{1-x^2}$

$\sqrt{n}(g(\hat{\rho})-g(\rho)) \xrightarrow{d} N(0, \underbrace{(1-\rho^2)^2 \cdot (g'(\rho))^2}_{=1}) \quad (= N(0,1))$

$\therefore g(\hat{\rho})-g(\rho) \xrightarrow{d} N(0, \frac{1}{n})$

$\therefore$ 證明完成

(b) 期望值 $\frac{\rho}{2(n-1)}$, 變異: $\frac{1}{n-1} + \frac{4-\rho^2}{2(n-1)^2}$

$\rho \approx 0$: mean ≈ 0

$\text{variance} \approx \frac{1}{n-1} + \frac{4}{2(n-1)^2} = \frac{1}{(n-1)} + \frac{2}{(n-1)^2}$

$= \frac{n+1}{(n-1)^2} = \frac{1}{(n-1)^2/(n+1)}$

$= \frac{1}{(n-3) + \frac{4}{n+1}}$

$n-3 \gg \frac{4}{n+1}$ (n 夠大) 故 $\text{Variance} \approx \frac{1}{n-3}$

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Date 資料科學的統計基礎(I)　　11.12, 11.14, 11.33, 11.35

HW10　R05246013 森元俊成

[1] (#11.12) $Y_{ij} \sim N(\theta_i, \sigma^2)$ independent $(i=1,2,\dots,k,\ j=1,2,\dots,l)$ $(n=kl)$

$$\begin{cases} H_0: \theta_1=\theta_2 \\ H_1: \theta_1 \neq \theta_2 \end{cases}$$ $(i=1\sim k)$ $(i\to1\ \ i\to2)$

利用概似比檢定來建立拒絕域

$$L(\theta_1,\dots,\theta_k,\sigma^2) = \prod_{i=1}^{k}\prod_{j=1}^{l} f(x_{ij}\mid\theta_i,\sigma^2)$$

$$= \prod_{i=1}^{k}\prod_{j=1}^{l} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x_{ij}-\theta_i)^2\right)$$

$$l(\theta_1,\dots,\theta_k,\sigma^2) = \sum_{i=1}^{k}\sum_{j=1}^{l}\left\{-\frac{1}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}(x_{ij}-\theta_i)^2\right\}$$

$$\frac{\partial l}{\partial\theta_i} = \sum_{j=1}^{l}\frac{1}{\sigma^2}(x_{ij}-\theta_i) = 0 \quad \therefore \hat{\theta}_i = \bar{x}_i$$

$$\frac{\partial l}{\partial\sigma^2} = \sum_{i=1}^{k}\sum_{j=1}^{l}\left(-\frac{1}{2\sigma^2} + \frac{1}{2\sigma^4}(x_{ij}-\theta_i)^2\right) = 0$$

$$\Leftrightarrow n\sigma^2 = \sum_{i=1}^{k}\sum_{j=1}^{l}(x_{ij}-\theta_i)^2$$

$$\therefore \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{k}\sum_{j=1}^{l}(x_{ij}-\bar{x}_{i})^2$$

接下來考慮 H_0 下的 MLE

$$\theta^* \overset{\text{def}}{=} \theta_1=\theta_2$$

$$l(\theta^*, \theta_3 \cdots \theta_k, \sigma^2) = \sum_{i=3}^{k}\sum_{j=1}^{l}\left\{-\frac{1}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}(x_{ij}-\theta_i)^2\right\}$$

$$+ \sum_{j=1}^{l}\left\{-\frac{1}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}(x_{1j}-\theta^*)^2\right\}$$

$$+ \sum_{j=1}^{l}\left\{-\frac{1}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}(x_{j}-\theta^*)^2\right\}$$

$$\frac{\partial l}{\partial \theta^*} = \frac{1}{\sigma^2}\sum_{j=1}^{l}\left\{(x_{1j}+x_{2j}-2\theta^*)\right\} = 0$$

$$\hat{\theta}^* = \frac{\bar{x}_{1\cdot}+\bar{x}_{2\cdot}}{2}$$

$$\frac{\partial l}{\partial \theta_i} = 0 \Rightarrow \hat{\theta}_i = \bar{x}_{i\cdot} \quad (i \geq 3)$$

$$\frac{\partial l}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \sum_{i=3}^{k}\sum_{j=1}^{l}\frac{1}{2\sigma^4}(x_{ij}-\theta_i)^2 + \sum_{j=1}^{l}\left\{\frac{1}{2\sigma^4}(x_{1j}-\theta^*)^2 + \frac{1}{2\sigma^4}(x_{2j}-\theta^*)^2\right\} = 0$$

$$\hat{\sigma}_0^2 = \frac{1}{n}\sum_{j=1}^{l}\left\{(x_{1j}-\hat{\theta}^*)^2 + (x_{2j}-\hat{\theta}^*)^2\right\} + \frac{1}{n}\sum_{i=3}^{k}\sum_{j=1}^{l}(x_{ij}-\hat{\theta}_i)^2$$

考慮 $\Lambda < C$ 的範圍：（注意 $\exp(\sim)$ 的部分變成 $\exp(-n)$）

$$\therefore \left(\frac{\hat{\sigma}^2}{\hat{\sigma}_0^2}\right)^{\frac{n}{2}} < C \iff \frac{\hat{\sigma}_0^2}{\hat{\sigma}^2} > C'$$

$$\frac{\sum_{j=1}^{l}(x_{1j}-\hat{\theta}^*)^2 + \sum_{j=1}^{l}(x_{2j}-\hat{\theta}^*)^2 + \sum_{i=3}^{k}\sum_{j=1}^{l}(x_{ij}-\bar{x}_{i\cdot})^2}{\sum_{i=1}^{k}\sum_{j=1}^{l}(x_{ij}-\bar{x}_{i\cdot})^2} > C'$$

$$\iff \frac{\sum_{i=1}^{k}\sum_{j=1}^{l}(x_{ij}-\bar{x}_{i\cdot})^2 + \sum_{j=1}^{l}\left\{(\bar{x}_{1\cdot}-\hat{\theta}^*)^2 + (\bar{x}_{2\cdot}-\hat{\theta}^*)^2\right\}}{\sum_{i=1}^{k}\sum_{j=1}^{l}(x_{ij}-\bar{x}_{i\cdot})^2} > C'$$

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$$1+\frac{\sum_{j=1}^{\ell} 2\cdot \frac{1}{4}(\bar{x}_{1\cdot}-\bar{x}_{2\cdot})^2}{\sum_{i=1}^{k}\sum_{j=1}^{\ell}(x_{ij}-\bar{x}_{i\cdot})^2} > c'$$

故 $$\frac{\frac{\ell}{2}(\bar{x}_{1\cdot}-\bar{x}_{2\cdot})^2}{\sum_{i=1}^{k}\sum_{j=1}^{\ell}(x_{ij}-\bar{x}_{i\cdot})^2} > c''$$

$\Leftrightarrow$ $$\frac{(\bar{x}_{1\cdot}-\bar{x}_{2\cdot})^2}{\underbrace{S_p^2\left(\frac{2}{n}\right)}_{t_{12}}} > c'''$$

(b) 證明 $\frac{1}{h(h-1)}\sum_{(i,i')\in\{1,2,\dots,h\}} t_{ii'}$ 為F統計量 $(t_{ii'}=0)$

我們利用 $\frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X})^2=\frac{1}{2n(n-1)}\sum_{i=1}^{n}\sum_{i'=1}^{n}(X_{i'}-X_i)^2$

故 $$\frac{1}{k-1}\sum_{i=1}^{k}(\bar{x}_{i\cdot}-\bar{x}_{\cdot\cdot})^2=\frac{1}{2k(k-1)}\sum_{i=1}^{k}\sum_{i'=1}^{\ell}\underbrace{(\bar{x}_{i\cdot}-\bar{x}_{i'\cdot})^2}_{t_{ij}^2\cdot S_p^2\cdot\frac{2}{\ell}}$$

$$=\frac{S_p^2}{k(k-1)\ell}\sum_{(i,j)} t_{ij}^2$$

故 $$F=\frac{\frac{1}{k-1}\sum_{i=1}^{k}(\bar{x}_{i\cdot}-\bar{x}_{\cdot\cdot})^2}{\frac{1}{\ell}S_p^2}$$

由此可知 F確實為 ANOVA 的 F統計量.

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$\boxed{2}$ (Casella 11.14)

$$\sum_{i=1}^{k} a_i \bar{Y}_{i\cdot} - M\sqrt{S_p^2 \sum_{i=1}^{k} \frac{a_i^2}{n_i}} \le \sum_{i=1}^{k} a_i \theta_i \le \sum_{i=1}^{k} a_i \bar{Y}_{i\cdot} + M\sqrt{S_p^2 \sum_{i=1}^{k} \frac{a_i^2}{n_i}}$$

$$\Leftrightarrow M \le \frac{\sum_{i=1}^{k} a_i(\bar{Y}_{i\cdot} - \theta_i)}{\sqrt{S_p^2 \sum_{i=1}^{k} \frac{a_i^2}{n_i}}} \quad \left(\text{for all } \begin{pmatrix} a_1 \\ \vdots \\ a_k \end{pmatrix} \in \mathbb{R}^k\right)$$

$$\Leftrightarrow M^2 \le \frac{\left(\sum_{i=1}^{k} a_i(\bar{Y}_{i\cdot} - \theta_i)\right)^2}{S_p^2 \sum_{i=1}^{k} \frac{a_i^2}{n_i}} \quad \left(\text{for all } \begin{pmatrix} a_1 \\ \vdots \\ a_k \end{pmatrix} \in \mathbb{R}^k\right)$$

$$\Leftrightarrow M^2 \le \sup_{a \in \mathbb{R}^k} \left\{ \frac{\left(\sum_{i=1}^{k} a_i(\bar{Y}_{i\cdot} - \theta_i)\right)^2}{S_p^2 \sum_{i=1}^{k} \frac{a_i^2}{n_i}} \right\}$$

我們利用提示: $\sup_{a} \frac{\left(\sum_{i=1}^{k} a_i v_i\right)^2}{\sum_{i=1}^{k} \frac{a_i^2}{c_i}} = \sum_{i=1}^{k} c_i v_i^2$

故 $M^2 \le \frac{1}{S_p^2} \cdot \sum_{i=1}^{k} n_i (\bar{Y}_{i\cdot} - \theta_i)^2$

$\Leftrightarrow k \cdot F_{k,N-k,\alpha} \le \frac{1}{S_p^2} \sum_{i=1}^{k} n_i (\bar{Y}_{i\cdot} - \theta_i)^2$

$\Leftrightarrow F_{k,N-k}(\alpha) \le \frac{1}{S_p^2} \cdot \frac{1}{k} \sum_{i=1}^{k} n_i (\bar{Y}_{i\cdot} - \theta_i)^2 \sim F_{k,N-k}$

故 $Pr\left(F_{k,N-k}(\alpha) \le \frac{1}{S_p^2} \cdot \frac{1}{k} \sum_{i=1}^{k} n_i (\bar{Y}_{i\cdot} - \theta_i)^2\right)$

$= 1-\alpha$

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$\boxed{3}$ (Casella 11.33)

X_i 與 Y_i 原為相互獨立

(a) $\mathrm{cov}[X_i, Y_i] = \mathrm{cov}[X_i, \alpha + \beta X_i + \varepsilon_i] = \beta \sigma_X^2$

$\rho = 0 \Leftrightarrow \mathrm{cov}[X_i, Y_i] = 0 \Leftrightarrow \beta = 0$

(b) $\dfrac{\hat\beta}{S/\sqrt{S_{xx}}} = \sqrt{n-2}\,\dfrac{r}{\sqrt{1-r^2}}$

$$\begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix} = \begin{pmatrix} 1 & X_1 \\ \vdots & \vdots \\ 1 & X_n \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} + \begin{pmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_n \end{pmatrix}$$

$Y = X\boldsymbol{\beta} + \varepsilon \qquad \left(\boldsymbol{\beta} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \right.$

$Y|X \sim N(X\beta, \sigma^2 I)$

$$f(\vec{y}|\vec{x}) = \left(\frac{1}{\sqrt{2\pi}}\right)^n \frac{1}{(\det(\sigma^2 I))^{\frac{1}{2}}} \exp\left(\frac{-1}{2\sigma^2} (y - X\beta)^t (\sigma^2 I)^{-1} (y - X\beta) \right)$$

$$\ell(\beta, \sigma^2) = \frac{-n}{2}\ln(2\pi) - \frac{n}{2}\ln(\sigma^2) - \frac{1}{2\sigma^2}(y - X\beta)^t (y - X\beta)$$

$$\frac{\partial \ell}{\partial \sigma^2} = 0 \Rightarrow \frac{-n}{2\sigma^2} + \frac{1}{2\sigma^4}(y - X\beta)^t (y - X\beta) = 0$$

$$\Rightarrow \hat{\sigma}^2_{MLE} = \frac{1}{n}(y - X\hat\beta)^t (y - X\hat\beta)$$

$$\frac{\partial \ell}{\partial \beta^t} = \frac{1}{2\sigma^2}(2X^t y - 2X^t X\beta) = 0$$

$$\hat{\beta}_{MLE} = (X^t X)^{-1} X^t y$$

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$$(X^tX)=\begin{pmatrix} n & n\bar{x} \\ n\bar{x} & S_{xx}+n\bar{x}^2 \end{pmatrix} \qquad (X^tX)^{-1}=\frac{1}{nS_{xx}}\begin{pmatrix} S_{xx}+n\bar{x}^2 & -n\bar{x} \\ -n\bar{x} & n \end{pmatrix}$$

$$X^tY=\begin{pmatrix} n\bar{y} \\ S_{xy}+n\bar{x}\bar{y} \end{pmatrix}$$

$$\hat{\beta}=\begin{pmatrix} \frac{1}{n}+\frac{\bar{x}^2}{S_{xx}} & \frac{-\bar{x}}{S_{xx}} \\ \frac{-\bar{x}}{S_{xx}} & \frac{1}{S_{xx}} \end{pmatrix}\begin{pmatrix} n\bar{y} \\ S_{xy}+n\bar{x}\bar{y} \end{pmatrix}=\begin{pmatrix} \bar{y}-\frac{S_{xy}\bar{x}}{S_{xx}} \\ \frac{S_{xy}}{S_{xx}} \end{pmatrix}=\begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \end{pmatrix}$$

$$\begin{aligned} SS_E &= \sum_{i=1}^{n}(y_i-\hat{y}_i)^2=\sum_{i=1}^{n}(y_i-\hat{\alpha}-\hat{\beta}x_i)^2 \\ &= \sum_{i=1}^{n}\left(y_i-\bar{y}+\frac{S_{xy}}{S_{xx}}\bar{x}-\frac{S_{xy}}{S_{xx}}x_i\right)^2 \\ &= \sum_{i=1}^{n}(y_i-\bar{y})^2-2\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})\frac{S_{xy}}{S_{xx}}+\left(\frac{S_{xy}}{S_{xx}}\right)^2\sum_{i=1}^{n}(x_i-\bar{x})^2 \\ &= S_{yy}-\frac{S_{xy}^2}{S_{xx}}=\frac{S_{xx}S_{yy}-S_{xy}^2}{S_{xx}} \end{aligned}$$

故 $$\frac{\hat{\beta}}{\left(\frac{\sqrt{SS_E}}{\sqrt{S_{xx}}}\right)}=\frac{S_{xy}}{\sqrt{S_{xx}S_{yy}-S_{xy}^2}}$$

$$r=\frac{S_{xy}}{\sqrt{S_{xx}S_{yy}}} \qquad 1-r^2=\frac{S_{xx}S_{yy}-S_{xy}^2}{S_{xx}S_{yy}} \Rightarrow \frac{r}{\sqrt{1-r^2}}=\frac{S_{xy}}{\sqrt{S_{xx}S_{yy}-S_{xy}^2}}$$

另外 $S^2=\frac{1}{n-2}SS_E$

$$\therefore \frac{\hat{\beta}}{\frac{\sqrt{(n-2)}\cdot S}{\sqrt{S_{xx}}}}=\frac{r}{\sqrt{1-r^2}} \Leftrightarrow \frac{\hat{\beta}}{S/\sqrt{S_{xx}}}=\sqrt{n-2}\cdot\frac{r}{\sqrt{1-r^2}}$$

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$$Y \sim N(X\beta, \sigma^2 I)$$

(c) $SS_E = Y^t(I-H)Y = (Y-X\beta)^t(I-H)(Y-X\beta)$

$(\because (I-H)X\beta = X\beta - X(X^tX)^{-1}X^tX\beta = 0)$

另外 $(I-H)$ 為 idampotent 且對稱的矩陣

$$\text{rank}(I-H) = tr(I-H) = trI - tr\,X(X^tX)^{-1}X^t$$
$$= trI - tr\underbrace{(X^tX)^{-1}X^tX}_{I_{2\times 2}}$$
$$= n-2$$

故 $Y^t(I-H)Y/\sigma^2 \sim \chi^2_{(n-2)}$

另外 $\begin{pmatrix}(X^tX)^{-1}X^t \\ I-H\end{pmatrix} Y$ 的變異為 $\left(= \begin{pmatrix}\hat{\beta} \\ SS_E\end{pmatrix}\right)$

$$= \begin{pmatrix}(X^tX)^{-1}X^t \\ I-H\end{pmatrix}_{2\times n} \sigma^2 I \cdot \left(X(X^tX)^{-1} \quad I-H\right)$$

$= \sigma^2 \begin{pmatrix}(X^tX)^{-1} & 0 \\ 0 & I-H\end{pmatrix}$ 故 $\hat{\beta} = \begin{pmatrix}\hat{\alpha} \\ \hat{\beta}\end{pmatrix}$ vs SS_E 為獨立的統計量

$\hat{\beta} \sim N(\beta, \sigma^2(X^tX)^{-1})$ $\quad (X^tX)^{-1} = \frac{1}{nS_{xx}}\begin{pmatrix}S_{xx}+n\bar{x}^2 & -n\bar{x} \\ -n\bar{x} & n\end{pmatrix}$

$\therefore \hat{\beta} \sim N\left(\beta, \frac{\sigma^2}{S_{xx}}\right)$

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故 $\frac{\sigma}{\sqrt{S_{xx}}}\hat{\beta} \sim N(0,1)$ $\Rightarrow$ $\dfrac{\frac{\sigma\hat{\beta}}{\sqrt{S_{xx}}}}{\sqrt{\frac{1}{n-2}\frac{SS_E}{\sigma^2}}} \sim t_{n-2}$

$= \dfrac{\hat{\beta}}{S/\sqrt{S_{xx}}} \sim t_{n-2}$

$\therefore \sqrt{n-2}\,\dfrac{r}{\sqrt{1-r^2}} \sim t_{n-2}$

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4 (casella 11.35)

(a) $Y_i = \theta x_i^2 + \varepsilon_i \quad \varepsilon_i \overset{iid}{\sim} N(0, \sigma^2)$

故 $Y_i \sim N(\theta x_i^2, \sigma^2) \quad E[Y_i] = \theta x_i^2$

$l(\theta) \triangleq \sum_{i=1}^{n} (y_i - \theta x_i^2)^2$

$l'(\theta) = \sum_{i=1}^{n} (-2x_i^2 y_i + 2\theta x_i^4) = 0$

$\therefore \hat\theta_{LSE} = \dfrac{\sum_{i=1}^{n} x_i^2 y_i}{\sum_{i=1}^{n} x_i^4}$

(b) $f(y_1 \cdots y_n \mid \theta, x_1^2 \cdots x_n^2) = \prod_{j=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(y_j - \theta x_j^2)^2\right)$

$l(\theta) = \sum_{j=1}^{n} -\frac{1}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}(y_i - \theta x_i^2)^2 = 0$

$l'(\theta) = -\frac{1}{\sigma^2}\sum_{j=1}^{n}(y_j - \theta x_i^2)(-x_i^2) = 0$

$\therefore \hat\theta_{MLE} = \hat\theta_{LSE} = \dfrac{\sum_{i=1}^{n} x_i^2 y_i}{\sum_{i=1}^{n} x_i^4}$

(c) 求θ的完備充份統計量

$f(y_1 \cdots y_n \mid \theta, \sigma^2) = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n} y_i^2 + \frac{\theta}{\sigma^2}\sum_{j=1}^{n} x_j^2 y_j - \frac{\theta^2}{2\sigma^2}\sum_{j=1}^{n} x_j^2\right)$

R^2 的子集合 $\left\{\left(-\frac{1}{2\sigma^2}, \frac{\theta}{\sigma^2}\right) \middle| \sigma^2 > 0, \theta \in R\right\}$

這題然包含開集合，故 $\left(\sum_{i=1}^{n} y_i^2, \sum_{i=1}^{n} x_i^2 y_i\right)$ 為 (θ, σ^2) 之完備充份統計量.

$E\left[\sum_{i=1}^{n} x_i^2 y_i\right] = \theta \sum_{i=1}^{n} x_i^4$

故 $E\left[\dfrac{\sum_{i=1}^{n} x_i^2 y_i}{\sum_{i=1}^{n} x_i^4}\right] = \theta$

$\therefore \dfrac{\sum_{i=1}^{n} x_i^2 y_i}{\sum_{i=1}^{n} x_i^4}$ 為 θ 之 UMVUE

(Rao-Blackwell & Lehman-Scheffe)

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$\boxed{5}$
$$\begin{pmatrix} Y_{1,1} \\ Y_{2,1} \\ \vdots \\ Y_{n,1} \\ Y_{1,2} \\ \vdots \\ Y_{n,2} \end{pmatrix} = \begin{pmatrix} 1 & x_{1,1} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_{n,1} & 0 & 0 \\ 0 & 0 & 1 & x_{1,2} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & x_{n,2} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \beta_1 \\ \alpha_2 \\ \beta_2 \end{pmatrix} + \begin{pmatrix} \varepsilon_{1,1} \\ \vdots \\ \varepsilon_{n,1} \\ \varepsilon_{1,2} \\ \vdots \\ \varepsilon_{n,2} \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} \vec{y_1} \\ \vec{y_2} \end{pmatrix} = \begin{pmatrix} X_1\vec{\beta_1} \\ X_2\vec{\beta_2} \end{pmatrix} + \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix} \qquad \begin{matrix} \varepsilon_1 \sim N(0, \sigma^2 I) \\ \varepsilon_2 \sim N(0, 2\sigma^2 I) \end{matrix}$$

$$f(\vec{y_1}, \vec{y_2} \mid \vec{\beta_1}, \vec{\beta_2}, \sigma^2) = \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \cdot \left(\frac{1}{\sigma^2}\right)^{\frac{n}{2}} \cdot \exp\left(\frac{-1}{2\sigma^2}(y_1 - X_1\beta_1)^t(y_1 - X_1\beta_1)\right)$$
$$\cdot \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \left(\frac{1}{2\sigma^2}\right)^{\frac{n}{2}} \exp\left(\frac{-1}{4\sigma^2}(y_2 - X_2\beta_2)^t(y_2 - X_2\beta_2)\right)$$

$$\hat{\beta_1} = X_1^t(X_1^tX_1)^{-1}X_1Y_1 \qquad (\because \frac{\partial}{\partial \beta_1^t}\ln f = 0)$$

$$\hat{\beta_2} = X_2^t(X_2^tX_2)^{-1}X_2Y_2$$

跟 $\boxed{3}$ 同理， $\hat{\beta_1} \sim N(\beta_1, \frac{\sigma^2}{S_{x_1x_1}})$

$$\hat{\beta_2} \sim N(\beta_2, \frac{2\sigma^2}{S_{x_2x_2}})$$

$$\therefore \hat{\beta_1} - \hat{\beta_2} \sim N\left(\beta_1 - \beta_2, \frac{\sigma^2}{S_{x_1x_1}} + \frac{2\sigma^2}{S_{x_2x_2}}\right)$$

$(\because \hat{\beta_1}, \hat{\beta_2}$ 為獨立$)$

另外 $\frac{1}{\sigma^2}\left(SS_{E1} + \frac{SS_{E2}}{2}\right) \sim \chi^2_{(2n-4)}$ 且與 $\begin{pmatrix} \hat{\beta_1} \\ \hat{\beta_2} \end{pmatrix}$ 獨立

$$\frac{\dfrac{(\hat{\beta}_1-\hat{\beta}_2)-(\beta_1-\beta_2)}{\sqrt{\dfrac{1}{S_{x_1x_1}}+\dfrac{2}{S_{x_2x_2}}}}}{\sqrt{\dfrac{1}{2n-4}\left(SS_{E_1}+\dfrac{1}{2}SS_{E_2}\right)}} \sim t(2n-4)$$

$\therefore H_0: \beta_1=\beta_2$ 下,

$$T = \frac{(\hat{\beta}_1-\hat{\beta}_2)\Big/\sqrt{\dfrac{1}{S_{x_1x_1}}+\dfrac{2}{S_{x_2x_2}}}}{\sqrt{\dfrac{1}{2n-4}\left(SS_{E_1}+\dfrac{1}{2}SS_{E_2}\right)}} \sim t_{2n-4}$$

$\therefore |T| \geq t_{2n-4}(1-\frac{\alpha}{2})$ 时拒绝

$\boxed{6}$ $Y = \underbrace{[\overset{\longleftrightarrow r}{X_1}\ X_2]}_{X}^{\longleftrightarrow p} \underbrace{\begin{bmatrix}\beta_1\\ \beta_2\end{bmatrix}}_{\beta} + \varepsilon \sim N(X\beta, \sigma^2 I)$

$SS_{E_0} = Y^t(I-H_1)Y^t \qquad H_1 = X_1(X_1^T X_1)^{-1} X_1^T$

$SS_{E_1} = Y^t(I-H)Y^t$

検定統計量 $F = \dfrac{(SS_{E_0} - SS_{E_1}) / (\underset{\underset{(n-r)}{\parallel}}{df_1} - \underset{\underset{(n-p)}{\parallel}}{df_0})}{SS_{E_1} / df_1}$

$\sim F_{df_1 - df_0,\ df_0}$

於 H_0: $\beta_2 = 0$ 之下,

$SS_{E_0} - SS_{E_1} = Y^T(H-H_1)Y$

$= (Y - X\beta)^T (H - H_1)(Y - X\beta)$

$tr(H - H_1) = p - r$,

(由 $\beta_2 = 0$ ∴ $X\beta = X_1\beta_1$

$HX\beta = X\beta$, $H_1 X\beta = H_1 X_1 \beta_1 = X_1\beta_1$
$= X\beta$

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$$(H-H_1)^2 = H^2 - HH_1 - H_1H + H_1^2$$
$$= H - HH_1 - H_1H + H_1$$

$$HH_1 = H X_1 (X_1^T X_1)^{-1} X_1^T = X_1 (X_1^T X_1)^{-1} X_1^T = H_1$$

∵ H 為 $col(X) = col(X_1 X_2)$ 的投影矩陣

故 $\forall x \in col(X)$ $Hx = x$

∴ $HX_1 = X_1$

同理 $H_1 H = H_1$

∴ $(H-H_1)^2 = H - H_1$

$H-H_1$ 為對稱矩陣

∴ $H-H_1$: idempotent

∴ $Y^T(H-H_1)Y \sim \chi^2_{(p-r)} \cdot \sigma^2$

同樣 $SSE_1 \sim \chi^2_{(n-p)} \cdot \sigma^2$ $tr(I-H) = n-p$

$\parallel$ $(Y-X\beta)^T(I-H)(Y-X\beta)$ $I-H$: idempotent

symmetric

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另外 $(H-H_1)(I-H) = H - H_1 - H^2 + H_1 H$

$= H - H_1 - H + H_1 = 0$

故 $SSE_0 - SSE_1$ & SSE_1 為獨立

$$\therefore F = \frac{\dfrac{(SSE_0 - SSE_1)}{\sigma^2} / (p-r)}{\dfrac{SSE_1}{\sigma^2} / (n-p)} \sim F_{p-r,n-p}$$

$\therefore F \geq F^{-1}_{p-r,n-p}(1-\alpha) \Rightarrow$ reject