No. 實分析

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Chapter 1. 作業 R05246013 森元俊成

3.

(a) $(\limsup E_k)^c = \Big(\bigcap_{k=1}^{\infty} \underbrace{\bigcup_{j\geq k} E_j}_{F_k}\Big)^c = \Big(\bigcap_{k=1}^{\infty} F_k\Big)^c$

$= \bigcup_{k=1}^{\infty} F_k^c = \bigcup_{k=1}^{\infty}\Big(\bigcup_{j\geq k} E_j\Big)^c = \bigcup_{k\geq 1}^{\infty} \bigcap_{j\geq k} E_j^c$

$= \liminf E_k^c$

(b) 我們注意 $\limsup_n E_n = \bigcap_{n=1}^{\infty} \bigcup_{m\geq n}^{\infty} E_m$

$\liminf_n E_n = \bigcup_{n=1}^{\infty} \bigcap_{m\geq n}^{\infty} E_m$

$\forall n \in \mathbb{N}$ 「$\underbrace{\bigcap_{m\geq n}^{\infty} E_m}_{=A_n} \subseteq E_n \subseteq \underbrace{\bigcup_{m\geq n}^{\infty} E_m}_{=\tilde{A}_n}$」成立.

且「$A_n \subseteq A_{n+1} \subseteq \hat{A}_{n+1} \subseteq \hat{A}_n$」成立

$\therefore \bigcup_{n=1}^{\infty} \underbrace{\bigcap_{m\geq n}^{\infty} E_m}_{=A_n} \subseteq \bigcap_{n=1}^{\infty} \underbrace{\bigcup_{m\geq n}^{\infty} E_n}_{=\tilde{A}_n}$

$\Rightarrow \liminf E_n \subseteq \limsup E_n$ 隨時都成立

$\therefore$ 我們的目標是證明:「$\liminf E_n \supseteq \limsup E_n$」

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① $E_n \nearrow E$ 時，取 $x \in \limsup E_n = \bigcap_{n=1}^{\infty}\bigcup_{m\geq n}^{\infty} E_m$

$\Rightarrow$「$\forall n \in \mathbb{N}$ $x \in \bigcup_{m\geq n}^{\infty} E_m$」$\Rightarrow$「$x \in \bigcup_{m=1}^{\infty} E_m$」($n=1$ 代入) … ⓧ1

$\liminf E_n = \bigcup_{n=1}^{\infty}\bigcap_{m\geq n} E_m = \bigcup_{n\geq 1}^{\infty} E_n$ $\left(\because \bigcap_{m\geq n} E_m = E_n\right)$

ⓧ1 ∴ $x \in \bigcup_{n\geq 1}^{\infty} E_n = \liminf E_n$

② $E_n \searrow E$ 時，取 $\forall x \in \limsup E_n = \bigcap_{n=1}^{\infty}\bigcup_{m\geq n}^{\infty} E_m$

$(\because E_n \searrow E)$

$\Rightarrow$「$\forall n \in \mathbb{N}$ $x \in \bigcup_{m\geq n}^{\infty} E_m = E_n$」∴「$\forall n \in \mathbb{N}$ $x \in E_n$」→ ⓧ2

$\liminf E_n = \bigcup_{n=1}^{\infty}\bigcap_{m\geq n} E_m$ 我們注意 $\forall n \in \mathbb{N}$ $x \in \bigcap_{m\geq n} E_m$ ($\because$ ⓧ2)

∴ $x \in \liminf_n E_n$

∴證明完成

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4.

(a) $\limsup(-a_n) = \lim_{n\to\infty} \sup_{m\ge n}\{-a_m\} = \lim_{n\to\infty}\{-\inf_{m\ge n}\{a_m\}\}$

$= -\lim_{n\to\infty}\inf_{m\ge n}\{a_m\} = -\liminf a_n$

跟m無關

(b) $\lim_{n\to\infty}\sup_{m\ge n}\{a_m+b_m\} \le \lim_{n\to\infty}\sup_{m\ge n}\{a_m + \sup_{k\ge n}\{b_k\}\}$

$= \lim_{n\to\infty}\left(\sup_{m\ge n}\{a_m\} + \sup_{k\ge n}\{b_k\}\right) = \limsup a_n + \limsup b_n$

跟m無關

(c) $\limsup_n a_n b_n = \lim_n \sup_{m\ge n}\{a_m b_m\} \le \lim_n \sup_{m\ge n}\{a_m \cdot \sup_{k\ge n} b_k\}$

$= \lim_n \sup_{m\ge n}\{a_m\} \cdot \sup_{k\ge n}\{b_k\} = \lim_{n\to\infty}\sup_{m\ge n}\{a_m\} \cdot \lim_{n\to\infty}\sup_{m\ge n}\{b_m\}$

$= \limsup a_n \cdot \limsup b_n$

(d) 例如取 $a_n \overset{def}{=} n \bmod 2$ ∴ $\{a_1, a_2, \dots\} = \{1,0,1,0,1,0\dots\}$
$b_n \overset{def}{=} n+1 \bmod 2$ $\{b_1, b_2, \dots\} = \{0,1,0,1\dots\}$

- $(a_n+b_n)=1 \Rightarrow \limsup(a_n+b_n)=1$
  $\limsup a_n = \limsup b_n = 1$ ∴ $\limsup a_n + \limsup b_n = 2$
  等號不成立
- $a_n b_n = 0 \Rightarrow \limsup a_n b_n = 0$
  $\limsup a_n \cdot \limsup b_n = 1$ ∴ 等號不成立

• 另外 $\lim\limits_{n} \sup\limits_{m\ge n}\{a_m+b_m\} \ge \lim\limits_{n} \sup\limits_{m\ge n}\{a_m + \inf\limits_{k\ge n} b_k\}$ 跟m無關

$= \lim\limits_{n}\left(\sup\limits_{m\ge n}\{a_m\} + \inf\limits_{k\ge n}\{b_k\}\right) = \limsup a_n + \liminf b_n$

∴「$\limsup a_n + \liminf b_n \le \limsup(a_n+b_n) \le \limsup a_n + \limsup b_n$」…⊛

若 $\{b_n\}_{n\ge 1}$ 收斂，則 $\liminf b_n = \limsup b_n$ $(=b)$ ∴上述⊛的等號成立

• 同樣 $\lim\limits_{n} \sup\limits_{m\ge n}\{a_m b_m\} \ge \lim\limits_{n} \sup\limits_{m\ge n}\{a_m \cdot \inf\limits_{k\ge n} b_k\} = \lim\limits_{n} \sup\limits_{m\ge n}\{a_m\} \cdot \inf\limits_{k\ge n}(b_k)$

$= \limsup\limits_{n} a_n \cdot \liminf\limits_{n} b_n$

∴「$\limsup\limits_{n} a_n \liminf b_n \le \limsup a_n b_n \le \limsup a_n \cdot \limsup b_n$」…⊛'

若 $\{b_n\}_{n\ge 1}$ 收斂，則 $\liminf b_n = \limsup b_n = b$ ∴上述⊛'的等號成立。

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12、

(a) $E_1 = \{(x,y) \mid x>0,\ y \geq \frac{1}{x}\}$
$E_2 = \{(x,y) \mid x>0\ \ y \leq -\frac{1}{x}\}$

E_1, E_2 均為閉集合，(不交) $\text{dist}(E_1, E_2) = 0$

(b) ① 「$\exists\, x \in E_1,\ y \in E_2$ s.t. $\text{dist}(E_1, E_2) = |x-y|$」

$d \overset{\text{def}}{=} \text{dist}(E_1, E_2) = \inf\{|x-y| \mid x \in E_1,\ y \in E_2\}$

由於 $\text{dist}(E_1,E_2)$ 的定義，可取 $\{x_n\}_{n\geq 1} \subseteq E_1$，$\{y_n\}_{n\geq 1} \subseteq E_2$

使得 $d \leq \|x_n - y_n\| \leq d + \frac{1}{n}$ (for all n) …… (*1)

E_2 為緊空間，故 $\{y_n\}_{n\geq 1} \subseteq E_2$ 為有界序列。(緊緻 $\Leftrightarrow$ 有界・閉)
$\Rightarrow$ 根據 Bolzano–Wierstrass theorem，存在子序列 $\{y_{n_k}\} \subseteq E_2$
$y_{n_k} \to y \in E_2$ (as $k \to \infty$)。 (且 E_2 為閉集合)

考慮 $\|x_{n_k}\| = \|x_{n_k} - y_{n_k} + y_{n_k}\| \leq \underbrace{\|x_{n_k} - y_{n_k}\|}_{\because (*1)} + \underbrace{\|y_{n_k}\|}_{\leq M < \infty} \leq (d+1) + M$

($\because \{y_{n_k}\}_{k\geq 1}$：有界)

由此可知 $\{x_{n_k}\}_{k\geq 1}$ 亦為有界序列。

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Bolzano-Wierstrass

子序列

同理，存在 $\{x_{n_{k_\ell}}\}_{\ell\geq 1}$，$x_{n_{k_\ell}} \to x \in E_1$.（$E_1$：閉集合）

∴ 考慮 $\{x_{n_{k_\ell}}\}_{\ell\geq 1}$ $\{y_{n_{k_\ell}}\}_{\ell\geq 1}$（注意，$x_{n_{k_\ell}} \to x \in E_1$
$y_{n_{k_\ell}} \to y \in E_2$）

$d \leq |x_{n_{k_\ell}} - y_{n_{k_\ell}}| \leq d + \frac{1}{n_{k_\ell}}$（∵ ※1）（↑ $\ell \to \infty \Rightarrow k_\ell \to \infty$）

$\ell \to \infty$ 時 $n_{k_\ell} \to \infty$. ∴ $|x-y| = d$ 且 $x \in E_1$, $y \in E_2$.

∴ 證明完成

② 「$E_1 \cap E_2 = \phi \Rightarrow d > 0$」… 利用反證法.

假設 $d=0$. $\exists\, x \in E_1, y \in E_2$ s.t. $|x-y|=0$

$\Rightarrow x=y \Rightarrow \underset{(=\{y\})}{\{x\}} \subseteq E_1 \cap E_2$. ∴ 矛盾（$E_1 \cap E_2 \neq \phi$）

∴ 證明完成

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13. E為極限點的集合，故 $\forall x\in\bar{E}$ $\exists\{a_n\}\subseteq E$ s.t. $a_n\to x$.

令 $\bar{f}(x)\ (x\in\bar{E})\overset{def}{=}\lim_{n\to\infty}f(a_n)$ where $\{a_n\}\subseteq E$ $a_n\to x$.

① 「$\lim_{n\to\infty}f(a_n)$ 存在」 設 $\{a_n\}\subseteq E$ $a_n\to x\in\bar{E}$

$\{f(a_n)\}_{n\ge1}$ 為 Cauchy 序列. $\because \lim_{m,n\to\infty}|f(a_n)-f(a_m)|=0$

(原因) $f(x)\ (x\in E)$ 為均勻連續. m,n 很大時,

$|a_m-a_n|\le\delta$ $\therefore |f(a_m)-f(a_n)|<\varepsilon$

(我們假設 f為實數函數. $f:E\to R$)

$\{f(a_n)\}_{n\ge1}\subseteq R$ 為 Cauchy 序列，且實數具完備性，

故 Cauchy 序列收斂 $\therefore \lim_n f(a_n)$ 存在.

② 「$\bar{f}=f\ (x\in E)$」

我們暫時假設 $\bar{f}$為 well-defined.

設$x\in E$, $\{a_n\}\subseteq E$ $a_n=x$ (for all $n\in N$)

$\bar{f}(x)=\lim_n f(a_n)=\lim_n f(x)=f(x)$ $\therefore \bar{f}(x)=f(x)$ when $x\in E$

③「$f(x\in E)$ 為連續函數」($\Rightarrow$ 由此可知，f 為 well-defined)

設 $\{x,y\}\subseteq E$ 且滿足 $|x-y|\le\frac{\delta}{3}$.

設 $\{x_n\}\subseteq E$，$x_n\to x$；$\{y_n\}_{n\ge1}\subseteq E$ $y_n\to y$.

$|f(x)-f(y)|=|f(x)-f(x_n)+f(x_n)-f(y_n)+f(y_n)-f(y)|$

$\le\underbrace{|f(x)-f(x_n)|}_{(\mathrm{I})}+\underbrace{|f(x_n)-f(y_n)|}_{(\mathrm{II})}+\underbrace{|f(y_n)-f(y)|}_{(\mathrm{III})}$

・(I), (III) $<\frac{\varepsilon}{3}$ ∵ $f(x)=\lim_{n} f(x_n)$ 且其極限存在
很大的 n 使得 (I), (III) $<\frac{\varepsilon}{3}$

・(II) $|x_n-y_n|=|x_n-x+x-y+y-y_n|\le\underbrace{|x_n-x|}_{\le\frac{\delta}{3}}+\underbrace{|y_n-y|}_{\le\frac{\delta}{3}}+\underbrace{|x-y|}_{\le\frac{\delta}{3}}\le\delta$

(∵ $x_n\to x$，$y_n\to y$ ∴ n 很大時 $|x_n-x|\le\frac{\delta}{3}$ $|y_n-y|\le\frac{\delta}{3}$)

f 於 E 上為均勻連續，$|x_n-y_n|\le\delta\Rightarrow|f(x_n)-f(y_n)|<\frac{\varepsilon}{3}$

∴ (II) $<\frac{\varepsilon}{3}$

綜合 (I)(II)(III) 可得 $|f(x)-f(y)|<\varepsilon$ ∴ f 為連續

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15、先證明 1(r)：Rieman 可積分 $\Leftrightarrow$「$\inf_P U_P = \sup_P L_P = A$」

① 「Rieman 可積分 $\Rightarrow$ $\inf U_P = \sup_P L_P = A$」的證明

令 $A \overset{def}{=} \int_I f(x)dx$. $\lim_{|P|\to 0} R_P = A \Rightarrow \forall \varepsilon>0\ \exists \delta>0\ \forall P\ (|P|<\delta)$
$|R_P - A| < \frac{\varepsilon}{2}$. $\left(\therefore A-\frac{\varepsilon}{2} < R_P < A+\frac{\varepsilon}{2}\right)$

(I) 固定 $P = P_1 = \{I_1, \dots I_{N_1}\}$ $|P_1| < \delta$

取 $\{\xi_1, \dots \xi_{N_1}\}$ 使得 $\xi_j \in I_j$ $(j=1\sim N_1)$ 且 $\sup_{x\in I_j} f(x) - f(\xi_j) < \frac{\varepsilon}{2N_1} \cdot \frac{1}{v(I_j)}$

$\Rightarrow U_{P_1} - R_{P_1} = \sum_{j=1}^{N_1}\left(\sup_{x\in I_j} f(x) - f(\xi_j)\right)\cdot v(I_j) < \sum_{j=1}^{N_1} \frac{\varepsilon}{2N_1} = \frac{\varepsilon}{2}$

$\Rightarrow U_{P_1} < R_{P_1} + \frac{\varepsilon}{2} < A+\varepsilon$ 。 $\inf_P U_P \leq U_{P_1} < A+\varepsilon$

(II) 同樣固定 $P = P_2 = \{I_1', \dots I_{N_2}'\}$ $|P_2| < \delta$
取 $\{\xi_1, \dots \xi_{N_2}\}$ 使得 $\xi_j \in I_j$ $(j=1\sim N_2)$ 且 $f(\xi_j) - \inf_{x\in I_j} f(x) < \frac{\varepsilon}{2N_2} \cdot \frac{1}{v(I_j)}$
$\Rightarrow R_{P_2} - L_{P_2} < \frac{\varepsilon}{2} \Rightarrow A - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} < R_P - \frac{\varepsilon}{2} < L_{P_2}$

$\therefore A - \varepsilon < L_{P_2} \leq \sup_P L_P$

(I)+(II) $\Rightarrow$ $A-\varepsilon < \sup_P L_P \leq \inf_P U_P < A+\varepsilon$

② 「$\inf_P U_P = \sup_P L_P = A \in (-\infty, \infty)$」$\Rightarrow$「Rieman 可積分」的證明

存在分割 P_1 與 P_2 分別滿足 $U_{P_1} < A + \frac{\varepsilon}{4}$　$L_{P_2} > A - \frac{\varepsilon}{4}$，

現在取一個分割 P_0，P_0 為 P_1 與 P_2 的 refinement

· $U_{P_0} \leq U_{P_1}$；$L_{P_0} \geq L_{P_2}$　故 $U_{P_0} - L_{P_0} \leq U_{P_1} - L_{P_2} < (A+\frac{\varepsilon}{4}) - (A-\frac{\varepsilon}{4}) = \frac{\varepsilon}{2}$

設 $P_0 = \{I_1, I_2, \ldots I_N\}$。

我們接著證明 $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $\forall P\ (|P| < \delta)$　$|R_P - A| < \varepsilon$。

取一個任意的分割 $P = \{J_k\}_k$，$|P| < \delta$

$$U_P - L_P = \sum_{k=1} \left(\sup_{x\in J_k} f(x) - \inf_{x\in J_k} f(x) \right) \cdot v(J_k)$$

$$= \underbrace{\sum_{\substack{J_k:\\ \exists I \in P_0 \text{ s.t. } I \supset J_k}} \left(\sup_{x\in J_k} f(x) - \inf_{x\in J_k} f(x) \right) v(J_k)}_{①} + \underbrace{\sum_{\substack{J_k:\\ \nexists I \in P_0 \text{ s.t. } I \supset J_k}} \left(\sup_{x\in J_k} f(x) - \inf_{x\in J_k} f(x) \right) \cdot v(J_k)}_{②}$$

$$① \leq \sum_{I\in P_0} \left(\sup_{x\in I} f(x) - \inf_{x\in I} f(x) \right) \cdot v(I_k) \leq \frac{\varepsilon}{2}$$

「$\nexists I \in P_0$ s.t. $I \supset J_k$」：這樣子的 J_k 至多 $N-1$ 個

$$② \leq \sum_{\substack{J_k:\\ \nexists I \in P_0 \text{ s.t. } I \supset J_k}} (2M) \cdot \delta \leq 2M \cdot \underbrace{(N-1)}_{\#P_0 = N} \cdot \delta \qquad (|f| \leq M < \infty)$$

$\therefore$ 取 $\delta < \dfrac{\varepsilon}{4M(N-1)} \Rightarrow \underset{(U_P - L_P < \varepsilon)}{① + ② < \varepsilon} \Rightarrow |R_P - A| < \varepsilon$

(三明治定理)

$\therefore$ 1-(1) 證明完成

③ 最後證明：「$\forall \varepsilon > 0\ \exists P$ s.t. $0 \leq U_P - L_P < \varepsilon$」
$\Leftrightarrow$「$\inf_P U_P = \sup_P L_P = A$」　證明這兩個是等價的

· $\Rightarrow$　$0 \leq \inf_P U_P - \sup_P L_P \leq U_P - L_P < \varepsilon$　(for all $\varepsilon > 0$)

· $\Leftarrow$　可以重複利用 ② 的證明；
我們已知：$\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $\forall P\ |P| < \delta$　$U_P - L_P < \varepsilon$
所以取其中一個 P 即可.

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17. $\begin{cases} ① \text{函數} f \text{…均勻連續} \\ ② \ \omega(\delta) \to 0 \ (\text{as } \delta \downarrow 0) \end{cases}$

(1) 證明 $\delta \downarrow 0 \Rightarrow \omega(\delta)$ 減少

$0 < \delta_1 < \delta_2 \Rightarrow \{|f(x)-f(y)| \mid |x-y| \leqq \delta_1\} \subseteq \{|f(x)-f(y)| \mid |x-y| \leqq \delta_2\}$

$\therefore \sup\{|f(x)-f(y)| \mid |x-y| \leqq \delta_1\} \leqq \sup\{|f(x)-f(y)| \mid |x-y| \leqq \delta_2\}$

$\therefore \omega(\delta_1) \leqq \omega(\delta_2) \quad (\delta_1 < \delta_2)$

由此可知，$\omega(\delta)$ $(\delta > 0)$ 爲遞增函數。
若 $\delta \downarrow 0$，則 $\omega(\delta)$ 減少。

(2) ② ⇒ ① 的證明…

$\omega(\delta) \to 0 \ (\text{as } \delta \downarrow 0) \Rightarrow \forall \varepsilon > 0 \ \exists \delta > 0 \ \text{st} \ 0 \leqq \omega(\delta) < \varepsilon$

$\begin{cases} (\text{注}) \ \text{由} \ \omega \ \text{的定義可知，} \omega(\delta) \geqq 0 \\ \cdot \ 0 \leqq \omega(\delta) < \varepsilon \Rightarrow \forall \delta_0 \in [0, \delta] \ \ 0 \leqq \omega(\delta_0) < \varepsilon \ \text{亦成立} \end{cases}$

換言之，$\forall \varepsilon > 0. \ \exists \delta > 0 \ \text{st} \ 0 \leqq \sup\{|f(x)-f(y)| \mid |x-y| \leqq \delta\} < \varepsilon$

$\Rightarrow \forall \varepsilon > 0 \ \exists \delta > 0, \ \text{st} \ \forall \{x, y\} \subseteq \mathbb{R}^n \ |x-y| \leqq \delta \Rightarrow |f(x)-f(y)| < \varepsilon$

$\therefore f$ 為均勻連續的函數

①⇒② 的證明：

f 為均勻連續 → $\forall \varepsilon>0 \ \exists \delta>0$ s.t. $\forall \{x,y\} \subseteq \mathbb{R}^n$ $|x-y| \leq \delta$

$\Rightarrow |f(x)-f(y)|<\varepsilon$.

換言之，$\forall \varepsilon>0 \ \exists \delta>0$ s.t. $\sup\{|f(x)-f(y)| \mid |x-y| \leq \delta\} < \varepsilon$

$$\Updownarrow$$

$$0 \leq \omega(\delta) < \varepsilon$$

由於 $\omega(\delta)$ 為遞增函數，故 $\forall \varepsilon>0 \ \exists \delta>0$ s.t. $\forall z \in (0,\delta)$

$0 \leq \omega(z) \ (\leq \omega(\delta)) < \varepsilon$

這表示 $\lim\limits_{z \searrow 0} \omega(z) = 0$.

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$\boxed{1}$

① 先證明連續性

(I)「$x=0$: 右連續」:

$x>0$ 時 $|f(x)| = |x| \cdot |\sin\frac{1}{x}| \le |x|$ ($\because 0 \le \sin\frac{1}{x} \le 1$)

$\therefore \lim_{x\searrow 0} |f(x)| \le \lim_{x\searrow 0} |x| = 0$ $\therefore \lim_{x\searrow 0} f(x) = 0 = f(0) = 0$

(II)「$x>0$: 連續」:

$x, \sin x, \frac{1}{x}$ 皆為連續函數 $(0<x<\infty)$

$\therefore f(x)$ 於 $(0,\infty)$ 連續 ($\Rightarrow$ 於 $(0,1]$ 連續)

$\therefore$ (I)+(II) $\Rightarrow$ $f(x)$ 於 $[0,1]$ 為連續函數

② $f(x)$ 於 $[0,1]$ 有界.

$f(x)$ 為 $[0,1]$ 上的連續函數, 且 $[0,1]$ 為有界閉集合

$\therefore f(x)$ 於 $[0,1]$ 上有界.

③ $V[f;a,b] = +\infty$

考慮 $P_M = \left\{ \frac{1}{(k-\frac{1}{2})\pi} \right\}_{k=1,2,\dots,M} \cup \{0,1\}$

$$S_{P_M} \ge \sum_{k=1}^{M} \left| \frac{1}{(k+\frac{1}{2})\pi} \sin(k+\tfrac{1}{2})\pi - \frac{1}{(k-\frac{1}{2})\pi} \sin(k-\tfrac{1}{2})\pi \right|$$

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$$= \sum_{k=1}^{M}\left\{\frac{1}{(k+\frac{1}{2})\pi}+\frac{1}{(k-\frac{1}{2})\pi}\right\} \geq \frac{1}{\pi}\cdot\sum_{k=1}^{M}\frac{1}{k}$$

注意 $\sum_{k=1}^{\infty}\frac{1}{k}=\infty$.

$\therefore M\to\infty$ 時 $S_{P_M}\to\infty$. $\therefore \sup_{P} S_P=\infty=V[f;0,1]$

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$\{f_n\}_{n\geq 1}$ … Bounded Variation ($V[f_n;a,b] \leq M$)

① $V[f;a,b] \leq M$ 的證明…

取一個 $\forall P$, 證明 $S_P[f;a,b] \leq M$ 即可

設 $P=\{x_0, \cdots x_m\}$ $S_P[f;a,b] = \sum_{i=1}^{m} |f(x_i) - f(x_{i-1})|$

$= \sum_{i=1}^{m} |f(x_i) - f_k(x_i) + f_k(x_i) - f(x_{i-1}) + f_k(x_{i-1}) - f_k(x_{i-1})|$

$\leq \underbrace{\sum_{i=1}^{m} |f(x_i) - f_k(x_i)|}_{①} + \underbrace{\sum_{i=1}^{m} |f(x_{i-1}) - f_k(x_{i-1})|}_{②} + \underbrace{\sum_{i=1}^{m} |f_k(x_i) - f_k(x_{i-1})|}_{③}$

①, ② … 利用:「$f_k \to f$」(for all $x \in [a,b]$)

若取很大的 N_i, 則 $\forall k \geq N_i$ $|f(x_i) - f_k(x_i)| \leq \frac{\varepsilon}{2m}$

∴ 取 $N = \max\{N_1, N_2, \dots N_m\}$.. $\forall k \geq N$ ①+② $< \varepsilon$

③ $= S_P[f_k;a,b] \leq M$

①+②+③ $< M+\varepsilon$

∴ $\forall P$ $S_P[f;a,b] < M+\varepsilon$

∴ $\varepsilon \downarrow 0$ $S_P \leq M$ ∴ $\underset{P}{\sup} S_P \leq M$

$\sup_P S_P = V[f;a,b]$

(注) $\mathbb{Q}=\{$有理數$\}$

② $f_n = I_{A_n}$ $\quad A_n=\left\{\frac{m}{k} \,\middle|\, 1\le k\le n,\ 1\le m\le k,\ k,m\text{ 整數}\right\}$

$f_n \nearrow I_{[0,1]\cap\mathbb{Q}}(x) = \begin{cases} 1 & \text{if } (0\le x\le 1,\ x\text{ 為有理數}) \\ 0 & \text{else} \end{cases}$

f_n 收斂到 Dirichlet 函數.

(P18. Example 4. Dirichlet Function 並非 Bounded Variation)

但 f_n 顯然是 Bounded Variation.
($\because \{x\in[0,1] \mid f_n(x)=1\}$ 是有限集合. 其他的 x 都使得 $f(x)=0$)

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① $V[f;a,b]<\infty$ 的證明

- $V[f;a+\varepsilon,b]<\infty \Rightarrow f$ 於 $[a+\varepsilon,b]$ 上有界.
- 接著證明: f 於 $(a,a+\varepsilon)$ 上有界.

利用反證法... 若 f 於 $(a,a+\varepsilon)$ 上非有界,
則 $\exists y\in(a,a+\varepsilon)$ s.t $|f(y)|>M+|f(a+\varepsilon)|$.
考慮 $[y,b]$ 的分割: $P_y=\{y,a+\varepsilon,\cdots b\}$.

$$S_{P_y}=\sum_{(x_{i-1},x_i)\in P}|f(x_i)-f(x_{i-1})|\ge|f(x_1)-f(x_0)|$$
$$\ge|f(x_0)|-|f(x_1)|=|f(y)|-|f(a+\varepsilon)|$$
$$>M.$$

$\Rightarrow$ 矛盾. $\therefore$ f 於 $(a,a+\varepsilon)$ 上有界.

- 利用Q4的結果

$$f_n=\begin{cases}f(a) & (x=a)\\ 0 & (a<x<a+\frac{1}{n})\\ f(x) & (a+\frac{1}{n}\le x\le b)\end{cases}$$

$f_n\to f\quad(a\le x\le b)$

$V[f_n;a,b]\le|f(a)|+M$

$\Rightarrow V[f;a,b]\le|f(a)|+M<\infty\quad(\because Q4)$

$\therefore$ ① 證明完成

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② $V[f;a,b]\leq M$ 不一定成立：例如考慮：

$$f(x)=\begin{cases}1 & (x=a)\\ 0 & (a<x\leq b)\end{cases}$$

$$V[f;a,b]=1\ ,\ V[f;\underset{(\varepsilon>0)}{a+\varepsilon},b]=0$$

③ 若 $f(x)$ 於 $x=a$ 連續（右連續）$\Rightarrow V[f;a,b]\leq M$.

$\boxed{\text{證明}}$ $\forall\eta>0\ \exists\delta>0$ s.t. $a<x<a+\delta\Rightarrow|f(x)-f(a)|<\eta$ （∵連續）

考慮任意 P：$\underbrace{x_1-x_0<\delta}_{}$ ㊕ where $P=\{\overset{a}{\overset{\shortparallel}{x_0}},x_1,\dots,\overset{b}{\overset{\shortparallel}{x_m}}\}$

$$S_P=\underbrace{|f(x_1)-f(x_0)|}_{}+\underbrace{\sum_{i\geq 2}^{m}|f(x_i)-f(x_{i-1})|}_{}$$

$$=\underbrace{|f(x_1)-f(a)|}_{<\eta}\qquad\leq M\quad(\because V[f;a+\varepsilon,b]\leq M\ \text{for all}\ \varepsilon^{*})$$

∴ $S_P<M+\eta$ (for all $\forall\eta>0$)

$\eta\downarrow 0$ 可得：$\underset{\overset{\shortparallel}{V}}{\sup_P S_P}\leq M$

㊟ 分割越細，S_P 越大，所以 ㊕ 的條件並不影響到結論.

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7. $V[a,b] = V[a,x_{i-1}] + V[x_{i-1},x_i] + V[x_i,b]$ ($\because$ Theorem 2.12)

① 首先證明：$V[x_{i-1},x_i] - |f(x_i)-f(x_{i-1})| \le V[a,b] - S_P$

$\Leftrightarrow V[a,b] - V[x_{i-1},x_i] + |f(x_i)-f(x_{i-1})| \ge S_P$

$\Leftrightarrow (V[a,x_{i-1}] + V[x_{i-1},x_i] + V[x_i,b]) - V[x_{i-1},x_i] + |f(x_i)-f(x_{i-1})| \ge S_P$

$\Leftrightarrow \underbrace{V[a,x_{i-1}]}_{①} + \underbrace{V[x_i,b]}_{②} + \underbrace{|f(x_i)-f(x_{i-1})|}_{③} \ge S_P$

$$\begin{cases} ①\ V[a,x_{i-1}] \ge \sum_{j=1}^{i-1} |f(x_j)-f(x_{j-1})| \\ ②\ V[x_i,b] \ge \sum_{j=i+1}^{m} |f(x_j)-f(x_{j-1})| \end{cases}$$

$$①+②+③ \ge \sum_{j=1}^{i-1} |f(x_j)-f(x_{j-1})| + \sum_{j=i+1}^{m} |f(x_j)-f(x_{j-1})| + |f(x_i)-f(x_{i-1})|$$

$$= \sum_{j=1}^{m} |f(x_j)-f(x_{j-1})| = S_P$$

$\therefore$ ① 證明完成。

② 接著證明 $V(x)$ 於 $x=\bar{x}$ 連續

$\sup_P S_P = V[a,b]$，我們取一個 P_0 滿足 $0 \le V[a,b] - S_{P_0} < \frac{\varepsilon}{2}$

f 於 $\bar{x}$ 連續 故 $\forall \varepsilon>0\ \exists \delta_0>0$ s.t. $|x-\bar{x}|<\delta_0 \Rightarrow |f(x)-f(\bar{x})| < \frac{\varepsilon}{2}$

$P \overset{def}{=} P_0 \cup \{x,\bar{x}\}$ where $|x-\bar{x}|<\delta_0$

但若 $\exists r \in P_0$ s.t. $r \in (x,\bar{x})$ (or $(\bar{x},x)$) 則將 x 再拉近 $\bar{x}$。

令 $\delta \overset{def}{=} \|P\| = \min\{\mathrm{dist}(x,y) \mid x,y \in P\}$

利用①的公式 $|V(x)-V(\bar{x})| = |V[a,x]-V[a,\bar{x}]|$
$= V[x,\bar{x}]$ (or $V[\bar{x},x]$) $\leqq \underbrace{|f(x)-f(\bar{x})|}_{<\frac{\varepsilon}{2}} + \underbrace{V[a,b]-S_P}_{<\frac{\varepsilon}{2}}$

$(\because |x-\bar{x}|<\delta\leqq\delta_0)$ $(\because S_P \geqq S_{P_0})$

$\therefore \forall\varepsilon>0\ \exists\delta>0\quad |x-\bar{x}|<\delta \Rightarrow |V(x)-V(\bar{x})|<\varepsilon$

由此可知 $V(x)$ 於 $\bar{x}$ 連續.

③ 至於 $P(x)$, $N(x)$ 的連續性

$P(x)=\frac{1}{2}(V(x)+f(x)-f(a))\quad N(x)=\frac{1}{2}(V(x)-f(x)+f(a))$

$V(x)$, $f(x)$ 均於 $x=\bar{x}$ 連續 $\Rightarrow$ $P(x)$, $N(x)$ 亦於 $\bar{x}$ 連續

~~由~~ $|P(x)-P(\bar{x})| = \left|\frac{1}{2}(V(x)-V(\bar{x})+f(x)-f(\bar{x}))\right|$
$\leqq \frac{1}{2}|V(x)-V(\bar{x})| + \frac{1}{2}|f(x)-f(\bar{x})| < \varepsilon$

$\begin{pmatrix} |x-\bar{x}|<\delta_1 & \Rightarrow & |V(x)-V(\bar{x})|<\varepsilon \\ |x-\bar{x}|<\delta_2 & \Rightarrow & |f(x)-f(\bar{x})|<\varepsilon \end{pmatrix}$

$\Rightarrow |x-\bar{x}|<\min\{\delta_1,\delta_2\} \Rightarrow |P(x)-P(\bar{x})|<\varepsilon.$

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$$f(x) = \begin{cases} 0 & : \quad a \leq x \leq c \\ 1 & : \quad c < x \leq b \end{cases}$$

$$\phi(x) = \begin{cases} 0 & a \leq x < c \\ 1 & c \leq x \leq b \end{cases}$$

① $\int_a^c f(x)\, d\phi \cdots \qquad \Gamma = \{\underset{\overset{\parallel}{a}}{x_0}, \ldots \underset{\overset{\parallel}{c}}{x_m}\}$

$$\sum_{i=1}^{m} \underbrace{f(\xi_i)}_{\to 0} (\phi(x_i) - \phi(x_{i-1}))$$

$$\therefore \int_a^c f\, d\phi = 0$$

② $\int_c^b f(x)\, d\phi \quad \therefore \qquad \Gamma = \{\underset{\overset{\parallel}{c}}{x_0}, \ldots \underset{\overset{\parallel}{b}}{x_m}\}$

$$\sum_{i=1}^{m} f(\xi_i) (\underbrace{\phi(x_i) - \phi(x_{i-1})}_{\to 0 \ (\phi(x_i) = 1,\ \phi(x_{i-1}) = 1)})$$

$$\therefore \int_c^b f(x)\, d\phi = 0$$

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③ $\int_a^b f \cdot d\phi$ 不存在,

$P_1 \cdots \quad \exists\, x_{k-1}, x_k \in P_1 \quad s.t \quad (x_{k-1}, x_k) \ni c$

$P_2 \cdots \quad \nexists\, x_{k-1}, x_k \in P_2 \quad s.t \quad (x_{k-1}, x_k) \ni c$

尤其是 P_1, $\sum_{i=1}^{m} f(\xi_i)(\phi(x_i) - \phi(x_{i-1}))$

$$= f(\xi_k)(\underbrace{\phi(x_k)}_{1} - \underbrace{\phi(x_{k-1})}_{0}) = f(\xi_k)$$

若取 $\xi_k > c$, 则 $f(\xi_k) = 1$.

$P_2 \cdots$ 跟 ①, ② 同理, $\sum_{i=1}^{m} f(\xi_i)(\phi(x_i) - \phi(x_{i-1})) = 0$

$\therefore$ 由此可知, $\int_a^b f d\phi$ 不存在

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① $\lim_{|P|\to\infty} S_P[f;a,b]=\infty$ 的 case

$\forall M>0 \ \exists \delta>0$ st $\forall P: |P|<\delta \quad S_P>M$

$\therefore \sup_P S_P \geq S_P > M \quad \therefore M\nearrow\infty \Rightarrow \sup_P S_P=\infty$

$\therefore V[f;a,b]=\infty \qquad \lim_{|P|\to\infty} S_P[f;a,b]=V[f;a,b]$

② $\lim_{|P|\to\infty} S_P[f;a,b]=S<\infty$ 的 case

$\forall \varepsilon>0 \ \exists \delta_0>0$ st $\forall P: |P|<\delta_0, \ |S-S_P|<\frac{\varepsilon}{2}$

$V \overset{def}{=} V[f;a,b]=\sup_P S_P$

我們欲證明 $S=V$.

取一個 $P_0 \quad 0\leq V-S_{P_0}<\frac{\varepsilon}{2}$

(I) 若 $|P_0|<\delta_0$, 則 $P \overset{put}{=} |P_0|$.

(II) 若 $|P_0|\geq\delta_0$, 則 考慮 P_0 的 refinement P 滿足 $|P|<\delta_0$.
(只要對 P_0 插入一些點即可)

這樣子的 P 使得 $S_P \geq S_{P_0}$，故 $0 \leq V - S_P < \frac{\varepsilon}{2}$

且 $|S - S_P| < \frac{\varepsilon}{2}$.

考慮 $|S - V| = |S - S_P + S_P - V| \leq |S - S_P| + (V - S_P)$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$\therefore \varepsilon \downarrow 0$ 得 $S = V$ $\quad \therefore \lim\limits_{|P| \to 0} S_P = \sup\limits_{P} S_P = V[f; a, b]$

$\therefore$ 證明完成

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ch2. 22

22 首先整理 Exercise 5 的命題…
「$\exists M>0\ \forall \varepsilon>0\ V[a+\varepsilon,b]\leq M<\infty$
$\Rightarrow V[a,b]<\infty$」
其命題的 Contraposition 亦為真
「$V[a,b]=\infty \Rightarrow \forall M>0\ \exists \varepsilon>0\ V[a+\varepsilon,b]>M$」…※1

$x_0 \overset{def}{=} \inf\{x\in[a,b] \mid V[x,b]<\infty\}$

① $V[x_0,b]=\infty$ 的 case…

(若 $\varepsilon_n \downarrow \exists\varepsilon>0$, 則違反 x_0 的定義)

利用 ※1 取 $\{\varepsilon_n\}_{n\geq 1}\subseteq(0,\infty)$ $\varepsilon_n\downarrow 0$.

s.t. $V[x_0+\varepsilon_1,b]\geq 2$
&
· $V[x_0+\varepsilon_{n+1},b]\geq V[x_0+\varepsilon_n,b]+2 \quad (n\geq 1)$

$(\Rightarrow V[x_0+\varepsilon_n, x_0+\varepsilon_{n+1}]\geq 2)$

$P_1 \overset{def}{=}$ partition of $[x_0+\varepsilon_1,b]$ where $S_{P_1}\geq 1$

$P_n \overset{def}{=}_{(n\geq 2)}$ partion of $[x_0+\varepsilon_n, x_0+\varepsilon_{n+1}]$ where $S_{P_n}\geq 1$

定義序列 $\{x_{k,l}\}_{k\in\mathbb{N},\ l=0,1\ldots n_k}$ …… $\{x_{k,l}\} \overset{def}{=} P_k\setminus\{x_0+\varepsilon_n\}$ (where $x_{k,1}>x_{k,2}>\cdots x_{k,n_k}$)

$$\sum_{k=1}^{M}\sum_{l=1}^{n_k}|\phi(x_{k,l})-\phi(x_{k,l-1})| = \sum_{k=1}^{M} S_{P_k} \geq M$$

$\therefore M \uparrow \infty$ 時 $\sum_{k=1}^{\infty}\sum_{l=1}^{n_k}|\phi(x_{k,l})-\phi(x_{k,l-1})| = +\infty$

$\{x_{k,l}\}_{\substack{k\in\mathbb{N}\\ l=1\sim n_k}}$ 為可數序列.

定義 $\{x_n\}_{n=1}$: $\{x_1, x_2, \cdots\} = \{x_{1,1}, x_{1,2}, \cdots x_{2,1}, x_{2,2} \cdots\}$

$x_n \downarrow x_0$ ($\because \varepsilon_n \downarrow 0$)

② $V[x_0, b] < \infty$ 的 case.

$x_0 > a$. ($\because$ 若 $x_0 = a$ 則違反假設)

取一個 $\bar{x}_0 \in (a, x_0) \Rightarrow V[\bar{x}_0, b] = \infty$

(可以利用跟① 同樣方法)

ch2. $\boxed{24}$ f：連續，ϕ：有界變動 on $[a,b]$

證明 $\lim_{\delta\to 0}\int_a^{a+\delta} f d\phi = 0 \Rightarrow f(a)=0$ or $\phi(x)$ 於 a 連續.

根據 Jourdan's theorem, $\exists\{\phi_1,\phi_2\}$ s.t. $\phi=\phi_1-\phi_2$
where ϕ_1,ϕ_2 遞增且有界.

$$\int_a^{a+\delta} f d\phi = \int_a^{a+\delta} f d\phi_1 - \int_a^{a+\delta} f d\phi_2$$

由於 f 為連續函數，故 $\forall\varepsilon>0$ $\exists\delta_\varepsilon>0$ s.t.
$a<x<a+\delta_\varepsilon \Rightarrow f(a)-\varepsilon<f(x)<f(a)+\varepsilon$

- $\int_a^{a+\delta_\varepsilon} f d\phi_1$ 的 Riemann Stiltjes sum：$R_1,[a,a+\delta_\varepsilon]$

$$\sum_{i=1}^{m} f(\xi_i)(\phi_1(x_i)-\phi_1(x_{i-1})) \ge (f(a)-\varepsilon)(\phi_1(a+\delta_\varepsilon)-\phi_1(a))$$
(注：ϕ_1：遞增) $\le (f(a)+\varepsilon)(\phi_1(a+\delta_\varepsilon)-\phi_1(a))$

- $\int_a^{a+\delta_\varepsilon} f d\phi_2$ 的 Riemann Stiltjes sum：$R_2[a,a+\delta_\varepsilon]$

$$\sum_{i=1}^{m} f(\eta_i)(\phi_2(x_i)-\phi_2(x_{i-1})) \ge (f(a)-\varepsilon)(\phi_2(a+\delta_\varepsilon)-\phi_2(a))$$
(注：ϕ_2：遞增) $\le (f(a)+\varepsilon)(\phi_2(a+\delta_\varepsilon)-\phi_2(a))$

由此可知，$\forall\varepsilon>0$ $\exists\delta_\varepsilon>0$ s.t. $|\phi| \overset{def}{=} \phi_1+\phi_2$

$$f(a)\cdot(\phi(a+\delta_\varepsilon)-\phi(a))-\varepsilon(|\phi|(a+\delta_\varepsilon)-|\phi|(a)) \le \int_a^{a+\delta_\varepsilon} f d\underbrace{(\phi_1-\phi_2)}_{\phi}$$
$$f(a)(\phi(a+\delta_\varepsilon)-\phi(a))+\varepsilon(|\phi|(a+\delta_\varepsilon)-|\phi|(a)) \ge$$

我們注意 Φ_1, Φ_2 為有界. $\Rightarrow |\Phi|$ 亦為有界.
$\Rightarrow \varepsilon \searrow 0$ 時, $\varepsilon(|\Phi|(a+\delta_\varepsilon)-|\Phi|(a)) \searrow 0$.

$\therefore \lim_{\delta_\varepsilon \searrow 0} \int f d\phi = \lim_{\delta_\varepsilon \searrow 0} f(a)(\phi(a+\delta_\varepsilon)-\phi(a)) = 0$

$\Leftarrow f(a)=0$ or $\lim_{\delta_\varepsilon \searrow 0} \phi(a+\delta_\varepsilon)-\phi(a)=0$
(中於 a 右連續)

$\therefore$ 證明完成

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ch2. $\boxed{31}$

① $\int_a^b df$ 存在 且 $\int_a^b df = f(b)-f(a)$：

P 為 $[a,b]$ 的分割：$P=\{x_0, x_1, \dots x_m\}$, $x_0 = a$, $x_m = b$

$$R_P = \sum_{j=1}^{m} 1\cdot(f(x_j)-f(x_{j-1})) = \sum_{j=1}^{m}(f(x_j)-f(x_{j-1}))$$

$= f(x_m)-f(x_0) = f(b)-f(a)$. ($\because x_m = b \quad x_0 = a$ for any P)

由此可知，無論 P 為何，R_P 總是等於 $f(b)-f(a)$.

($\forall \varepsilon>0\ \exists \delta>0$ s.t $\forall P$: $|P|<\delta \quad |R_P-(f(b)-f(a))|<\varepsilon$ 顯然成立)

$\therefore \int_a^b df$ 存在 且 $\int_a^b df = f(b)-f(a)$.

② 若 f' 存在 且 於 $[a,b]$ 上 Riemann 可積分，則 $\int_a^b f' dx = f(b)-f(a)$.

$$R_P = \sum_{j=1}^{m} f'(\xi_j)(x_j - x_{j-1}) \quad \text{where } P=\{x_0, \dots x_m\},\ x_{j-1}\le \xi_j \le x_j$$

由於 f' 可微，根據 均值定理，$\exists \xi_j \in (x_{j-1}, x_j)$

s.t $\dfrac{f(x_j)-f(x_{j-1})}{x_j - x_{j-1}} = f'(\xi_j)$

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$\int_a^b f'dx$ 存在，所以不管如何取 ξ_j R_P 都趨近於 $I=\int_a^b f dx$.

我們已知 $\lim_{|P|\to 0} R_P = \int_a^b f'dx$，所以我們可以將均值定理中的 ξ_j，代入 R_P 中的 ξ_j.

$$R_P = \sum_{j=1}^{m} f'(\xi_j)(x_j - x_{j-1}) = \sum_{j=1}^{m} \frac{f(x_j)-f(x_{j-1})}{x_j - x_{j-1}} \cdot (x_j - x_{j-1})$$

$$= \sum_{j=1}^{m} \left(f(x_j) - f(x_{j-1})\right) = f(b) - f(a)$$

由此可知 $\lim_{|P|\to 0} R_P = f(b) - f(a)$ $\therefore \int_a^b f'dx = f(b) - f(a)$.

ch2. $\boxed{32}$

根據 Theorem 2.21，若 $\int_a^b f d\phi$ 存在，則 $\int_a^b \phi df$ 亦存在.

我們證明 $\int_a^b x df$ 存在即可.

f 為有界變動函數，故根據 Jordan's theorem
(Bounded Variation)

$\exists \{f_1, f_2\}$ s.t. $f = f_1 - f_2$ where f_1, f_2: 有界、遞增函數

若 $\int_a^b x df_1$, $\int_a^b x df_2$ 皆存在，則 $\int_a^b x df = \int_a^b x df_1 - \int_a^b x df_2$.

所以我們證明 $\int_a^b x df_1$, $\int_a^b x df_2$ 存在.

利用 Theorem 2.24. 由於 $x \mapsto x$ 為連續函數,
且 f_1, f_2: 有界變動 (∵ 有界且遞增
$\Rightarrow V(f_1; [a,b]) = f_1(b) - f_1(a) < \infty$)

故 $\int_a^b x df_1$, $\int_a^b x df_2$ 皆存在.

∴ $\int_a^b x df$ 存在. $\left(= \int_a^b x df_1 - \int_a^b x df_2\right)$

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ch3. $\boxed{1}$

① 存在性

$c_k \overset{\text{def}}{=} \max\{0,1,2,\dots,b-1 \mid x-\sum_{j=1}^{k}\frac{c_j}{b^j} \geq 0\}$

這樣子的 $\{c_j\}_{j=1,2..k}$ 滿足 $0 \leq x-\sum_{j=1}^{k}\frac{c_j}{b^j} < \frac{1}{b^k}$

故 $k \to \infty$ $\sum_{j=1}^{k}\frac{c_j}{b^j} \nearrow x$. $\therefore x=\sum_{k=1}^{\infty}\frac{a_k}{b^k}$.

② 唯一性.

假設 $\{c_k\}_k \{d_k\}_k \subseteq \{0,1,2,\dots,b-1\}$ 皆滿足:

$x=\sum_{k=1}^{\infty}\frac{c_k}{b^k}=\sum_{k=1}^{\infty}\frac{d_k}{b^k}$.

若非「$c_k=d_k$ for all $k\in\mathbb{N}$」, 令 $k_0=\min\{k \mid c_k\neq d_k\}$

$\therefore c_1=d_1, c_2=d_2, \dots c_{k_0-1}=d_{k_0-1}$, $c_{k_0}\neq d_{k_0}$.

假設 $c_{k_0}<d_{k_0}$.

$\sum_{j=1}^{k_0}\left(\frac{d_j}{b^j}-\frac{c_j}{b^j}\right) \geq \frac{1}{b^{k_0}}$. 我們注意 $x=\sum_{k=1}^{\infty}\frac{c_k}{b^k}=\sum_{k=1}^{\infty}\frac{d_k}{b^k}$.

$\Rightarrow \sum_{j=k_0+1}^{\infty}\left(\frac{c_j}{b^j}-\frac{d_j}{b^j}\right) \geq \frac{1}{b^{k_0}}$

$$\because \frac{1}{b^{k_0}} \leq \sum_{j=k_0+1}^{\infty} \frac{c_j-d_j}{b^j} \overset{\text{三角不等式}}{\leq} \sum_{j=k_0+1}^{\infty} \frac{|c_j-d_j|}{b^j} \overset{\text{原因如下}}{\left(\leq \frac{1}{b^{k_0}}\right)}$$

另外 $|c_j-b_j| \leq b-1$，只有 $|c_j-b_j| = b-1$ (for all $j \geq k_0+1$)

才能滿足 $\sum_{j=k_0+1}^{\infty} \frac{|c_j-d_j|}{b^j} \geq \frac{1}{b^{k_0}}$ (而且等號成立) $\left(\because \sum_{j=k_0+1}^{\infty} \frac{b-1}{b^j} = \frac{b-1}{b^{k_0+1}} \cdot \frac{1}{1-\frac{1}{b}} = \frac{1}{b^{k_0}}\right)$

故可能會發生的狀況是：

「$c_1=d_1$, $c_2=d_2$, … $c_{k_0-1}=d_{k_0-1}$, $d_{k_0}-c_{k_0}=1$ $(\because c_{k_0}<d_{k_0})$

$c_k=b-1$, $d_k=0$ $(k\geq k_0+1)$ 」

$$\therefore x \in \left\{ \sum_{k=1}^{k_0} \frac{d_k}{b^k} \,\middle|\, \{d_1 \sim d_{k_0-1}\} \subseteq \{0,1,\dots,b-1\},\ d_{k_0} \in \{1,2,\dots,b-1\} \right\}_{k_0\geq 1}$$

$$\begin{cases} \cdot\ k_0=1 \text{ 時}\ldots \left\{\frac{1}{b}, \frac{2}{b}, \dots, \frac{b-1}{b}\right\} \\ \cdot\ k_0=2 \text{ 時}\ldots \left\{\frac{1}{b^2}, \frac{2}{b^2}, \dots, \frac{b^2-1}{b^2}\right\} \end{cases}$$

$$x \in \left\{ \frac{1}{b^k}, \frac{2}{b^k}, \dots, \frac{b^k-1}{b^k} \right\}_{k\geq 1}$$

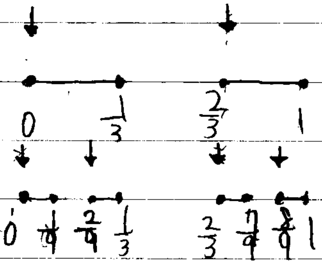
∴ 證明完成

No.

Date . .

ch3、$\boxed{2}$

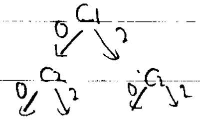
(a) 我們先考慮該如何表達 C_k.



• $k=1$: $=\{x=\frac{c_1}{3}+a \mid c_1=0 \text{ or } 2 \quad a\in[0,\frac{1}{3}]\}$

• $k=2$: $=\{x=\frac{c_1}{3}+\frac{c_2}{9}+a \mid c_1,c_2=0 \text{ or } 2 \quad a\in[0,\frac{1}{9}]\}$

我們注意圖中「↓」的位置可以寫成：



$\{\frac{c_1}{3}+\frac{c_2}{9}+\dots+\frac{c_k}{3^k} \mid (c_1,\dots,c_k)\in\{0,2\}^k\}$

另外，每個區塊的大小是 $\frac{1}{3^k}$.

故 $C_k=\{\sum_{j=1}^{k}\frac{c_j}{3^j}+a_k \mid (c_1,\dots,c_k)\in\{0,2\}^k \quad a_k\in[0,\frac{1}{3^k}]\}$

$C=\bigcap_{k=1}^{\infty}C_k$. 取一個點 $x\in C$.

$\forall k\in\mathbb{N} \quad \exists\{c_j\}_{j=1}^{k}\subset\{0,2\}$

s.t. $|x-\sum_{j=1}^{k}\frac{c_j}{3^j}|=a\leq\frac{1}{3^k}$

∴ $k\to\infty$ 時 $x=\sum_{k=1}^{\infty}\frac{c_k}{3^k}$

(b) 接下來證明 $f(x)\ (x\in C) = \sum_{k=1}\left(\frac{c_k}{2}\right)\cdot 2^{-k}$.

$x\in C \Rightarrow \exists\{c_k\}_{k=1\sim\infty}\subseteq\{0,2\}$ s.t. $x=\sum_{k=1}^{\infty}\frac{c_k}{3^k}$.

令 $x_k \overset{\text{def}}{=} \sum_{j=1}^{k}\frac{c_j}{3^j}$.

$$\lim_{k\to\infty}|f_k(x_k)-f(x)| = \lim_{k\to\infty}|f_k(x_k)-f(x_k)+f(x_k)-f(x)|$$

$$\leq \lim_{k\to\infty}\underbrace{|f_k(x_k)-f(x_k)|}_{\to 0}+\underbrace{|f(x_k)-f(x)|}_{\to 0}$$

f 為連續函數, k 很大的時候, $|x_k-x|<\delta \Rightarrow |f(x_k)-f(x)|<\frac{\varepsilon}{2}$

(f_k 均勻收斂至 f, 無論 x_k 在哪裡, 只要 k 很大即可小於 $\frac{\varepsilon}{2}$)

$\therefore f(x)=\lim_{k} f_k(x_k)$

接下來, 我們研究一下 $f_k(x_k)$:

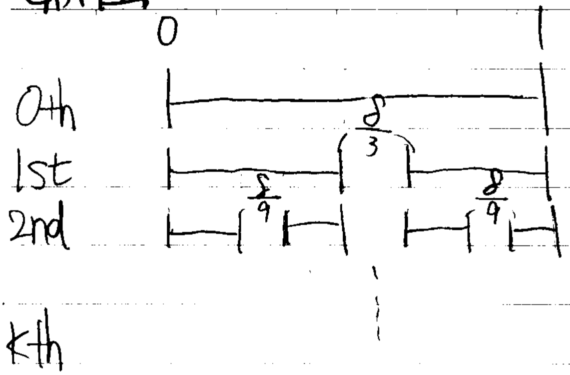
由於 $x\in C$, 所以只要考慮 (a) 的圖中「↓」的點即可. 我們容易發現, 若 $c_k=2$, 則 $f_k(x_k)$ 會增加 $\frac{1}{2^k}$. 故此 $f_k(x_k)=\sum_{j=1}^{k}\frac{1}{2^j}\cdot I_{(c_j=2)}$

$I_{(c_j=2)}$ 為 Indicator function. $I_{(c_j=2)}=\begin{cases}1 & \text{if } c_j=2\\ 0 & \text{if } c_j=0\end{cases}$
所以可寫成 $\frac{1}{2}c_j$.

$\therefore f_k(x_k)=\sum_{j=1}^{k}\left(\frac{c_j}{2}\right)\cdot\frac{1}{2^j}$

$\therefore f(x)=\lim_{k\to\infty}f_k(x_k)=\sum_{k=1}^{\infty}\frac{c_k}{2}\cdot\frac{1}{2^k}$.

ch2. 5



測度

1

$1-\frac{\delta}{3}$

$1-\frac{\delta}{3}-\frac{2\delta}{9}$

$1-\sum_{j=1}^{k}\frac{1}{3}\left(\frac{2}{3}\right)^{j-1}\delta$

令 $C_{\delta,k}$ 表示第k次操作完後的集合.

$C_\delta \overset{\text{def}}{=} \bigcap_{k=1}^{\infty} C_{\delta,k}$.　C_δ 為我們所求的集合.

① C_δ 為 perfect set 的證明

(I) C_δ 為閉集合…　$C_\delta^c = \bigcup_{k=1}^{\infty} C_{\delta,k}^c$ … 開集合

(II) C_δ 上的點都是極限點;

令 $I_{k,j}$ 表示第k次操作完後的閉區間 ($j=1\sim 2^k$)

$\Rightarrow C_{\delta,k} = \sum_{j=1}^{2^k} I_{k,j}$　(Σ: disjoint union)

C_δ 包含 $I_{k,j}$ 兩端的點, (觀察操作即可知) … ※1

且 $I_{k,j}$ 的 Lebesgue 測度 $\mu(I_{k,j}) \to 0$ (as $k\to\infty$)

換言之 $I_{k,j}$ 兩端的距離趨近於 0.

取一個任意的點 $x \in C_\delta = \bigcap_{k=1}^{\infty} C_{\delta,k}$

$\Rightarrow x \in C_{\delta,k}$ (for all $k \in \mathbb{N}$) $C_{\delta,k} = \sum_{j=1}^{2^k} I_{k,j}$ (disjoint union)

$\Rightarrow \forall k \ \exists j \in \{1,2,3 \ldots 2^k\}$ s.t. $x \in I_{k,j}$

我們取一個任意正數 $\varepsilon > 0$，並考慮 $B_\varepsilon(x)$。

$\forall \varepsilon > 0 \ \exists k$ s.t. $\varepsilon > \mu(I_{k,j}) = \frac{1}{2^k}\left(1 - \sum_{j=1}^{k} \frac{1}{3}\left(\frac{2}{3}\right)^{j-1} \delta\right)$
(夠大)

$\therefore B_\varepsilon(x) \setminus \{x\}$ 至少包含 $I_{k,j}$ 兩端的點的其中一個。 它在 C_δ 裡面 ($\because$ (1))

$\Rightarrow B_\varepsilon(x) \setminus \{x\} \cap I_{k,j} \supseteq \{I_{k,j}$ 的 end points$\} \subseteq C_\delta$

$\Rightarrow B_\varepsilon(x) \setminus \{x\} \cap C_\delta \neq \phi \ (\forall \varepsilon)$ $\therefore x$ 為極限點

② C_δ 的測度：利用測度的連續性

$\{C_{\delta,k}\}_{k \geq 1}$ 均為可測集合（閉區間的 disjoint union）($\Rightarrow C_\delta$ 為可測)

且 $\mu(C_{\delta,1}) < \infty$，$C_{\delta,k} \downarrow C_\delta$。$\Rightarrow \mu(C_\delta) = \mu(\bigcap C_{\delta,k}) =$

$= \lim_k \mu(C_{\delta,k}) = \lim_k \left(1 - \sum_{j=1}^{k} \frac{1}{3}\left(\frac{2}{3}\right)^{j-1} \cdot \delta\right) = 1 - \sum_{k=1}^{\infty} \frac{\delta}{2} \cdot \left(\frac{2}{3}\right)^k = 1 - \delta$

③ C_δ 不包含閉區間。若 $[a,b] \subseteq C_\delta$ 則 $[a,b] \subseteq C_{\delta,k}$ ($\forall k$)

$\Rightarrow \forall k \ \exists j \in \{1,2 \ldots 2^k\}$ s.t. $[a,b] \subseteq I_{k,j}$ 但 $\mu(I_{k,j}) \to 0$ (as $k \to \infty$)
$\therefore$ 不可能包含。

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ch3、$\boxed{9}$ Borel-Cantelli's lemma

為了方便起見，將外測度 $|\cdot|_e$ 寫為 $\mu(\cdot)$.

外測度 $\mu: 2^S \to [0,\infty]$，具有 ① 非負性 ② 單調性 以及
③ 次可加性

- $\limsup_n E_n = \bigcap_{n=1}^{\infty} \bigcup_{m \geq n}^{\infty} E_m = \bigcap_{n=1}^{\infty} A_n$ (where $A_n = \bigcup_{m \geq n}^{\infty} E_m$)

$\therefore A_n \downarrow \limsup_n E_n$ $(\limsup E_n \subseteq A_n \subseteq A_{n+1})$

- $\mu(\limsup_n E_n) \leq \mu(A_n)$ $(\forall n \in \mathbb{N})$

(② 單調性)

$= \mu(\bigcup_{m \geq n}^{\infty} E_m) \leq \sum_{m=n}^{\infty} \mu(E_m)$

(③ 次可加性)

- 由於 $\sum_{m=1}^{\infty} \mu(E_m) < \infty$，故 $\lim_n \sum_{m \geq n}^{\infty} \mu(E_m) = 0$. … ⊛ ↗

$\therefore 0 \leq \mu(\limsup_n E_n) \leq \sum_{m \geq n}^{\infty} \mu(E_m)$ $(\forall n \in \mathbb{N})$

(① 非負性)

$\therefore n \to \infty$ 得 $\mu(\limsup E_n) = 0$

$\liminf E_n \subseteq \limsup E_n$

$0 \leq \mu(\liminf E_n) \leq \mu(\limsup E_n) = 0$
(② 單調性)

∴ 證明完成

註 ⊛ 的部份：

$S_n \overset{def}{=} \sum_{m=1}^{n} \mu(E_m)$ $\{S_n\}_{n\geq 1}$ 為遞增且收斂的序列

$S_n \uparrow S < \infty$. $\sum_{m\geq n+1}^{\infty} \mu(E_m) = S - S_n$

由於 $S_n \uparrow S < \infty$, 故 $\forall \varepsilon > 0$ $\exists N$ s.t. $\forall n \geq N$

$|S - S_n| < \varepsilon \Leftrightarrow \sum_{m\geq n+1}^{\infty} \mu(E_m) < \varepsilon$

$\therefore \lim_{n\to\infty} \sum_{m\geq n+1}^{\infty} \mu(E_m) = 0$

實分析(I) 作業4　R05246013 森元俊成

3.12 (No.1)

① $E_1 \times E_2 \subseteq \mathbb{R}^2$ 為 Lebesgue 可測集合

根據 Theorem 3.28, $E_1 = F_1 \times N_1$　$E_2 = F_2 \times N_2$　where $\{F_1, F_2\} \subseteq \mathcal{F}_\sigma$ and $|N_1|, |N_2| = 0$。 $E_1 \times E_2 = (F_1 \times F_2) \cup (N_1 \times F_2) \cup (F_1 \times N_2) \cup (N_1 \times N_2)$

- $F_1 = \bigcup_{n\ge1} F_{1,n}$, $F_2 = \bigcup_{n\ge1} F_{2,n}$　$F_1 \times F_2 = \bigcup_{n=1, m=1}^{\infty} F_{1,n} \times F_{2,m}$

$\{F_{1,n}\} \cup \{F_{2,n}\}$ 為 R^1 上的閉集合, $F_{1,n} \times F_{2,n} \subset R^2$ 為閉集合
$\therefore F_1 \times F_2$ 為 Lebesgue 可測集合.

- 接著證明, $^\forall E \subseteq \mathbb{R}^1$　$^\forall N \subset R^1$: $|N| = 0$　$|E \times N|_2 = 0$.
(所以 $E \times N$ 為 R^2 上的 Lebesgue 可測集合)

$E_n \overset{def}{=} E \cap [\frac{-n}{2}, \frac{n}{2}]$, $\{I_m\}_{m\ge1} \subseteq \{R$上的閉區間$\}$ 為 N 的覆蓋.

$E_n \times N \subseteq [\frac{-n}{2}, \frac{n}{2}] \times \bigcup_{m=1}^{\infty} I_m = \bigcup_{m=1}^{\infty} \underbrace{[\frac{-n}{2}, \frac{n}{2}] \times I_m}_{R^2\text{上的閉區間}}$

$|E_n \times N|_{2e} \le \sum_{m=1}^{\infty} v_2([\frac{-n}{2}, \frac{n}{2}] \times I_m) = \sum_{m=1}^{\infty} n \cdot |I_m|$

由於 $|N| = 0$, 故 $\{I_m\}_{m\ge1}$ 可滿足 $\sum_{m\ge1}^{\infty} |I_m| < \varepsilon$ (for all $\varepsilon > 0$)

$\therefore |E_n \times N|_{2e} < n\varepsilon$. $\therefore$ $\varepsilon \downarrow 0$ 得 $|E_n \times N|_2 = 0$

$\lim_n |E_n \times N|_2 = |\bigcup_{n\ge1} E_n \times N| = |E \times N|_2 = 0$

由此可知, $E \times N$ 為 Lebesgue 可測 (on R^2). $\therefore E_1 \times E_2 = (F_1 \times F_2) \cup (N_1 \times F_2) \cup (F_1 \times N_2) \cup (N_1 \times N_2)$ 為可測.
($N \times E$ 也是)

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② $|E_1 \times E_2|_2 = |E_1||E_2|$ (When E_1, E_2 為有界)

由於 E_1, E_2 為 Lebesgue 可測，故存在 $\{G_{1,n}\}_{n\geq1}$ $\{G_{2,n}\}_{n\geq1}$
$|G_{1,n} \setminus E_1|_e < \frac{1}{n}$，$|G_{2,n} \setminus E_2|_e < \frac{1}{n}$ (開集合的序列)

我們不妨假設 $G_{1,n} \supseteq G_{1,n+1}$，$G_{2,n} \supseteq G_{2,n+1}$，且 $|G_{1,n}| < \infty$
(若 $G_{1,n} \not\supseteq G_{1,n+1}$ 則 $G_{1,n+1} \overset{re\text{-}def}{=} G_{1,n} \cap G_{1,n+1}$) $|G_{2,n}| < \infty$
($\because E_1, E_2$ 有界)

$G_1 \overset{def}{=} \bigcap_{n\geq1}^{\infty} G_{1,n}$，$G_2 \overset{def}{=} \bigcap_{n\geq1}^{\infty} G_{2,n}$ $\therefore G_1, G_2$ 可測 (G_δ 集合)

- $|G_1 \setminus E_1| \leq |G_{1,n} \setminus E_1| < \frac{1}{n}$ (for all n) $\therefore |G_1 \setminus E_1| = 0$
根據 Corollary 3.25，$|G_1| = |E_1|$. $|G_2| = |E_2|$.

- $|G_1 \times G_2| = |G_1||G_2|$ ...

根據 Theorem 1.11，$\exists \{I_{1,n,k}\}_{k\geq1}$，$\{I_{2,n,\ell}\}_{\ell\geq1}$ (non-overlapping 的閉區間)
$G_{1,n} = \bigcup_{k\geq1} I_{1,n,k}$，$G_{2,n} = \bigcup_{\ell\geq1} I_{2,n,\ell}$

根據 Corollary 3.24 $|G_{1,n}| = \sum_{k\geq1} |I_{1,n,k}|$，$|G_{2,n}| = \sum_{\ell\geq1} |I_{2,n,\ell}|$
而且 $|G_{1,n} \times G_{2,n}| = |\bigcup_{k,\ell} I_{1,n,k} \times I_{2,n,\ell}| = \sum_k \sum_\ell |I_{1,n,k} \times I_{2,n,\ell}|_2$
$= \sum_k \sum_\ell |I_{1,n,k}||I_{2,n,\ell}| = (\sum |I_{1,n,k}|)(\sum |I_{2,n,\ell}|) = |G_{1,n}||G_{2,n}|$

($\because \{I_{1,n,k} \times I_{2,n,\ell}\}_{k,\ell}$ 亦為 non-overlapping 的 closed-interval)

(遞減集合的測度連續性)
$\therefore |G_{1,n} \times G_{2,n}|_2 = |G_{1,n}||G_{2,n}|$ $(<\infty)$
$\Rightarrow |\bigcap_{n\geq1}^{\infty} G_{1,n} \times G_{2,n}|_2 = \lim_{n\to\infty} |G_{1,n} \times G_{2,n}| = \lim_{n} |G_{1,n}||G_{2,n}| = |\bigcap_{n\geq1}^{\infty} G_{1,n}||\bigcap_{n\geq1}^{\infty} G_{2,n}|$
$\therefore |G_1 \times G_2| = |G_1||G_2|$

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$\boxed{3.12}$ (NO.2)

- $|G_1 \times G_2| = |E_1 \times E_2|$

Corollary 3.25: $G_1\times G_2$, $E_1 \times E_2$ 均可測, ($\because$ ①)

$\Rightarrow |G_1\times G_2 \setminus E_1\times E_2| = |G_1\times G_2| - |E_1 \times E_2|$

另外. $|G_1\times G_2 \setminus E_1\times E_2| \leq |(G_1\setminus E_1)\times G_2| + |G_1 \times (G_2\setminus E_2)|$

($|G_1\setminus E_1| = 0$
根據①, $\forall E\ \forall N: |N|=0 \quad |E\times N|_2 = 0$)

$\therefore |G_1\times G_2| - \underbrace{|E_1\times E_2|}_{<\infty} = 0 \qquad \therefore |G_1\times G_2| = |E_1\times E_2|$.

(我們考慮有界的情況. 測度應為有限.)

- 綜合上述結論, $|G_1\times G_2| = |E_1\times E_2|$

$$\begin{aligned} &\| \\ &|G_1|\cdot|G_2| = |E_1||E_2| \end{aligned}$$

③ $|E_1\times E_2|_2 = |E_1||E_2|$ (E_1, E_2 不一定有界)

$$\begin{cases} E_{1,n} \overset{\text{def}}{=} E_1 \cap [-n,n] \\ E_{2,n} \overset{\text{def}}{=} E_2 \cap [-n,n] \end{cases}$$

$E_{1,n}$. $E_{2,n}$ 為可測集合

$$|E_1 \times E_2| = \left| \bigcup_{n \geq 1} E_{1,n} \times E_{2,n} \right| = \lim_{n \to \infty} |E_{1,n} \times E_{2,n}|_2$$

$$(E_{1,n} \times E_{2,n} \nearrow E_1 \times E_2)$$

$$= \lim_n |E_{1,n}| \, |E_{2,n}| = \left| \bigcup_{n \geq 1} E_{1,n} \right| \left| \bigcup_{n \geq 1} E_{2,n} \right| = |E_1| \, |E_2|$$

$$(E_{1,n} \nearrow E_1, \quad E_{2,n} \nearrow E_2)$$

(Theorem 3.26)

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3.17

設 $A \subseteq [0,1]$ 為 Lebesgue 非可測集合.

$E \overset{def}{=} f^{-1}(A)$ where f 為 Cantor-Lebesgue 函數.

$f(E) = A$ 顯然為 Lebesgue 非可測集合

我們證明 E 為 Lebesgue 可測集合.

$$E = f^{-1}(A) = \underbrace{(f^{-1}(A) \cap C)}_{①} \cup \underbrace{(f^{-1}(A) \cap [0,1] \setminus C)}_{②} \in \mathcal{L}^1 \text{ (Lebesgue 可測)}$$

∵

① ∵ $f^{-1}(A) \cap C \subseteq C$ 且 $|C| = 0$, 故 $f^{-1}(A) \cap C$ 亦為 measure-zero 集合. 故 $f^{-1}(A) \cap C \in \mathcal{L}^1$ (Lebesgue 可測)

② ∵ $f^{-1}(A) \cap [0,1] \setminus C$ 為由可數個開區間所構成的集合. 故 $f^{-1}(A) \cap [0,1] \setminus C$ 為可測集合. ∎

原因 令 C_k 為 $\left\{ \sum_{j=1}^{k} \frac{c_j}{3^j} + a \;\middle|\; \{c_j\} \subseteq \{0,2\} \quad a \in [0, \frac{1}{3^k}] \right\}$

那麼 $C = \bigcap_{k=1}^{\infty} C_k$.

$$f^{-1}(A) \cap [0,1] \setminus C = \bigcup_{k=1}^{\infty} \left(f^{-1}(A) \cap [0,1] \setminus C_k \right)$$

$$[0,1] \setminus C_k = \bigcup_{l=1}^{2^k-1} I^{\circ}_{k,l} \quad (I^{\circ}_{k,l} \text{ 為開區間})$$

$\forall x, y \in I^{\circ}_{k,l} \quad f(x) = f(y)$ (x 於 $I^{\circ}_{k,l}$ 時, $f(x)$ 只取一個值)

故 $f^{-1}(A) \cap [0,1] \setminus C_k = \bigcup_{l=1}^{2^k-1} I^{\circ}_{k,l} \in \mathcal{L}^1$ (Lebesgue 可測)

(∵ $f(x) \in A$ when $x \in I^{\circ}_{k,l}$)

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故 E 為 Lebesgue 可測.

證明完成 ▣

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$\boxed{3.21}$

[0,1] 的 Lebesgue 外測度為 1.
子集合存在 Lebesgue 非可測集合 A. (利用跟課本同樣方法)
($A \subset [0,1]$)

$A_n \overset{def}{=} \{x+\frac{1}{n} \mid x\in A\}$ $(n=1,2,3,\dots)$

我們注意, $A_n\cap A_m=\phi$ for all $(m,n)\in \mathbf{N}\times \mathbf{N}$ $(m\neq n)$
(若存在 $x_1\in A_n$, $x_2\in A_m$ s.t. $x_1=x_2$, 則 $x_1-\frac{1}{n}, x_2-\frac{1}{m} \in A$.
$(x_1-\frac{1}{n})-(x_2-\frac{1}{m}) = \frac{1}{m}-\frac{1}{n}\in \mathbb{Q}$ ∴ $x_1-\frac{1}{n} = x_2-\frac{1}{m} \Rightarrow n=m$.
∴ $m\neq n \Rightarrow A_n\cap A_m=\phi$)

$E_n \overset{def}{=} \bigcup_{m\geq n} A_m$ $(E_{n+1}\subseteq E_n)$

$A_n\subseteq E_n\subseteq[0,2]$ ∴ $0<|A|_e=|A_n|_e\leq|E_n|_e<2$
$(|E_n|_e<\infty)$

由此可知, $\liminf_n |E_n| = \lim_n |E_n| >0$ ($|E_n|$ 遞減且有界) ⇒ 極限存在

另外 $E_n\downarrow \limsup_n A_n = \phi$

(∵ $\{A_n\}_{n\geq1}$ 為 disjoint. 故不存在一個點 infinitely often 出現在 A_n $(n=1,2,\dots)$.

∴ $|\bigcap E_n| = |\phi| = 0$ ∴

$\therefore$ $\varlimsup_{n} |E_n|_e > 0$ | $\bigcap_{n\geq 1}^{\infty} E_n|_e = 0$

where $E_n \supseteq E_{n+1} \cdots$

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$\boxed{3.23}$

$Z_n \subseteq \mathbb{Z}$

$Z_n \overset{def}{=} [-n, n] \cap \mathbb{Z}$. $(|Z_n| \leq |\mathbb{Z}| = 0)$

我們先證明 $\{x^2 \mid x \in Z_n\}$ 的 Lebesgue 測度為零.

考慮 $\{I_m\}_{m\geq 1} \subseteq \{\mathbb{R}$ 上的閉區間$\}$ where $Z_n \subseteq \bigcup_{m=1}^{\infty} I_m$.

若 $I_m = [a_m, b_m]$, 則 $\{x^2 \mid x \in I_m\} =$

$[a_m^2, b_m^2]$, $[0, \max\{a_m^2, b_m^2\}]$, $[b_m^2, a_m^2]$
$(0 \leq a_m \leq b_m)$ $(a_m \leq 0 \leq b_m)$ $(a_m \leq b_m \leq 0)$

若 $a_m \leq 0 \leq b_m$, 則將 I_m 分為兩個 non-overlapping 的閉區間: $[a_m, 0] \cup [0, b_m]$.
根據 lemma 3.15, 這樣子的操作並不會改變 $\sum_{m=1} |I_m|$

$J_m \overset{def}{=} \{x^2 \mid x \in I_m\}$. ($\Rightarrow$ 我們只要考慮 $0 \leq a_m \leq b_m$ or $a_m \leq b_m \leq 0$ 即可)

$\{x^2 \mid x \in Z_n\} \subseteq \bigcup_{m=1}^{\infty} J_m$ $(\because J_m = [a_m^2, b_m^2]$ or $[b_m^2, a_m^2])$

$|\{x^2 \mid x \in Z_n\}| \leq \sum_{m=1}^{\infty} |J_m| = \sum_{m=1}^{\infty} |a_m^2 - b_m^2| = \sum_{m=1}^{\infty} |a_m - b_m||a_m + b_m|$

$\leq \sum_{m=1}^{\infty} 2n(b_m - a_m) = (2n) \sum_{m=1}^{\infty} (b_m - a_m) = (2n) \sum_{m=1}^{\infty} |I_m|$
$(Z_n \subseteq [-n, n])$

$< 2n\varepsilon$ ($\because Z_n$ 的 Lebesgue 測度為零)

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$\therefore \varepsilon \downarrow 0$ 得: $|\{x^2 \mid x \in Z_n\}| = 0$

$\therefore |\{x^2 \mid x \in Z\}| = |\bigcup_{n \geq 1}^{\infty} \{x^2 \mid x \in Z_n\}|$

$\leq \sum_{n=1}^{\infty} |\{x^2 \mid x \in Z_n\}| = 0$ $\therefore$ 證明完成

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$\boxed{3.25}$

令 $\{U_n\}_{n\geq 1} \subseteq \{(a,b) \subseteq [0,1] \mid \{a,b\} \subseteq \mathbb{Q}\}$.
(換言之, U_n 為 $[0,1]$ 之子集合的開區間, 而且其兩端的點均為有理數.) 我們注意 $\{U_n\}_{n\geq 1}$ 為可數集合.

① $U_1 \overset{\text{put}}{=} (a_1, b_1)$
我們於 (a_1,b_1) 中構造一個 fat-Cantor Set A_1. (其測度非零) $(a_1,b_1)\setminus A_1$ 為可數個開區間的聯集. 我們取其中一個開區間, 並於其中構造一個 fat-Cantor Set A_2.

② $U_2 \overset{\text{put}}{=} (a_2, b_2)$ 但 $U_2 \subseteq [0,1] \setminus \bigcup_{n=1}^{2} A_n$
(有理數的稠密性, $[0,1]\setminus \bigcup_{n=1}^{2} A_n$ 為可數個區間的聯集。所以一定可以取這樣子的 U_2.)
透過跟① 同樣的操作, 我們得 A_3, A_4.

③ 接著繼續同樣的操作, 得 $\{A_{2n-1}\}_{n\geq 1}, \{A_{2n}\}_{n\geq 1}$

$E \overset{\text{put}}{=} \bigcup_{n=1}^{\infty} A_{2n} \Rightarrow \bigcup_{n=1}^{\infty} A_{2n-1} \subseteq E^c$ ($\because A_{2n-1}, A_{2n}$ 為 disjoint.)

④ $\forall I \subseteq [0,1]$ $\exists m$ st $U_m \subseteq I$.

$\because U_m$ 包含 A_{2m-1} 與 A_{2m}, 且其測度為正.

$\therefore$ 證明完成

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$\boxed{3.31}$

令 $g(x) = x + f(x)$ $(x \in [0,1])$
where $f(x)$ 為 Cantor-Lebesgue 函數

$g(x): [0,1] \to [0,2]$ 為連續遞增函數.
故 $g(x)$ 為 one-to-one & onto.

以下是接下來使用的符号… ↗ $(S,\mathcal{A})$ $(T,\mathcal{B})$：可測空間

- $M((S,\mathcal{A}) \to (T,\mathcal{B}))$：可測函數 $f: S \to T$ 的集合
- $\sigma[\mathcal{G}]$：包含 $\mathcal{G}$ 最小的 σ-algebra.
- $\mathcal{B}([0,1])$ 為 $[0,1]$ 上所有 Borel 集合的集合.
  $(\mathcal{B}([0,1]) = \sigma[\mathcal{O}^1]|_{[0,1]} = \sigma[\mathcal{O}^1|_{[0,1]}]_{[0,1]}$
- $\mathcal{O}^1$ … $\mathbb{R}^1$ 上的所有開集合的集合,
- $|_{[0,1]}$ … 限制於 $[0,1]$ 上. $\mathcal{A}|_{[0,1]} = \{A \cap [0,1] \mid A \in \mathcal{A}\}$

① $g \in M(([0,1], \mathcal{B}([0,1])) \to ([0,2], \mathcal{B}([0,2])))$

$\because$ g 為連續函數, $\therefore$ $\forall G \in \mathcal{O}^1|_{[0,2]}$ ($[0,2]$ 上的開集合)
$g^{-1}(G) \in \mathcal{O}^1|_{[0,1]} \subseteq \sigma[\mathcal{O}^1|_{[0,1]}]_{[0,1]} = \mathcal{B}([0,1])$

考慮 σ-algebra：$\mathcal{F} = \{B \subseteq [0,2] \mid f^{-1}(B) \in \mathcal{B}([0,1])\}$
由於 $\mathcal{F}$ 為 σ-algebra ($\because \mathcal{B}([0,1])$ 為 σ-algebra.)
故 $\mathcal{O}^1|_{[0,2]} \subseteq \mathcal{F} \Rightarrow \sigma[\mathcal{O}^1|_{[0,2]}]_{[0,2]} \subseteq \mathcal{F}$.
$\therefore \mathcal{B}([0,2]) \subseteq \mathcal{F}$.
$\therefore \forall B \in \mathcal{B}([0,2])$, $g^{-1}(B) \in \mathcal{B}([0,1])$ (可測函數的定義)

同理, $g^{-1}: [0,2] \to [0,1]$ 亦為可測函數. ($\because$ 連續函數)
$\therefore g^{-1} \in M(([0,2], \mathcal{B}([0,2])) \to ([0,1], \mathcal{B}([0,1])))$

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② $g(C)$ 為 Borel 可測集合. (C 為 Cantor-Set)

$\because\ g^{-1} \in \mathcal{M}(([0,2], \mathcal{B}([0,2])) \to ([0,1], \mathcal{B}([0,1])))$
$C \in \mathcal{B}([0,1]) \Rightarrow (g^{-1})^{-1}(C) = g(C) \in \mathcal{B}([0,2])$

③ $g(C)$ 的 Lebesgue 測度非零, 故能取一個 Lebesgue 非可測集合 $N \subseteq g(C)$. ($\because$ Corollary 3.39)

證明 $|g(C)| = 1$ … 我們考慮 $g([0,1] \setminus C)$
$[0,1] \setminus C$ 可寫成開區間的聯集. $[0,1] \setminus C = \sum_{n=1}^{\infty} (a_n, b_n)$ (disjoint)
$g([0,1] \setminus C) = \bigcup_{n=1}^{\infty} (a_n + f(a_n), b_n + f(b_n))$
$x \in [0,1] \setminus C$ 時 $f(x)$ 只取一個值, 故 $f(a_n) = f(b_n)$.
故 $|g([0,1] \setminus C)| = |[0,1] \setminus C| = 1$
$\therefore |g([0,1]) \setminus g(C)| = |[0,2] \setminus g(C)| = |[0,2]| - |g(C)| = 1$
$\therefore |g(C)| = 1$

④ $g^{-1}(N) \subseteq C$ 為 Lebesgue 可測, 但並非 Borel 可測.

C 為 Lebesgue 測度為零的集合, 故其子集合亦為測度零的集合,
$\therefore g^{-1}(N)$ 為 Lebesgue 可測.

利用 g^{-1} 為 Borel 可測函數

N 不是 Lebesgue 可測, 故也不是 Borel 可測.
假如 $g^{-1}(N)$ 為 Borel 可測, 則 $(g^{-1})^{-1}(g^{-1}(N)) = g \circ g^{-1}(N) = N$
應為 Borel 可測. (矛盾) 故 $g^{-1}(N)$ 為我們所求的集合.

注) $\mathcal{L}^d$… $\mathbb{R}^d$上的Lebesgue可測集合
$\mathcal{B}(\mathbb{R})$… $\mathbb{R}$上的Borel可測集合

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(1)
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$\boxed{4.3}$ $f = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}$ where $f_1 : \mathbb{R}^d \to \mathbb{R}$, $f_2 : \mathbb{R}^d \to \mathbb{R}$
$(f : \mathbb{R}^d \to \mathbb{R}^2)$

我們證明「$\forall G \in \mathcal{O}^2 \quad f^{-1}(G) \in \mathcal{L}^d$」⇔
「$\{f_1, f_2\} \subseteq \mathcal{M}((\mathbb{R}^d, \mathcal{L}^d) \to (\mathbb{R}, \mathcal{B}(\mathbb{R})))$」

$G = \bigcup_{n=1}^{\infty} I_n \quad (I_n = I_{1n} \times I_{2n})$ where $\{I_n\}_{n \geq 1}$ 為 $\mathbb{R}^2$上的閉區間.
($\because$ Theorem. 1.11)

$f^{-1}(G) = f^{-1}\left(\bigcup_{n=1}^{\infty} I_n\right) = \bigcup_{n=1}^{\infty} f^{-1}(I_n) = \bigcup_{n=1}^{\infty} \left(f_1^{-1}(I_{1n}) \cap f_2^{-1}(I_{2n})\right)$
$= \left(\bigcup_{n=1}^{\infty} f_1^{-1}(I_{1n})\right) \cap \left(\bigcup_{n=1}^{\infty} f_2^{-1}(I_{2n})\right) \quad (I_{1n} = [a_n, b_n] \quad I_{2n} = [c_n, d_n])$

① $\Rightarrow$
$G \overset{\text{put}}{=} (a, \infty) \times (-\infty, \infty) \quad (\subset \mathcal{O}^2) \quad (a \in \mathbb{R})$
$f^{-1}(G) = f_1^{-1}((a, \infty)) \cap f_2^{-1}((-\infty, \infty)) = f_1^{-1}((a, \infty)) \in \mathcal{L}^d \quad (\forall a \in \mathbb{R})$
由此可知, $f_1 \in \mathcal{M}((\mathbb{R}^d, \mathcal{L}^d) \to (\mathbb{R}, \mathcal{B}(\mathbb{R})))$ (可測函數)
同理, $f_2 \in \mathcal{M}((\mathbb{R}^d, \mathcal{L}^d) \to (\mathbb{R}, \mathcal{B}(\mathbb{R})))$

② $\Leftarrow$
$f^{-1}(G) = \bigcup_{n=1}^{\infty} f_1^{-1}(I_{1n}) \cap \bigcup_{n=1}^{\infty} f_2^{-1}(I_{2n})$

f_1, f_2 均為可測函數 $\therefore$ $f_1^{-1}(I_{1n}) = f_1^{-1}(\overset{I_{1n}}{[a_n, b_n]})$
$= \{f_1 \geq a_n\} \setminus \{f_2 > b_n\} \in \mathcal{L}^d$
$\therefore \bigcup_{n=1}^{\infty} f_1^{-1}(I_{1n}) \in \mathcal{L}^d$
同理 $\bigcup_{n=1}^{\infty} f_2^{-1}(I_{2n}) \in \mathcal{L}^d$

$f^{-1}(G) \in \mathcal{L}^d$

$\therefore$

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4.5 令 $f_0(x)$ 為 Cantor-Lebesgue 函數.

$g(x) \stackrel{def}{=} x + f_0(x) \quad (g : [0,1] \to [0,2])$

不難驗證, $g(x) : [0,1] \to [0,2]$ 為連續且嚴格遞增的函數, 故為 one-to-one & onto.

如 Exercise 3.31 (Homework 4) 所證明過的. $g(C)$ 的 Lebesgue 外測度為正, 故其子集合存在 Lebesgue 非可測集合. (C: Cantor Set)

設 $A \subseteq g(C)$ 為 Lebesgue 非可測集合.

$f \stackrel{def}{=} g^{-1}, \quad \phi \stackrel{def}{=} I_{g^{-1}(A)}(x)$

- 如 Exercise 3.31 所提到的, g^{-1} 亦為連續函數. ($\stackrel{g^{-1}}{\overset{\|}{f}}$ 為連續)

  故 $g^{-1} \in M(([0,2], \mathcal{B}([0,2])) \to ([0,1], \mathcal{B}([0,1])))$ (可測函數)

- $g^{-1}(A) \subseteq C$, C 為 Lebesgue 測度 0 的集合, 故 $g^{-1}(A)$ 亦為 Lebesgue 測度 0 的集合, 故可測. (Lebesgue 可測集合的完備性)

  $g^{-1}(A) \in \mathcal{L}^1([0,1]) \Rightarrow \phi \in M(([0,1], \mathcal{L}^1([0,1])) \to ([0,1], \mathcal{B}([0,1])))$

由此可知, f, ϕ 分別為 $\mathcal{B}([0,2]) / \mathcal{B}([0,1])$ 可測,

$\mathcal{L}^1([0,1]) / \mathcal{B}([0,1])$ 可測

註 $\mathcal{L}^1$ 表示 R上的 Lebesgue 可測集合族. (跟 L^p 無關)

然而 $\phi \circ f$ 並非可測函數

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$$\phi\circ f = I_{g^{-1}(A)}(g^{-1}(x)) = \begin{cases} 1 & g^{-1}(x) \in g^{-1}(A) \\ 0 & g^{-1}(x) \in g^{-1}(A) \end{cases}$$

$$\Leftrightarrow \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases} \quad (\because g \text{為 one-to-one})$$

$\therefore \phi\circ f = I_A(x)$. $A \notin \mathcal{L}^1$ (Lebesgue 非可測)

證明完成.

4.8 假設 $f,g: E\to\mathbb{R}$

(a) (i) f與g均為 upper semi continuous $\Rightarrow$ $f+g$ 亦為 upper semi continuous

$\lim\limits_{\delta\downarrow 0}\sup\limits_{x\in B(x_0,\delta)\cap E} f(x) \le f(x_0)$, $\lim\limits_{\delta\downarrow 0}\sup\limits_{x\in B(x_0,\delta)\cap E} g(x) \le g(x_0)$

$$\sup_{x\in B(x_0,\delta)\cap E}(f(x)+g(x)) \le \sup_{x\in B(x_0,\delta)\cap E} f(x) + \sup_{x\in B(x_0,\delta)\cap E} g(x)$$

$$\therefore \lim_{\delta\downarrow 0}\sup_{x\in B(x_0,\delta)\cap E}(f(x)+g(x)) \le \lim_{\delta\downarrow 0}\sup_{x\in B(x_0,\delta)\cap E} f(x) + \lim_{\delta\downarrow 0}\sup_{x\in B(x_0,\delta)\cap E} g(x) \le f(x_0)+g(x_0)$$

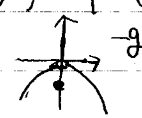
(或 $\forall M_0 > f(x_0)+g(x_0)$ $M_0=M_1+M_2$ $M_1>f(x_0)$ $M_2>g(x_0)$.

$\exists\delta_1>0, \delta_2>0$ st $\sup\limits_{x\in B(x_0,\delta)\cap E} f(x) < M_1$ $(0<\delta\le\delta_1)$ $\sup\limits_{x\in B(x_0,\delta)\cap E} g(x) < M_2$ $(0<\delta\le\delta_2)$

$\therefore 0<\delta\le\min(\delta_1,\delta_2) \Rightarrow \sup\limits_{x\in B(x_0,\delta)\cap E}(f(x)+g(x)) \le \sup\limits_{x\in B(x_0,\delta)\cap E} f(x) + \sup\limits_{x\in B(x_0,\delta)\cap E} g(x) < M_1+M_2=M_0$)

(ii) f跟g均為 upper semi continuous $\Rightarrow$ $f-g$是否為 upper semi continuous ?

$\because$ 不一定成立, 例如 $f=0$. $g= x^2\ (x\neq 0)\ ;\ -1\ (x=0)$

$f-g=-g$ 顯然不為 upper semi continuous. 

$\sup\limits_{x\in B(\delta,0)}(-g)=0>-1\ (=-g(0))$

(iii) 例如, $f,g\ge 0$ $(x\in E)$. $(0\le)\sup\limits_{x\in B(x_0,\delta)\cap E} f(x)g(x) \le \sup\limits_{x\in B(x_0,\delta)\cap E} f(x)\cdot\sup\limits_{x\in B(x_0,\delta)\cap E} g(x)$

跟(i)同樣, $\delta\downarrow 0$ 得 $\lim\limits_{\delta\downarrow 0}\sup\limits_{x\in B(x_0,\delta)\cap E} f(x)g(x) \le f(x_0)g(x_0)$

(b) $\{f_n\}_{n\ge 1}$ 均為 Upper Semi Continuous.

$g(x) \overset{\text{def}}{=} \inf\limits_{n\ge 1} f_n(x)$

我們欲證: $\lim\limits_{\delta\downarrow 0}\sup\limits_{x\in B_\delta(x_0)\cap E} g(x) \le g(x_0)$. $(x_0\in E)$

$\Rightarrow$ $\forall\varepsilon>0$ $\exists\delta_0>0$ st $0<\delta\le\delta_0 \Rightarrow \sup\limits_{x\in B_\delta(x_0)\cap E} g(x) < g(x_0)+\varepsilon$.

(1) $g(x_0)=\inf\limits_{n\ge 1} f_n(x_0)$ $\therefore$ $\forall\varepsilon>0$ $\exists m\in\mathbb{N}$ st $g(x_0)\le\inf\limits_{1\le n\le m} f_n(x_0) < g(x_0)+\varepsilon$

(ii) $\{f_n\}_{n\geq 1}$ 為 Upper Semi Continuous.

$\forall \varepsilon>0\ \exists \delta_n>0$ st $\sup\limits_{x\in B_{\delta_n}(x_0)} f_n(x) < f_n(x_0)+\varepsilon$

$\Rightarrow \inf\limits_{1\leq n\leq m}\{\sup\limits_{x\in B_{\delta_n}(x_0)} f_n(x)\} < \inf\limits_{1\leq n\leq m} f_n(x_0) < g(x_0)+\varepsilon$

(iii) $\delta_0 \overset{def}{=} \min\{\delta_1, \delta_2, \dots \delta_m\}$

$\inf\limits_{1\leq n\leq m}\{\sup\limits_{x\in B_{\delta_0}(x_0)} f_n(x)\} \leq \inf\limits_{1\leq n\leq m}\{\sup\limits_{x\in B_{\delta_n}(x_0)} f_n(x)\} < g(x_0)+\varepsilon$

註

$\inf\limits_{x\in X}\sup\limits_{y\in Y} f(x,y) \geq \sup\limits_{y\in Y}\inf\limits_{x\in X} f(x,y)$

$\sup\limits_{x\in B_{\delta_0}(x_0)}\{\inf\limits_{1\leq n\leq m} f_n(x)\} \leq \inf\limits_{1\leq n\leq m}\{\sup\limits_{x\in B_{\delta_0}(x_0)} f_n(x)\} < g(x_0)+\varepsilon$

$\sup\limits_{x\in B_{\delta_0}(x_0)}\{\inf\limits_{n\geq 1} f_n(x)\} \leq \sup\limits_{x\in B_{\delta_0}(x_0)}\inf\limits_{1\leq n\leq m} f_n(x) < g(x_0)+\varepsilon$

$\sup\limits_{x\in B_{\delta_0}(x_0)} g(x) < g(x_0)+\varepsilon \Rightarrow \ulcorner \forall \varepsilon>0\ \exists \delta_0>0$ st $0<\delta\leq\delta_0$ $\sup\limits_{x\in B_{\delta}(x_0)} g(x) < g(x_0)+\varepsilon$

(c) ① $f_n\to f$ (uniformly near x_0) $\Rightarrow \exists \delta_1>0$ st $0<\delta\leq\delta_1$

$\lim\limits_{n\to\infty}\sup\limits_{x\in B(x_0,\delta)\cap E}|f_n(x)-f(x)| \leq \lim\limits_{n\to\infty}\sup\limits_{x\in B(x_0,\delta_1)\cap E}|f_n(x)-f(x)| = 0$

$\therefore \forall \varepsilon>0\ \exists N\in\mathbb{N}$ st $\ulcorner n\geq N \Rightarrow \sup\limits_{x\in B(x_0,\delta)\cap E}|f(x)-f_n(x)| < \frac{\varepsilon}{3}$ when $\delta\leq\delta_1 \lrcorner$

(①' $|f_n(x_0)-f(x_0)| \leq \sup\limits_{x\in B(x_0,\delta)\cap E}|f(x)-f_n(x)| < \frac{\varepsilon}{3}$)

② f_n 於 x_0 為 upper semi continuous $\Rightarrow \exists \delta_2>0$ st $0<\delta\leq\delta_2$

$\Rightarrow \sup\limits_{x\in B(x_0,\delta)\cap E} f_n(x) < f_n(x_0)+\frac{\varepsilon}{3}$

綜合 ①, ②: $0<\delta\leq\min\{\delta_1,\delta_2\}$ $\forall \varepsilon>0\ \exists N$. $n\geq N\Rightarrow$

$\sup\limits_{x\in B(x_0,\delta)\cap E} f(x) = \sup\limits_{x\in B(x_0,\delta)\cap E}(f(x)-f_n(x)+f_n(x)) \leq \sup\limits_{x\in B(x_0,\delta)\cap E}(f(x)-f_n(x)) +$

$\sup\limits_{x\in B(x_0,\delta)\cap E} f_n(x) < \underset{(①)}{\frac{\varepsilon}{3}} + \underset{(②)}{\left(f_n(x_0)+\frac{\varepsilon}{3}\right)} = \frac{2\varepsilon}{3} + f_n(x_0) < \frac{2\varepsilon}{3} + \underset{(①')}{f(x_0)+\frac{\varepsilon}{3}} = f(x_0)+\varepsilon$

$\therefore$ 證明完成

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Date $f: \mathbb{R}^n \to \mathbb{R}$ $\quad g(x) \overset{\text{def}}{=} \sup_{y \in B_r(x)} f(y)$

$\boxed{\text{4.11}}$ ① 證明 $g(x)$ 為 Lower Semi Continuous

我們的目標：$\lim_{\delta \downarrow 0} \inf_{x \in B_\delta(x_0)} g(x) \geq g(x_0)$

換言之 $\forall \varepsilon > 0 \ \exists \delta_0 > 0$ st $0 < \delta \leq \delta_0 \Rightarrow \inf_{x \in B_\delta(x_0)} g(x) > g(x_0) - \varepsilon$.

(1) $g(x_0) = \sup_{y \in B_r(x_0)} f(y)$

$\Rightarrow \forall \varepsilon > 0 \ \exists x_1 \in B_r(x_0)$ st $f(x_1) \leq \underline{g(x_0) < f(x_1) + \varepsilon}$

$(\therefore f(x_1) > g(x_0) - \varepsilon)$

(2) $|x_1 - x_0| < r \quad \therefore B_r(x_1) \cap B_r(x_0) \neq \phi$

設 $x \in B_r(x_1) \cap B_r(x_0)$

$f(x_1) \leq \sup_{y \in B_r(x)} f(y) \quad (\because |x - x_1| < r \ \therefore x_1 \in B_r(x))$

$\therefore f(x_1) \leq g(x)$ (for any $x \in B_r(x_1) \cap B_r(x_0)$)

$\Rightarrow f(x_1) \leq \inf_{x \in B_r(x_1) \cap B_r(x_0)} g(x)$

(3) $B_r(x_1) \cap B_r(x_0)$ 為開集合

故存在 $\delta_0 > 0$ st $B_{\delta_0}(x_0) \subseteq B_r(x_1) \cap B_r(x_0)$

$\inf_{x \in B_r(x_1) \cap B_r(x_0)} g(x) \leq \inf_{x \in B_{\delta_0}(x_0)} g(x)$

(4) 綜合上述結論，$\forall \varepsilon > 0 \ \exists \delta_0 > 0$ st $g(x_0) < f(x_1) + \varepsilon \leq \inf_{x \in B_r(x_0) \cap B_r(x_1)} g(x) + \varepsilon$

$\leq \inf_{x \in B_{\delta_0}(x_0)} g(x) + \varepsilon$

$\leq \inf\limits_{x\in B_\delta(x_0)} g(x) \quad (\delta \leq \delta_0)$ ∴ 證明完成

② $h(x)$ 為 Upper Semi Continuous

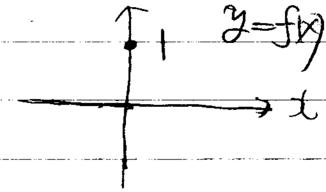
證明 $-h(x)$ 為 Lower Semi Continuous 即可

$-h(x) = -\inf\{f(y) \mid y\in B_r(x)\} = \sup\limits_{y\in B_r(x)}\{-f(y)\} \qquad \tilde{f} \overset{def}{=} -f$

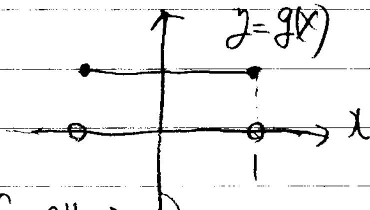
$= \sup\limits_{y\in B_r(x)}\{\tilde{f}(y)\}$

跟① 同理, $-h(x)$ 為 Lower Semi Continuous $\Rightarrow$ $h(x)$ 為 Upper Semi Continuous.

③ 例如, 取 $f(x) = \begin{cases} 1 & (x=0) \\ 0 & (x\neq 0) \end{cases}$, $r=1$



$g(x) = \sup\limits_{y\in B(x,1)} f(y)$



$\inf\limits_{x\in B(1,\delta)} g(x) = 0$ (for all $\delta>0$)

∴ $\lim\limits_{\delta\downarrow 0} \inf\limits_{x\in B(1,\delta)} g(x) = 0 \qquad g(1)=1$

∴ $g(x)$ 並不是 Lower Semi Continuous

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$(N \subseteq [a,b])$

$\boxed{4.12}$ 根據題意，$\exists N \in \mathcal{L}^1 : |N|=0$ s.t

f 於 $[a,b] \setminus N$ 上連續，令 c 為一個任意實數。

$$f^{-1}((c,\infty)) \overset{①}{=} \left(f^{-1}((c,\infty)) \cap [a,b] \setminus N\right) \cup \overset{②}{f^{-1}((c,\infty)) \cap N}.$$

①：f 於 $[a,b] \setminus N$ 上連續，故 $f^{-1}((c,\infty)) \cap [a,b] \setminus N$ 可寫成 $G \cap [a,b] \setminus N$.（G 為 $\mathbb{R}$ 上的開集合）

$G \in \mathcal{L}^1$，$[a,b] \setminus N \in \mathcal{L}^1$（$N : |N|=0$ 為 Lebesgue 可測）

故 $G \cap [a,b] \setminus N \in \mathcal{L}^1([a,b]) = \{[a,b] \cap A \mid A \in \mathcal{L}^1\}$

②：$f^{-1}((c,\infty)) \cap N$ 為 N 的子集合，且 N 為外測度 0 的集合，故 $f^{-1}((c,\infty)) \cap N$ 亦為外測度 0 的集合

$\therefore f^{-1}((c,\infty)) \cap N \in \mathcal{L}^1([a,b])$ $(N \subseteq [a,b])$

綜合①，②：$f^{-1}((c,\infty)) \in \mathcal{L}^1([a,b])$ (for all $c \in \mathbb{R}$)

故 $f \in \mathcal{M}\left(([a,b], \mathcal{L}^d([a,b])) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))\right)$

註 $\mathcal{M}((S,\mathcal{A}) \to (T,\mathcal{B}))$ 表示 $\mathcal{A} \setminus \mathcal{B}$ 可測函數。

$(S,\mathcal{A}), (T,\mathcal{B})$ 為可測空間

$\mathcal{L}^d$：$\mathbb{R}^d$ 上的 Lebesgue 集合族

$\mathcal{B}(\mathbb{R}^d)$：$\mathbb{R}^d$ 上的 Borel 集合族.

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$\boxed{4.15}$

我們不妨設 $M(x) \overset{\text{def}}{=} \sup_{k \geq 1} |f_k(x)|$.

令 $E_n \overset{\text{def}}{=} \{x \in E \mid M(x) \leq n\}$ $E_n \nearrow E$.

(E_n 亦為 Lebesgue 可測集合)

由於 $|E| < \infty$, $\forall \varepsilon > 0$ $\exists m$ s.t. $|E \setminus E_m| < \frac{\varepsilon}{2}$.

另外, E_m 為 Lebesgue 可測集合, 根據 Theorem,
存在 閉集合 F_m s.t. $|E_m \setminus F_m| < \frac{\varepsilon}{2}$.

F_m 滿足 $|E \setminus F_m| = |(E \setminus E_m) \cup (E_m \setminus F_m)|$
$\leq |E \setminus E_m| + |E_m \setminus F_m| < \varepsilon$

$x \in F_m \Rightarrow x \in E_m \Rightarrow \underbrace{M(x)}_{= \sup_k |f_k(x)|} \leq m < \infty$

∴ 證明完成

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(注 $\mathcal{M}(\cdot \to \cdot)$: 可測函數
$\mathcal{L}^d$ … Lebesgue 可測集合 (on $\mathbb{R}^d$) 的 Family (σ-代數)
| … 限制範圍 ($\mathcal{L}^d|_A = \{A \cap B \mid B \in \mathcal{L}^d\}$)

$\boxed{4.17}$ $\{f_n\} \cup \{f\} \cup \{g_n\} \cup \{g\} \subseteq \mathcal{M}((E, \mathcal{L}^d(E)) \to (\mathbb{R}, \mathcal{B}(\mathbb{R})))$

$f_n \xrightarrow{|\cdot|} f$ (in measure) $g_n \xrightarrow{|\cdot|} g$ (in measure)

① 證明 $f_n + g_n \xrightarrow{|\cdot|} f + g$ (in measure)

$$|\{|f_n + g_n - f - g| \geq \delta\}| \leq \left|\{|f_n - f| \geq \tfrac{\delta}{2}\} \cup \{|g_n - g| \geq \tfrac{\delta}{2}\}\right|$$
$$\leq |\{|f_n - f| \geq \tfrac{\delta}{2}\}| + |\{|g_n - g| \geq \tfrac{\delta}{2}\}|$$

$$\lim_{n\to\infty} |\{|f_n - f| \geq \tfrac{\delta}{2}\}| + |\{|g_n - g| \geq \tfrac{\delta}{2}\}| = 0$$
($\because$ $f_n \xrightarrow{|\cdot|} f$ (in measure), $g_n \xrightarrow{|\cdot|} g$ (in measure))

$\therefore \lim_{n} |\{|f_n + g_n - f - g| \geq \delta\}| = 0$ ($\forall \delta > 0$)

② 證明 $f_n g_n \xrightarrow{|\cdot|} fg$ (in measure) ($|E| < \infty$)

($\delta > 0$)

$$\{|f_n g_n - fg| \geq \delta\} \subseteq \{|f_n - f||g_n - g| + |f||g_n - g| + |g||f_n - f| \geq \delta\}$$
$$\subseteq \underbrace{\{|f_n - f||g_n - g| \geq \tfrac{\delta}{3}\}}_{A_1} \cup \underbrace{\{|f||g_n - g| \geq \tfrac{\delta}{3}\}}_{A_2} \cup \underbrace{\{|g||f_n - f| \geq \tfrac{\delta}{3}\}}_{A_3}$$

- $A_1 \subseteq \{|f_n - f| \geq \frac{\sqrt{\delta}}{\sqrt{3}}\} \cup \{|g_n - g| \geq \frac{\sqrt{\delta}}{\sqrt{3}}\}$

$$|A_1| \leq |\{|f_n - f| \geq \tfrac{\sqrt{\delta}}{\sqrt{3}}\}| + |\{|g_n - g| \geq \tfrac{\sqrt{\delta}}{\sqrt{3}}\}|$$

$$\therefore \lim_{n\to\infty} |A_1| \leq \lim_{n\to\infty} |\{|f_n - f| \geq \tfrac{\sqrt{\delta}}{\sqrt{3}}\}| + |\{|g_n - g| \geq \tfrac{\delta}{\sqrt{3}}\}| = 0$$

- $A_2 \subseteq \{|f| \geq M\} \cup \{|g_n - g| \geq \frac{\delta}{3M}\}$ ($M > 0$)

$|A_2| \leq |\{|f| \geq M\}| + |\{|g_n - g| \geq \frac{\delta}{3M}\}|$

$\lim_{n\to\infty} |\{|g_n - g| \geq \frac{\delta}{3M}\}| = 0$ (for any $M>0$)

$\because M \nearrow \infty \quad |\{|f| \geq M\}| = 0$ ($\because |E| < \infty$, $M \nearrow \infty \therefore |\{|f| < M\}| \nearrow |E|$)

$\therefore \lim_{n\to\infty} |A_2| = 0$

- A_3: 跟 A_2 同理 $\qquad \lim_{n\to\infty} |A_1| + |A_2| + |A_3| = 0$

③ $N \overset{def}{=} \{x \in E \mid g(x) = 0 \text{ or } g_n(x) = 0 \text{ for sufficiently large } n\}$, $|N| = 0$.

證明 $\frac{1}{g_n} \xrightarrow{|\cdot|} \frac{1}{g}$ (in measure) 即可 ($\because$ ②)

令 $\omega: \mathbb{R} \to \mathbb{R}$ 為連續函數 $\Rightarrow \omega \in \mathcal{M}((\mathbb{R}, \mathcal{B}(\mathbb{R})) \to (\mathbb{R}, \mathcal{B}(\mathbb{R})))$

$\Rightarrow \{\omega \circ g_n\}_{n\geq 1} \cup \{\omega \circ g\} \subset \mathcal{M}((E, \mathcal{L}^d|_E) \to (\mathbb{R}, \mathcal{B}(\mathbb{R})))$

我們先證明: $\omega \circ g_n \xrightarrow{|\cdot|} \omega \circ g$ (in measure)

利用反證法, 若 $\omega \circ g_n \xrightarrow{|\cdot|} \omega \circ g$ (in measure) 不成立, 則 $\exists \delta > 0$ $\exists \varepsilon > 0$

$\limsup_n |\{|\omega \circ g_n - \omega \circ g| \geq \delta\}| > \varepsilon > 0$

$\Rightarrow$ 故存在子序列 $\{n_k\}_{k\geq 1}$: $|\{|\omega \circ g_{n_k} - \omega \circ g| \geq \delta\}| > \varepsilon$ (for all k) ㊟

$g_n \xrightarrow{|\cdot|} g$ (in measure) $\Rightarrow g_{n_k} \xrightarrow{|\cdot|} g$ (in measure) 顯然成立.

根據 Theorem 4.22, 存在子序列 $\{n_{k_\ell}\}_{\ell\geq 1}$ 使得 $g_{n_{k_\ell}} \to g$ (a.e)

由於 ω 為連續函數, 故 $\omega \circ g_{n_{k_\ell}} \to \omega \circ g$ (a.e)

在有限測度空間上, 「(a.e) 收斂 $\Rightarrow$ 測度收斂.」

故 $\omega \circ g_{n_{k_\ell}} \xrightarrow{|\cdot|} \omega \circ g$ (in measure).

但這違反了 ㊟. ($\because \{n_{k_\ell}\}_\ell \subset \{n_k\}_k$) 故 $\omega \circ g_n \xrightarrow{|\cdot|} \omega \circ g$ (in measure)

最後 $\omega(x) \overset{def}{=} \frac{1}{x}$ $(x \neq 0)$. $\omega \circ g_n|_{E \setminus N}$, $\omega \circ g|_{E\setminus N} \in \mathcal{M}(\mathcal{L}^d|_{E\setminus N} \to \mathcal{B}(\mathbb{R}))$

$\omega \circ g_n \xrightarrow{|\cdot|} \omega \circ g$ (in measure, on $E \setminus N$)

由於 N 為 measure zero set, 故 $\omega \circ g_n$, $\omega \circ g$ 於 E 上 為 a.e 可測的函數

且 $\omega \circ g_n \xrightarrow{|\cdot|} \omega \circ g$ (in measure on E)

$\leq |\{|g_n - g| \geq \delta\}| + |\{|g| \leq \delta\}| \quad n \to \infty, \delta \downarrow 0$

(NOTE) $|\{g_n = 0\}| \leq |\{g_n = 0, |g_n - g| \geq \delta\}| + |\{g_n = 0, |g_n - g| < \delta\}|$

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$\boxed{4.18}$

① $f_k \uparrow f$

$$\lim_{k\to\infty} \omega_{f_k}(a) = \lim_{k\to\infty} \left|\{f_k > a\}\right| = \left|\bigcup_{k\geq 1}^{\infty} \{f_k > a\}\right| = \left|\{f > a\}\right|$$

$$(\because \{f_k > a\} \subseteq \{f_{k+1} > a\}) \qquad = \omega_f(a)$$

$$\therefore \lim_{k\to\infty} \omega_{f_k}(a) = \omega_f(a)$$

② 假設 $\omega_f(a)$ 於 $a=a_0$ 連續.

$\forall \varepsilon > 0\ \exists \delta > 0$ st $|a-a_0| \leq \delta \Rightarrow |\omega_f(a) - \omega_f(a_0)| < \varepsilon$,

$\omega_f(a)$ 為遞減函數. 「$\omega_f(a_0) - \varepsilon < \omega_f(a_0+\delta) \leq \omega_f(a_0-\delta) < \omega_f(a_0) + \varepsilon$」

證明 $\limsup_n \omega_{f_n}(a_0) \leq \omega_f(a_0-\delta) < \omega_f(a_0) + \varepsilon$

及 $\liminf_n \omega_{f_n}(a_0) \geq \omega_f(a_0+\delta) > \omega_f(a_0) - \varepsilon$

(I)
$$\begin{aligned}\omega_{f_n}(a_0) &= \left|\{f_n > a_0\}\right| = \left|\{f_n - f + f > a_0\}\right| \\ &\leq \left|\{f_n - f > \delta\} \cup \{f > a_0 - \delta\}\right| \\ &\leq \left|\{f_n - f > \delta\}\right| + \left|\{f > a_0 - \delta\}\right| \\ &\leq \left|\{|f_n - f| > \delta\}\right| + \omega_f(a_0-\delta)\end{aligned}$$

$$\therefore \limsup_n \omega_{f_n}(a_0) \leq \lim_n \left|\{|f_n - f| > \delta\}\right| + \omega_f(a_0-\delta) = \omega_f(a_0-\delta)$$

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(II) $\omega_f(a+\delta) = |\{f > a+\delta\}| = |\{f_k > a\} \cup \{f - f_k > \delta\}|$

$\leq |\{f_k > a\}| + |\{f - f_k > \delta\}|$

$|\{f_k > a\}| \geq \omega_f(a+\delta) - |\{|f - f_k| > \delta\}|$

$> \omega_f(a+\delta) - \frac{\varepsilon}{2}$ （k充分大）

$> \omega_f(a_0) - \varepsilon$

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$\boxed{4.19}$ ① $f_n \overset{\text{def}}{=} f(x, \frac{[ny]}{n})$, $f = \lim_{n\to\infty} f_n(x,y)$

(∵ 連續性)

$\forall y \in [0,1]$ $x \mapsto f(x,y)$ 為連續, $\Rightarrow$ $f(x,y) \in M(\mathcal{L}^2([0,1]^2)$

$\forall x \in [0,1]$ $y \mapsto f(x,y)$ 為連續, $\rightarrow \mathcal{B}(\mathbb{R}))$

$\{(x,y) \in [0,1] \times [0,1] \mid f_n > a\} =$

$= \left(\bigcup_{k=1}^{n} \{x \in [0,1] \mid f_n(x, \frac{k-1}{n}) > a\} \times [\frac{k-1}{n}, \frac{k}{n}) \right) \cup$

$\{x \in [0,1] \mid f_n(x, 1) > a\} \times \{1\}$

㊟ y 分為 $[\frac{0}{n}, \frac{1}{n}) \cup [\frac{1}{n}, \frac{2}{n}) \cdots \cup [\frac{n-1}{n}, 1) \cup \{1\}$

(i) $\{x \in [0,1] \mid f_n(x, \frac{k-1}{n}) > a\} \times [\frac{k-1}{n}, \frac{k}{n}) \in \mathcal{B}(\mathbb{R}^2)$

[原因] $\{x \in [0,1] \mid f_n(x, \frac{k-1}{n}) > a\}$ 為開集合

(∵ $f_n(x, \frac{k-1}{n})$ 為 x 的連續函數)

$(\frac{k-1}{n} - \frac{1}{m}, \frac{k}{n})$ 為開集合

∴ $\{x \in [0,1] \mid f_n(x, \frac{k-1}{n}) > a\} \times (\frac{k-1}{n} - \frac{1}{m}, \frac{k}{n}) \in \mathcal{B}(\mathbb{R}^2)$

∴ $\bigcap_{m=1}^{\infty} \{x \in [0,1] \mid f_n(x, \frac{k-1}{n}) > a\} \times (\frac{k-1}{n} - \frac{1}{m}, \frac{k}{n})$

$= \{x \in [0,1] \mid f_n(x, \frac{k-1}{n}) > a\} \times [\frac{k-1}{n}, \frac{k}{n})$

(ii) $\{x\in[0,1] \mid f_n(x,1)>a\}\times\{1\}$ 的測度為 0.

$\subseteq [0,1]\times\{1\} \subseteq [0,1]\times[1-\varepsilon,1]$ $(\forall \varepsilon>0)$

$\left|[0,1]\times[1-\varepsilon,1]\right| = \varepsilon$ ∵ $\varepsilon\downarrow 0$

∴ $\{(x,y)\in[0,1]\times[0,1] \mid f_n>a\} \in \mathcal{B}(\mathbb{R}^2)$
(for all $a\in\mathbb{R}$)

∴ $f_n \in \mathcal{M}((\mathbb{R}^2,\mathcal{B}(\mathbb{R}^2))\to(\mathbb{R},\mathcal{B}(\mathbb{R})))$ (or $\mathcal{M}((\mathbb{R}^2,\mathcal{B}(\mathbb{R}^2))\to(\overline{\mathbb{R}},\mathcal{B}(\overline{\mathbb{R}})))$)
∴ $\lim f_n \in \mathcal{M}((\mathbb{R}^2,\mathcal{B}(\mathbb{R}^2))\to(\mathbb{R},\mathcal{B}(\mathbb{R})))$
(假設 $f:[0,1]\times[0,1]\to\mathbb{R}$)
(若需要, 可改為 $f:[0,1]\times[0,1]\to\overline{\mathbb{R}}$)

② 只有假設 $\forall y\in[0,1]$ $x\mapsto f(x,y)$ 為連續的 case

$A\subseteq[0,1]$ 為 non-measurable set. $f(x,y)=I_A(y)=I_{[0,1]\times A}\binom{x}{y}$
$[0,1]\times A$ 並非 Lebesgue Measurable
$[0,1]\times[0,1]\subseteq [0,1]\times\bigcup_{r\in\mathbb{Q}\cap[0,1]} A+r \subseteq [0,1]\times[-1,2]$

⇒ 若 $[0,1]\times A$ 為可測, 則 $[0,1]\times A+r$ 的測度非零.
且 $[0,1]\times A+r$ 為 disjoint ⇒ $\left|[0,1]\times\bigcup_{r\in\mathbb{Q}\cap[0,1]} A+r\right| = +\infty$
∴ 矛盾

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$\boxed{4.22}$

(a) 考慮一般的情況. 令 $(S, \mathcal{A})$, $(T, \mathcal{B})$ 為可測空間.
設 $f \in M((S, \mathcal{A}) \to (T, \mathcal{B}))$
若 $\mathcal{B} = \sigma[\mathcal{g}]$ $(\mathcal{g} \subseteq 2^T)$, 則 $\forall B \in \mathcal{B}$ $f^{-1}(B) \in \mathcal{A}$
$\Leftrightarrow$ $\forall G \in \mathcal{g}$ $f^{-1}(G) \in \mathcal{A}$.

• $\Rightarrow$ 顯然成立 $(\because \mathcal{g} \subseteq \mathcal{B})$.
• $\Leftarrow$ 考慮 $\mathcal{F} = \{B \subseteq T \mid f^{-1}(B) \in \mathcal{A}\}$.
由於 $\mathcal{A}$ 為 σ-代數, 故 $\mathcal{F}$ 亦為 σ-代數.
$\mathcal{g} \subseteq \mathcal{F} \Rightarrow \sigma[\mathcal{g}] \subseteq \mathcal{F}$ $\therefore \mathcal{B} \subseteq \mathcal{F}$.
$\therefore$ 證明完成

$(S, \mathcal{A}) = (\mathbb{R}^d, \mathcal{L}^d)$ $(\mathcal{L}^d$ … $\mathbb{R}^d$ 上的 Lebesgue 可測集合 (的 Family)$)$
$(T, \mathcal{B}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ $(\mathcal{B}(\mathbb{R}) = \sigma[\mathcal{O}^1]$, $\mathcal{O}$ … $\mathbb{R}$ 上的開集合$)$
Borel 可測集合族

根據 Theorem 4.3 $\forall G \in \mathcal{O}^1$ $f^{-1}(G) \in \mathcal{L}^d$ $\Rightarrow$ f: Lebesgue 可測
$\Leftrightarrow$ $\forall B \in \mathcal{B}(\mathbb{R}) (= \sigma[\mathcal{O}^1])$ $f^{-1}(B) \in \mathcal{L}^d$

$\therefore$ 證明完成

(b) $\phi \in M((\mathbb{R}, \mathcal{B}(\mathbb{R})) \to (\mathbb{R}, \mathcal{B}(\mathbb{R})))$
$f \in M((\mathbb{R}^n, \mathcal{L}^n) \to (\mathbb{R}, \mathcal{B}(\mathbb{R})))$

$\phi \circ f: \mathbb{R}^n \to \mathbb{R}$ 證明 $\phi \circ f \in M((\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n)) \to (\mathbb{R}, \mathcal{B}(\mathbb{R})))$.

$\forall B \in \mathcal{B}(\mathbb{R})$, $(\phi \circ f)^{-1}(B) = f^{-1} \circ \phi^{-1}(B)$

$\phi^{-1}(B) \in \mathcal{B}(\mathbb{R})$ $\therefore f^{-1}(\phi^{-1}(B)) \in \mathcal{F}^{n}$

($\because$ ϕ的可測性) ($\because$ f的可測性)

$\therefore$ 證明完成

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$\boxed{5.3}$

$$\begin{cases} \{f_n\}_{n\geq 1} \subseteq \mathcal{M}((E, \mathcal{L}^d|_E) \to [0,\infty)) \\ f_n \to f, \quad f_n \leq f \ (a.e) \end{cases}$$

根據 Fatou's lemma

$$\int \liminf f_n \leq \liminf_n \int f_n$$

$$\int f \leq \liminf_n \int f_n \quad \cdots ①$$

$(\because f_n \to f)$

另外 $f_n \leq f$ (a.e)

故 $\int f_n \leq \int f$ (a.e: 不影響到 積分)

$$\Rightarrow \limsup_n \int f_n \leq \int f \quad \cdots ②$$

綜合 ① & ② 得:

$$\int f \leq \lim \int f_n \leq \int f$$

$$\therefore \lim_{n\to\infty} \int f_n = \int f$$

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$\boxed{5.4}$

$g_n(x) \overset{\text{def}}{=} x^n f(x) \quad (n=0,1,2\cdots) \quad (g_0(x) \overset{\text{def}}{=} f(x))$

$\{g_n\}_{n\geq 0} \subseteq M(([0,1], \mathcal{L}^1([0,1])) \to R) \cap L^1(E)$

- $|g_{n+1}(x)| \leq |g_n(x)| \quad (n=0,1,2\cdots) \quad (\because\ 0\leqq x\leqq 1)$

$\Rightarrow \sup_{n\geq 0} |g_n(x)| \underset{(=)}{\leqq} |f(x)| \in L^1(E)$

- $\lim_{n\to\infty} g_n(x) = 0 \quad$ (a.e. on $[0,1]$) (除了 $x=1$ 之外)

故根據 Lebesgue 控制收斂定理,

$$\lim_n \int_E g_n(x) = \int_E \lim_n g_n(x) = 0$$

$$\Rightarrow \lim_{n\to\infty} \int_{[0,1]} x^n f(x)\, dx = 0$$

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$\boxed{5.5}$

• 設 $\{f_n\}_{n\geq 1} \subseteq M((E, \mathcal{L}^d|_E) \to (\overline{\mathbb{R}}, \mathbb{B}(\overline{\mathbb{R}})))$

$f_n \to f$ (ae) 且 $|E| < \infty$ ($E \in \mathcal{L}^d$)

存在 $0 \leq M < \infty$ st $|f_n| \leq M < \infty$ (ae)

$$\left|\int_E f_n - \int f\right| \underset{\text{三角不等式}}{\leq} \int_E |f_n - f| = \int_F |f_n - f| + \int_{E\setminus F} |f_n - f|$$

三角不等式 (Egorov's Theorem $f_n \to f$ (uniformly on F))

$$\leq \int_F \sup_{x\in F} |f_n - f| + \int_{E\setminus F} 2M$$

$(\because f_n \to f$ (ae), $|f_n| \leq M$ (ae) $\Rightarrow |f_n - f| \leq |f_n| + |f| \leq 2M$ (ae))

$$= \underbrace{\left(\sup_{x\in F} |f_n - f|\right)\cdot |F|}_{①} + \underbrace{2M \cdot |E\setminus F|}_{②}$$

① ∴ $\lim_{n\to\infty} \sup_{x\in F} |f_n - f| = 0$ 且 $|F| < \infty$

$\exists N$ st $n \geq N$ $\sup_{x\in F} |f_n - f| < \frac{\varepsilon}{2|F|}$

$\Rightarrow$ ① $< \frac{\varepsilon}{2}$

② ∴ 取 F 使得 $|E\setminus F| < \frac{\varepsilon}{4M}$ $(M < \infty)$

$\Rightarrow$ ② $< \frac{\varepsilon}{2}$

∴ n 很大時 $\left|\int_E f_n - \int f\right| < \varepsilon$

(①+② $< \varepsilon$)

∴ $\limsup_n \left|\int f_n - \int f\right| = 0$

$\Rightarrow \lim \int f_n = \int f$

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$\boxed{5.6}$

考慮 $\{h_n\}_{n\geq 1} \subseteq \mathbb{R}\setminus\{0\}$ where $h_n \to 0$ (as $n\to\infty$)

$\frac{d}{dx}\int_0^1 f(x,y)\,dy = \lim_{n\to\infty}\frac{1}{h_n}\left(\int_0^1 f(x+h_n) - \int_0^1 f(x,y)\right)$

$= \lim_{n\to\infty}\int_0^1 \frac{f(x+h_n,y)-f(x,y)}{h_n}$

根據題意, 對於任意 $y\in[0,1]$, $\frac{\partial f}{\partial x}(x,y)$ 均存在。

$g_n \stackrel{def}{=} \frac{f(x+h_n,y)-f(x,y)}{h_n}$ (g_n 顯然為可測函數 (y的))

$\Rightarrow \lim_{n\to\infty} g_n = \frac{\partial f}{\partial x}$ 亦為可測函數 (y的)

根據均值定理, $\forall x\in(0,1)$ $\exists\theta\in(0,1)$ s.t. $g_n = \frac{\partial f}{\partial x}\Big|_{x=x+\theta h_n}$

(注: 我們在此取很大的 n, 所以滿足 $x+h_n\in(0,1)$)

由於 $\frac{\partial f}{\partial x}$ 為有界, 故此 $|g_n| = \left|\frac{\partial f}{\partial x}\Big|_{x=x+\theta h_n}\right| \leq M \in L^1([0,1])$

($\because |[0,1]| = 1 < \infty$)

有限測度

利用 Lebesgue 控制收斂定理, 得

$\lim_{n\to\infty}\int_0^1 g_n = \int_0^1 \lim_n g_n$

$\frac{d}{dx}\int_0^1 f(x,y)\,dy = \int_0^1 \frac{\partial f}{\partial x}(x,y)\,dy$

$\therefore$ 證明完成

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$\boxed{5.9}$

限制積分範圍

$$\limsup_n \int_E |f-f_n|^p \geq \limsup_n \int_{E\cap\{|f-f_n|\geq\varepsilon\}} |f-f_n|^p$$

$(\varepsilon>0)$

$$\geq \limsup_n \int_{\{x\in E \mid |f-f_n|\geq\varepsilon\}} \varepsilon^p$$

$$= \limsup_n \varepsilon^p \cdot \left|\{x\in E \mid |f-f_n|\geq\varepsilon\}\right|$$

$$\limsup_n \left|\{x\in E \mid |f-f_n|\geq\varepsilon\}\right| \leq \limsup_n \frac{1}{\varepsilon^p}\int_E |f-f_n|^p$$

$$= 0 \quad (\text{for all } \varepsilon>0)$$

由此可知，$f_n \xrightarrow{m} f$ (in measure)

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$\boxed{5.10}$

• 根據 Exercise 5.9, $\int |f-f_n|^p \to 0$ (as $n\to\infty$) $\Rightarrow$ $f_n \xrightarrow{m} f$ (in measure)

∵ 取一個子序列 $\{n_k\}_{k\geq 1}$ $f_{n_k} \to f$ (a.e.) (as $k\to\infty$)

• Fatou's Lemma:

$$\int_E \liminf_k |f_{n_k}|^p \leq \liminf_k \int_E |f_{n_k}|^p \leq \limsup_k \int_E |f_{n_k}|^p$$

$$\leq \sup_k \int_E |f_k|^p \leq M$$

$$\int_E |f|^p \;\; (= \int_E \liminf_k |f_{n_k}|^p)$$

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$\boxed{5.13}$

(1) 利用 Theorem 5.16 & 5.22

$$\int_E \sum_{n=1}^{\infty} |f_n| = \sum_{n=1}^{\infty} \int_E |f_n| < \infty$$

(monotone convergence theorem 的 Σ 版)

$\therefore \sum_{n=1}^{\infty} |f_n| < \infty$ (a.e in E) (Theorem 5.22)

$\left|\sum_{n=1}^{\infty} f_n\right| \leq \sum_{n=1}^{\infty} |f_n| < \infty$ (a.e in E)

由此可知，$\sum_{n=1}^{\infty} f_n$ converges almost everywhere in E.

(2) 利用(1)：$f_n \overset{\text{def}}{=} a_n |x-r_n|^{-\frac{1}{2}}$

f_n 於 $[0,1]$ 上 almost everywhere 連續，
故 $\{f_n\} \subseteq \mathcal{M}(([0,1], \mathcal{L}([0,1])) \to (\mathbb{R}, \mathcal{B}(\mathbb{R})))$

(R) $\int_0^1 |a_n| |x-r_n|^{-\frac{1}{2}} dx = \int_0^{r_n-0} |a_n| (r_n-x)^{-\frac{1}{2}} dx + \int_{r_n+0}^1 |a_n| (x-r_n)^{-\frac{1}{2}} dx$

$= \left[-2|a_n| (r_n-x)^{\frac{1}{2}}\right]_0^{r_n-0} + \left[2|a_n| (x-r_n)^{\frac{1}{2}}\right]_{r_n+0}^1$

$= 2|a_n| r_n^{\frac{1}{2}} + 2|a_n| (1-r_n)^{\frac{1}{2}} = 2|a_n| \left(r_n^{\frac{1}{2}} + (1-r_n)^{\frac{1}{2}}\right) \leq 2\sqrt{2} |a_n| < \infty$

($r_n = \frac{1}{2}$ 時最大)

根據 Theorem 5.53，$\{|f_n|\}_n$ Lebesgue 可測且 Lebesgue 可積分
($\therefore \{f_n\}_n$ 為可積分)

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(2)

$\because \int_{[0,1]^2} |a_n||x-r_n|^{-\frac{1}{2}} dx \leq 2\sqrt{2}|a_n|$

$\therefore \sum_{n=1}^{\infty} \int_{[0,1]^2} |a_n||x-r_n|^{-\frac{1}{2}} dx = \int_{[0,1]^2} \sum_{n=1}^{\infty} |a_n||x-r_n|^{-\frac{1}{2}} dx \leq \sum_{n=1}^{\infty} 2\sqrt{2}|a_n| < \infty$

故根據(1)，$\sum_{n=1}^{\infty} a_n |x-r_n|^{-\frac{1}{2}}$ converges almost everywhere.

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$\boxed{5.18}$

根據 ex5.16，若 $f \in \mathcal{M}((E, \mathcal{L}^d|_E) \to ([0,\infty], \mathcal{B}([0,\infty])))$
$(E \in \mathcal{L}^d)$

則 $(L)\int_E f^p = (L),(R)\ p\int_0^\infty \alpha^{p-1}\omega(\alpha)\,d\alpha$

以下將右邊的積分視為 Lebesgue 積分並表示為 I

$$I = p\int_{(0,\infty)} \alpha^{p-1}\omega(\alpha) = p\int_{\bigcup_{k\in\mathbb{Z}}(2^{k-1},2^k]} \alpha^{p-1}\omega(\alpha)$$

$$= p\sum_{k\in\mathbb{Z}}\int_{(2^{k-1},2^k]} \alpha^{p-1}\omega(\alpha)$$

• $p \geq 1$ … $p\sum (2^{k-1})^{p-1}\cdot\omega(2^k)\cdot|(2^{k-1},2^k]| \leq I \leq$
$p\sum (2^k)^{p-1}\cdot\omega(2^{k-1})\cdot|(2^{k-1},2^k]|$　　($\because \omega$: 遞減函數)

故 $p\sum_{k\in\mathbb{Z}} 2^{kp-p}\cdot\omega(2^k) \leq I \leq p\sum_{k\in\mathbb{Z}} 2^{kp-1}\cdot\omega(2^{k-1})$

• $p<1$ ·

同理 $p\sum_{k\in\mathbb{Z}} 2^{kp-1}\omega(2^k) \leq I \leq p\sum_{k\in\mathbb{Z}} 2^{kp-p}\omega(2^{k-1})$

① $f \in L^p(E) \Rightarrow I < \infty \Rightarrow$

$p\geq 1$ …, $p\sum_{k\in\mathbb{Z}} 2^{kp-p}\cdot\omega(2^k) < \infty \Rightarrow \sum_{k\in\mathbb{Z}} 2^{kp}\cdot\omega(2^k) < \infty$

$p<1$ …, $p\sum_{k\in\mathbb{Z}} 2^{kp-1}\omega(2^k) < \infty \Rightarrow \sum_{k\in\mathbb{Z}} 2^{kp}\omega(2^k) < \infty$

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② $\sum_{k\in\mathbb{Z}} 2^{kp} w(2^k) < \infty \Rightarrow \sum_{k\in\mathbb{Z}} 2^{(k-1)p} w(2^{k-1}) < \infty$

$(\because k \to k-1)$

$\Rightarrow \sum_{k\in\mathbb{Z}} 2^{kp-p} w(2^{k-1}) < \infty$

$p \geq 1$ $\because$ $p \cdot 2^{p-1} \left(\sum_{k\in\mathbb{Z}} 2^{kp-p} w(2^{k-1}) \right) < \infty \Rightarrow p \sum_{k\in\mathbb{Z}} 2^{kp-1} w(2^{k-1}) < \infty$

$\Rightarrow I < \infty \quad \therefore f \in L^p$

$p < 1$ $\because$ $p \left(\sum_{k\in\mathbb{Z}} 2^{kp-p} w(2^{k-1}) \right) < \infty \Rightarrow p \sum_{k\in\mathbb{Z}} 2^{kp-p} w(2^{k-1}) < \infty$

$\Rightarrow I < \infty \quad \therefore f \in L^p$

$\therefore$ 證明完成

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$\boxed{5.23}$

$$|f_n - f| \leq |f_n| + |f| \leq \varphi_n + |f| \quad (ae)$$

(三角不等式) $\quad (|f_n| \leq \varphi_n)$

故 $\varphi_n + |f| - |f_n - f| \geq 0 \quad (ae)$

利用 Fatou's 引理:

$$\int \liminf_n (\varphi_n + |f| - |f_n - f|) \leq \liminf_n \int (\varphi_n + |f| - |f_n - f|)$$

左 $= \int (\varphi + |f|) \quad (\because \varphi_n \to \varphi \text{ ae in } E$

$\qquad f_n \to f \text{ ae in } E)$

$= \int \varphi + \int |f| \quad (\int \varphi < \infty, \int |f| < \infty \quad (\because |f_n| \leq \varphi_n \ (ae)$

$\qquad \Rightarrow |f| \leq \varphi \ (ae), \ \varphi \in L^1(E)))$

右 $\displaystyle \lim_{n\to\infty} \inf_{m\geq n} \int (\varphi_m + |f| - |f_m - f|)$ $\quad (\because \varphi_m, |f| \in L^1(E))$

m: 很大时

$\displaystyle = \lim_{n\to\infty} \inf_{m\geq n} \left(\int \varphi_m + \int |f| - \int |f_m - f| \right)$

$\displaystyle \leq \lim_{n\to\infty} \inf_{m\geq n} \left(\sup_{k\geq n} \int \varphi_k + \int |f| - \int |f_m - f| \right)$

$\displaystyle = \lim_{n\to\infty} \left(\sup_{k\geq n} \int \varphi_k + \int |f| - \sup_{m\geq n} \int |f_m - f| \right)$

$\displaystyle = \limsup_n \int \varphi_n + \int |f| - \limsup_n \int |f_n - f|$

$\displaystyle = \int \varphi + \int |f| - \limsup_n \int |f_n - f|$

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左 $\leq$ 右 $\ni$ $\overline{\lim\sup_n} \int |f_n - f| = 0$

$\left(\left| \int f_n - \int f \right| \leq \int |f_n - f| \right.$

$\left. \therefore \overline{\lim\sup_n} \left| \int f_n - \int f \right| = 0 \right)$

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$\boxed{6.1}$

(a) 考慮 Indicator Function: $f(x,y) \overset{def}{=} \mathbb{I}_E\binom{x}{y}$ (≥ 0)
由於 $E \in \mathcal{L}^2$, 故 $f(x,y) \in \mathcal{M}((\mathbb{R}^2, \mathcal{L}^2) \to [0,\infty])$
(非負 Lebesgue 可測函數.)

根據 Tonelli's Theorem, $\iint_{\mathbb{R}^2} f = \int_R \{\int_R f dy\} dx$
$= \int_R \{\int_R f dx\} dy$.

由於 for almost every $x \in \mathbb{R}^1$, $\{y \mid (x,y) \in E\}$
為測度零, 故 for almost every $x \in \mathbb{R}^1$
$\int_R f(x,y)\, dy = 0$. $\Rightarrow \int_R \{\int_R f dy\} dx = 0$.

$g(y) \overset{def}{=} \int_R f dx$ 為非負且 a.e. 可測的函數.

根據 Theorem 5.11, $\int_R g(y)\, dy = 0 \Rightarrow g(y) = 0$ (a.e. $y \in \mathbb{R}^1$)
$\Rightarrow |E_y| = 0$ $(|\{x \mid (x,y) \in E\}| = 0)$

∴ 證明完成.

(b) $E \overset{def}{=} \{(x,y) \mid f(x,y) = +\infty\}$ 為 $\mathbb{R}^2$ 上的 Lebesgue 可測集合.
同樣考慮 $\mathbb{I}_E\binom{x}{y}$ (≥ 0) 且利用 Tonelli's Theorem

$$\int_R \underbrace{\int_R \mathbb{I}_E \, dy}_{\substack{\| \\ |\{y \mid f(x,y) = +\infty\}| \\ \| \\ 0 \ (\text{a.e. } x \in \mathbb{R})}} dx = \int_R \underbrace{\int_R \mathbb{I}_E \, dx}_{=0 \ (\text{a.e. } y \in \mathbb{R}^1)} dy = 0$$

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跟(a)同樣道理，非負可測函數：$\int_E I_E \, dx = 0$ $(ae\ y \in \mathbb{R}^1)$

$\therefore |\{x \mid f(x,y) = +\infty\}| = 0$ $(ae\ y \in \mathbb{R}^1)$

$\therefore$ 證明完成

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$\boxed{6.2}$ $\overset{\text{改為}}{n \to d}$ $\overline{R} = [-\infty, \infty]$ $\mathcal{B}(\cdot)$: Borel-Family

$F(x,y) \overset{\text{def}}{=} f(x)$
$G(x,y) \overset{\text{def}}{=} g(y)$

可測函數　R^{2d} 上的 Lebesgue 可測集合族

① $\{F(x,y), G(x,y)\} \subseteq \mathcal{M}((R^{2d}, \mathcal{L}^{2d}) \to (\overline{R}, \mathcal{B}(\overline{R})))$

(證明) $\{F(x,y) > a\} = \{(x,y) \in R^{2d} \mid F(x,y) > a\}$
$= \underbrace{\{x \in R^d \mid f(x) > a\}}_{\in \mathcal{L}^d} \times R^d$

(R^d 上的 Lebesgue 可測集合)

跟 Lemma 5.2 同理，$\{x \in R^d \mid f(x) > a\} \times R^1 \in \mathcal{L}^{d+1}$
$\Rightarrow \{x \in R^d \mid f(x) > a\} \times R^2 \in \mathcal{L}^{d+2}$ (重複利用 lemma 5.2)
$\cdots \Rightarrow \{x \in R^d \mid f(x) > a\} \times R^d \in \mathcal{L}^{2d}$

故 $F(x,y)$ 為 R^{2d} 上的可測函數
($G(x,y)$ 亦為如此)

② $F(x,y) \cdot G(x,y) \in \mathcal{M}((R^{2d}, \mathcal{L}^{2d}) \to (\overline{R}, \mathcal{B}(\overline{R})))$
(Theorem 4.10)

故 $f(x)g(y)$ $(= F \cdot G)$ 為 R^{2d} 上的 Lebesgue 可測函數。

③ 接著證明 $\{E_1, E_2\} \subseteq \mathcal{L}^d$
$\Rightarrow E_1 \times E_2 \in \mathcal{L}^{2d}$

(其實可以用跟 Exercise 5.12 一樣的方法)

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Indicator Function $I_{E_1}(x)$, $I_{E_2}(y)$ 為 R^d 上的 Lebesgue 可測函數.
$\Rightarrow I_{E_1}(x) I_{E_2}(y)$ 為 R^{2d} 上的 Lebesgue 可測函數
$\Rightarrow I_{E_1 \times E_2}\binom{x}{y}$ 為 R^{2d} 上的 Lebesgue 可測函數
$\therefore \{(x,y) \in R^{2d} \mid I_{E_1 \times E_2} = 1\} = E_1 \times E_2 \in \mathcal{L}^{2d}$

∴ 證明完成

④ 最後證明 $|E_1 \times E_2|_{(2d)} = |E_1|_{(d)} \cdot |E_2|_{(d)}$

根據 Tonelli's Theorem, $\iint_{R^{2d}} I_{E_1 \times E_2} = \int_{R^d} \int_{R^d} I_{E_1} I_{E_2} \, dy dx$

左 $= |E_1 \times E_2|$ 右 $= |E_1||E_2|$

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$\boxed{6.3}$

$g(x,y) \overset{\text{def}}{=} f(x)-f(y)$

跟 6.2 同理, $(x,y) \mapsto f(x) \in M(\mathcal{L}^2([0,1]\times[0,1]) \to \mathcal{B}(\mathbb{R}))$
$(x,y) \mapsto f(y) \in M(\mathcal{L}^2([0,1]\times[0,1]) \to \mathcal{B}(\mathbb{R}))$

且 f 為 almost everywhere 有限, 故 $f(x)-f(y)$ 為 almost everywhere 可測且有限的函數.

$g(x,y)$ 為可積分, 故 $|g(x,y)|$ 亦為可積分.

$\int_{[0,1]}\int_{[0,1]} |g(x,y)|\,dx\,dy < \infty$

$\Rightarrow \int_{[0,1]} |g(x,y)|\,dx < \infty \quad (\text{a.e } y\in[0,1])$

另外, f 為 almost everywhere 有限, 故存在 y_0 使得
$\int_{[0,1]} |g(x,y_0)|\,dx < \infty$ 且 $|f(y_0)|<\infty$.

$$\begin{aligned}\therefore \int_{[0,1]} |f(x)|\,dx &\le \int_{[0,1]} \left(|f(x)-f(y_0)| + |f(y_0)|\right) dx \\ &= \int_{[0,1]} |f(x)-f(y_0)|\,dx + \int_{[0,1]} |f(y_0)|\,dx \\ &= \underbrace{\int_{[0,1]} |g(x,y_0)|\,dx}_{<\infty} + \underbrace{|f(y_0)|}_{<\infty}\end{aligned}$$

由此可知 $f\in L^1([0,1])$

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$\boxed{6.4}$

- 利用 Exercise 6.3 的結果：我們證明 $(x,y)\mapsto f(x)-f(y)$ 為於 $[0,1]\times[0,1]$ 上可積分.

- $a=u$, $b=-u$. 得 $\int_0^1 |f(u+t)-f(-u+t)|\,dt \leq C$
  根據 Lemma 6.15, $|f(u+t)-f(-u+t)|$ 為 $\mathbb{R}^2$ 上的非負可測函數.

而且 $\int_0^1 |f(u+t)-f(-u+t)|\,dt \leq C$ (for all u)
根據 Tonelli's Theorem, $\int_0^1 |f(u+t)-f(-u+t)|\,dt$ 為 Lebesgue 可測且可積分的函數且

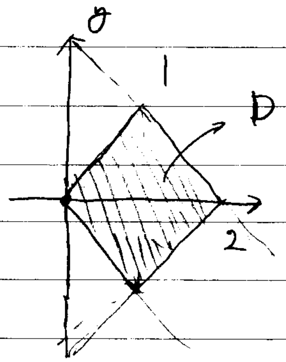
$$\iint_{[0,1]\times[0,1]} |f(u+t)-f(-u+t)| = \int du \int |f(u+t)-f(-u+t)|\,dt \leq C$$

- 依照題目提示，令 $\begin{cases} x=u+t \\ y=-u+t \end{cases}$，並變數轉換.

(在此假設，在 Lebesgue 積分亦可做這樣子的轉換)(線性的)

$u=\frac{x-y}{2}$, $t=\frac{x+y}{2}$ 故 $0\leq\frac{x-y}{2}\leq 1$, $0\leq\frac{x+y}{2}\leq 1$

$\Rightarrow$



$D=\{(x,y)\mid 0\leq x\leq 1,\ -x\leq y\leq x\}\cup$
$\{(x,y)\mid 1\leq x\leq 2,\ x-2\leq y\leq 2-x\}$

$J = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} u \\ t \end{pmatrix}$ $\quad dxdy = 2dudt$

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$C \geq \frac{1}{2}\int_D |f(x)-f(y)|\,dxdy \quad \therefore \int_D |f(x)-f(y)| \leq 2C$

$\therefore \int_{x=0}^{x=1}\int_{y=-x}^{y=x} |f(x)-f(y)| + \int_{x=1}^{x=2}\int_{y=x-2}^{y=2-x} |f(x)-f(y)| \leq 2C$

($\because$ Tonelli's Theorem)

$= \int_{x=0}^{x=1}\int_{y=-x}^{y=x} |f(x)-f(y)| + \int_{x'=0}^{x'=1}\int_{y=x'-1}^{y=1-x'} |f(x'+1)-f(y)| \quad (x-1=x')$

$= \int_{x=0}^{x=1}\int_{y=-x}^{y=x} |f(x)-f(y)| + \int_{x=0}^{x=1}\int_{y=x-1}^{y=1-x} |f(x)-f(y)|$ (利用 $f(y)=f(y+1)$)

$= \int_{x=0}^{x=1}\int_{y=0}^{y=x} |f(x)-f(y)| + \int_{x=0}^{x=1}\int_{y=1-x}^{1} |f(x)-f(y)|$

$+ \int_{x=0}^{x=1}\int_{y=0}^{y=1-x} |f(x)-f(y)| + \int_{x=0}^{x=1}\int_{y=x}^{y=1} |f(x)-f(y)|$

$= 2\int_{x=0}^{x=1}\int_{y=0}^{y=1} |f(x)-f(y)| \leq 2C$

故 $\int_{x=0}^{x=1}\int_{y=0}^{y=1} |f(x)-f(y)| \leq C$

故 $f(x)-f(y) \in L^1([0,1]\times[0,1])$

$\Rightarrow f(x) \in L^1([0,1])$

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$\boxed{6.5}$ $E \in \mathcal{L}^d$

(a) $\int_E f = |R(f,E)| = \iint_{R(f,E)} = \iint_{\mathbb{R}^{d+1}} \mathbb{I}\{(x,y) \mid x \in E,\ y \in [0,f(x)] \cap \mathbb{R}\}$

$= \iint_{\mathbb{R}^{d+1}} \mathbb{I}\{x \in E \mid f(x) \geq y\}(x) \cdot \mathbb{I}_{[0,\infty)}(y)$

$\downarrow$ Tonelli's Theorem

$= \int_{\mathbb{R}^1} dy \int_{\mathbb{R}^d} \mathbb{I}\{x \in E \mid f(x) \geq y\}(x)\, \mathbb{I}_{[0,\infty)}(y)\, dx$

$= \int_{\mathbb{R}^1} \omega(y-)\, \mathbb{I}_{[0,\infty)}(y)\, dy$

$= \int_{[0,\infty)} \omega(y-)\, dy = \int_{[0,\infty)} \omega(y)\, dy$

$\because$ $\omega(y-) = \omega(y)$ (ae)

$\omega(y)$ 為遞減函數, 故其不連續點應為可數.

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(b) 利用 (R) $\int_0^{f(x)} p y^{p-1} dy = f^p$ $(f(x) \geq 0)$

Riemann 可積分 $\Rightarrow$ Lebesgue 可積分

故 (L) $\int_{[0,f(x)]} p y^{p-1} dy = f^p$

故 $\int_E f^p = \int_{x \in E} \underbrace{\int_{[0,f(x)]} p y^{p-1} dy}_{= f^p} dx$ ... $\circledast$

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接下來，$\int_{R(f;E)} p y^{p-1} \, dx\,dy = \int_{x\in E}\int_{[0,f(x)]} p y^{p-1} \, dy\,dx$ } Tonelli's Theorem

可視為 R^{n+1} 上的非可測函數

$= \int_0^{\infty} p y^{p-1} \omega(y)\, dy$ （跟 (9) 同理，替換積分順序）

$= \int_E f^p$ $(\because \circledast)$

$\therefore$ 證明完成

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$$\int_{x\in E}\int_{[0,f(x)]} p y^{p-1}\, dy\,dx \quad \text{Tonelli's Theorem}$$
$$= \int_{y=0}^{y=\infty}\int_{\{x\in E \mid f(x)\geq y\}} p y^{p-1}\, dx\,dy$$
$$= \int_{y=0}^{y=\infty} p y^{p-1}\, \omega(y^-)\, dy$$
$$= \int_{y=0}^{y=\infty} p y^{p-1} \omega(y)\, dy \quad (\because \omega(y^-)=\omega(y) \ \text{(a.e)})$$

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6.17

$G \overset{\text{def}}{=} \mathbb{R}\setminus F$

- $y \in F \Rightarrow \delta(y) = 0$

$\Rightarrow \int_{\mathbb{R}^1} \frac{\delta(y)^\lambda f(y)}{|x-y|^{1+\lambda}} dy = \int_G \frac{\delta(y)^\lambda f(y)}{|x-y|^{1+\lambda}} dy$

- $\frac{\delta(y)^\lambda f(y)}{|x-y|^{1+\lambda}}$ 為非負可測函數. ($F \cap G = \phi$) $\therefore x \neq y$

根據 Tonelli's Theorem

$$\int_{x\in F}\int_{y\in G} \frac{\delta(y)^\lambda f(y)}{|x-y|^{1+\lambda}} dy\,dx = \int_{y\in G} \delta(y)^\lambda f(y) \int_{x\in F} \frac{dx}{|x-y|^{1+\lambda}} dy$$

利用 Theorem 6.17 證明中的不等式: $\int_F \frac{dx}{|x-y|^{1+\lambda}} \leq \frac{2}{\lambda} \delta(y)^{-\lambda}$

$\Rightarrow \leq \int_{y\in G} \delta(y)^\lambda f(y) \cdot \frac{2}{\lambda} \delta(y)^{-\lambda} = \int_{y\in G} \frac{2}{\lambda} f(y)$

$= \frac{2}{\lambda} \int_{y\in G} f(y) < \infty \quad (f \in L^1(G))$

$\therefore$ 證明完成

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$\boxed{6.10}$ $V_n = |\{(x_1,\dots,x_n) \mid x_1^2+x_2^2+\dots x_n^2 \le 1\}|_{(n)}$ 可測

$= \int_{x_1^2+\dots x_n^2\le 1} dx_1 \dots dx_d = \int_{\mathbb{R}^n} \mathbb{I}_{\{x_1^2+\dots+x_n^2\le 1\}}(x_1\dots x_n)$

$= \int_{\mathbb{R}^n} \mathbb{I}_{\{x_1^2+\dots x_{n-1}^2 \le 1-x_n^2\}}(x_1,\dots x_{n-1}) \cdot \mathbb{I}_{[-1,1]}(x_n)$

根據 Tonelli's theorem

$= \int_{[-1,1]} dx_n \int_{\mathbb{R}^{n-1}} \mathbb{I}_{\{x_1^2+\dots+x_{n-1}^2\le 1-x_n^2\}}(x_1,\dots,x_{n-1})\, dx_1 \dots dx_{n-1}$

$= \int_{[-1,1]} |B_{(1-x_n^2)}(\vec{0}_{n-1})|_{(n-1)}\, dx_n$

$$\left(T\begin{pmatrix} x_1 \\ \vdots \\ x_{n-1}\end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} \sqrt{1-x_n^2} & & O \\ & \sqrt{1-x_n^2} \ddots & \\ O & & \sqrt{1-x_n^2}\end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1}\end{pmatrix} \right)$$

$= \int_{[-1,1]} |TB_1(\vec{0}_{n-1})|\, dx_n$ (B ~ open ball)

$= \int_{[-1,1]} |\det T| \cdot |B_1(\vec{0}_{n-1})|\, dx_n$ ($\because$ Theorem 2.35)

$= \int_{[-1,1]} (1-x_n^2)^{\frac{n-1}{2}} \cdot V_{n-1}\, dx_n$

$= V_{n-1} \int_{[-1,1]} (1-x_n^2)^{\frac{n-1}{2}}\, dx_n$

(在有界閉區間上有界且 a.e. 連續 $\Rightarrow$ R 可積分)

$(1-x_n^2)^{\frac{n-1}{2}}$ 為於 $[-1,1]$ 上連續，故 Riemann 可積分

$\therefore (R)\int_{[-1,1]} (1-x_n^2)^{\frac{n-1}{2}}\, dx_n = (L)\int_{[-1,1]} (1-x_n^2)^{\frac{n-1}{2}}\, dx_n$

故 $(R)\int_{[-1,1]}(1-x_n^2)^{\frac{n-1}{2}}dx_n = {}^{(R)}2\int_{[0,1]}(1-x_n^2)^{\frac{n-1}{2}}dx_n$
$= {}^{(L)}2\int_{[0,1]}(1-x_n^2)^{\frac{n-1}{2}}dx_n$

(Rieman 可積分 ⇒ Lebesgue 可積分)

∴ $V_{n-1}\int_{[-1,1]}(1-x_n^2)^{\frac{n-1}{2}} = 2V_{n-1}\int_{[-1,1]}(1-x_n^2)^{\frac{n-1}{2}}$.

證明完成

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$\boxed{6.12}$

(a) 設 $A \subseteq [0,1]$ 為 $\mathbb{R}^1$ 的 非可測函數
並設 N 為 $\mathbb{R}^1$ 的 null-set. (Lebesgue 測度空間為完備, 所以 N 為測度 0 的可測集合)

$A \times N$ 為 $\mathbb{R}^2$ 上的可測集合, 但 $\operatorname{proj}_x(A \times N) = A \notin \mathcal{L}^1$ (非可測)
$(= \{(x,y) \mid x \in A, y \in N\})$

$\because A \times N \subseteq [0,1] \times N \subseteq \bigcup_{n=1}^{\infty} [0,1] \times I_n$ $\quad (N \subseteq \bigcup_{n=1}^{\infty} I_n,\ I_n$: 開區間$)$

$$|A \times N|_e \le \sum_{n=1}^{\infty} \underbrace{\big|[0,1] \times I_n\big|_e}_{\operatorname{vol}([0,1]\times I_n)} = \sum_{n=1}^{\infty} |I_n| < \varepsilon$$

$(\because N$ 為測度 0 的集合$)$

$\because \varepsilon \downarrow 0$ 得 $|A \times N|_{(e)} = 0$ $(\Rightarrow$ Lebesgue 可測$)$

(b) $G \in \mathcal{O}^d$ ($\mathbb{R}^d$ 上的開集合) $\exists \{I_n\}$ st

$G = \bigcup_{n=1}^{\infty} I_n$ $\quad \{I_n\}$ 為半開區間 $\quad$ (右頁有證明)

設 $I_n = \prod_{j=1}^{d} (a_{n,j}, b_{n,j}] \cap \mathbb{R}^1$ $\quad (-\infty \le a_{n,j} < b_{n,j} \le \infty)$

$\operatorname{proj}_x(G) = \bigcup_{n=1}^{\infty} (a_{n,1}, b_{n,1}] \cap \mathbb{R} \in \mathcal{B}(\mathbb{R}^1) \subseteq \mathcal{L}^1(\mathbb{R})$

($\mathbb{R}^1$ 上的 Borel 可測集合)

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$F \in \mathscr{F}^d$（$\mathbb{R}^d$上的閉集合）

$F_n \overset{\text{def}}{=} F \cap [-n, n]^d$ ‥ F_n為compact（緊緻集合）
且 $F_n \nearrow F$ (as $n \to \infty$)

$T\begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix} = \underset{x_1}{\text{proj}}\begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix} = x_1$ 顯然為連續函數

故 $T(F) = T\left(\bigcup_{n=1}^{\infty} F_n\right) = \bigcup_{n=1}^{\infty} T(F_n) \in \mathcal{B}(\mathbb{R}^d) \subseteq \mathcal{L}^d(\mathbb{R}^d)$

（$\because$ T：連續，F_n：緊緻集合
$\Rightarrow T(F_n)$ 亦為緊緻集合） ∎

㊀ $\forall G \in \mathcal{O}^d$（$\mathbb{R}^d$上的開集合）$\exists \{I_n\}$‥半開區間 st $G = \bigcup_{n=1}^{\infty} I_n$

$\boxed{\text{證明}}$ $I_{n,k}$ $(n \in \mathbb{N}, k \in \mathbb{Z}^d) \overset{\text{def}}{=} \prod_{j=1}^{d}\left(\frac{k_j}{2^n}, \frac{k_j+1}{2^n}\right]$ $k = \begin{pmatrix} k_1 \\ \vdots \\ k_d \end{pmatrix}$

「$G = \bigcup_{\substack{n \in \mathbb{N} \\ k \in \mathbb{Z}^d \\ I_{nk} \subseteq G}} I_{nk}$」 ① $G \supseteq \bigcup_{\substack{(n,k) \in \mathbb{N} \times \mathbb{Z}^d \\ I_{n,k} \subseteq G}} I_{nk}$ ‥ 顯然成立

② $\forall x \in G, \forall n \in \mathbb{N}, \exists k \in \mathbb{Z}^d$ st $x \in I_{nk}$
由於G為開集合 故 $\exists \varepsilon > 0$ st $B_\varepsilon(x) \subseteq G$.
$\forall \varepsilon > 0 \ \exists n$ st $\underset{\substack{\| \\ \left(\frac{\sqrt{d}}{2^n}\right)}}{\text{diam}|I_{nk}|} < \varepsilon$
（很大的n）

故 $\forall x \in G \ \exists I_{nk}$ st $x \in I_{nk} \subseteq G$.

$\therefore G \subseteq \bigcup_{\substack{(n,k) \in \mathbb{N} \times \mathbb{Z}^d \\ I_{nk} \subseteq G}} I_{nk}$

No. 實分析 (I)

Date HW10.

應數所： R05246013 森元俊成

$\boxed{7.1}$ 根據題意，我們定義 $E \overset{\text{def}}{=} \{x \in \mathbb{R}^d \mid |f(x)| > 0\}$ 從n改為d

$(|E| > 0)$

$Q_n(x)$ 為「中心為 x，邊長為 n」的立方體

$\bigcup_{n=1}^{\infty} Q_n(0) \cap E = E$

故存在 $m \in \mathbb{N}$ 使得 $|Q_m(0) \cap E|_d > 0$

$f^*(x) \overset{\text{def}}{=} \sup_{Q(x)} \frac{1}{|Q(x)|} \int_{Q(x)} |f(y)| dy \geq$ ①

$\frac{1}{(m+\|x\|)^d} \int_{Q_{m+\|x\|}(x)} |f(y)| dy \geq$ ②

$\frac{1}{(m\|x\|+\|x\|)^d} \int_{Q_m(0)} |f(x)| dy = \underbrace{\frac{\int_{Q_m(0)} |f(y)| dy}{(m+1)^d}}_{=c} \cdot \frac{1}{\|x\|^d}$

③

case I: $\int_{Q_m(0)} |f(y)| dy = +\infty \Rightarrow f^*(x) = +\infty$

∴ 顯然成立

case II: $\int_{Q_m(0)} |f(y)| dy < \infty \Rightarrow c \overset{\text{def}}{=} \frac{\int_{Q_m(0)} |f(y)| dy}{(m+1)^d}$

$f^*(x) \geq \frac{c}{\|x\|^d}$

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(注) ① $|Q_{m+\|x\|}(x)| = (m+\|x\|)^d$,

② $Q_m(0) \subseteq Q_{m+\|x\|}(x)$, $\|x\| \geq 1$

③ $\int_{Q_m(0)} |f(y)|\,dy \geq \int_{Q_m(0)\cap E} |f(y)|\,dy > 0$

$\because \int_A |f| = 0 \Leftrightarrow |f| = 0$ (ae in A)

$\therefore \int_A |f| > 0 \Leftrightarrow |\{x \in A \mid |f| > 0\}| > 0$

$\therefore C > 0$

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$\boxed{7.2}$

$$(f*\phi_\varepsilon)(x) - f(x) = \int_{\mathbb{R}^d} f(x-t)\,\phi_\varepsilon(t)\,dt - \int_{\mathbb{R}^d} f(x)\,\phi_\varepsilon(t)\,dt$$

($\because$ Lebesgue 積分的線性變數轉換: $A = \frac{1}{\varepsilon} \cdot \mathrm{Id}$

$$1 = \int_{\mathbb{R}^d} \phi(y)\,dy = \int_{\mathbb{R}^d} \phi(At) \cdot |A|\,dt = \int_{\mathbb{R}^d} \varepsilon^{-d} \phi\left(\frac{t}{\varepsilon}\right) dt$$
$$= \int_{\mathbb{R}^d} \phi_\varepsilon(t)\,dt \;)$$

$$\therefore \left|(f*\phi_\varepsilon)(x) - f(x)\right| \le \int_{\mathbb{R}^d} |f(x-t) - f(x)|\,|\phi_\varepsilon(t)|\,dt$$

$$= \varepsilon^{-d} \int_{\mathbb{R}^d} |f(x-t) - f(x)| \cdot \left|\phi\left(\frac{t}{\varepsilon}\right)\right| dt$$

$$\le \varepsilon^{-d} \int_{|t| \le \varepsilon} |f(x-t) - f(x)| \cdot M\,dt$$

($\because$ ϕ: Bounded & $\phi(y) = 0$ $|y| > 1$)

$$= \frac{M}{\varepsilon^d} \int_{|t| \le \varepsilon} |f(x-t) - f(x)|\,dt$$

$$= \frac{M}{\varepsilon^d} \int_{|t-x| \le \varepsilon} |f(t) - f(x)|\,dt$$

$$\le \frac{M}{\varepsilon^d} \int_{Q_{2\varepsilon}(x)} |f(t) - f(x)|\,dt$$

$$= \frac{2^d M}{|Q_{2\varepsilon}(x)|} \int_{Q_{2\varepsilon}(x)} |f(t) - f(x)|\,dt$$

x 為 f 的 Lebesgue point. 故 $\varepsilon \downarrow 0$ 時 $\to 0$.

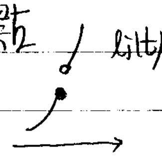
$$\therefore \lim_{\varepsilon \downarrow 0} (f*\phi_\varepsilon)(x) = f(x)$$

$\boxed{7.3}$ $K\subseteq \{Q\subseteq\mathbb{R}^d \mid Q:\text{rectangles}\}$, $E\subseteq \bigcup_{Q\in K} Q$, $|E|_e<\infty$

$Q(t)$ 表示長方體 Q 的第一個邊長為 t.

令 $l_j(t)$ 為其長方體第 j 個邊長. ($j=1\sim d$) ($l_1(t)=t$)

$l_j(t)$ 為遞增函數, 故其不連續點為可數個,
我們在此加一個條件, $l_j(t)$ 的不連續點
均為左連續



• $K_1 \overset{def}{=} K$, $t_1^* \overset{def}{=} \sup_t\{t \mid Q(t)\in K_1\}$ 若 $t_1^*=+\infty$, 則存在 $\{Q_n\}\subseteq K$ s.t. $|Q_n|\nearrow+\infty$ ($\because |Q|=\prod_{j=1}^{d} l_j(t)\ge t\, l_2(1)l_3(1)\cdots l_d(1)$ $\therefore t\nearrow\infty$ $|Q|\nearrow+\infty$)
故命題成立

以下假設 $t_1^*<\infty$. 我們能找到 $Q_1(t_1)\in K_1$, 滿足 $1\le \frac{l_j(t_1^*)}{l_j(t_1)}<2$, 另外 $Q_1^* \overset{def}{=} Q_1^*(\underset{5t_1}{\underset{\parallel}{5l_1(t_1)}}, 5l_2(t_1),\dots,5l_d(t_1))$ where Q_1^*, Q_1 的中心為相同

$K_2 \overset{def}{=}\{Q\in K_1 \mid Q\cap Q_1=\phi\}$, $K_2'\overset{def}{=}\{Q\in K_1\mid Q\cap Q_1\ne\phi\}$

我們注意 $\forall Q\in K_2'$: $Q\subseteq Q_1^*$

• 同樣定義 ($n\ge 2$)

$t_n^*\overset{def}{=}\sup_t\{t\mid Q=Q(t)\in K_n\}$ ($\sup_t(\phi)\overset{def}{=}0$)

$Q_n \overset{def}{=} Q_n(l_1(t_n), l_2(t_n),\dots l_d(t_n))\in K_n$

where $1\le\frac{l_j(t_n^*)}{l_j(t_n)}<2$ ($j=1\sim d$)

$Q_n^*\overset{def}{=}Q_n^*(5l_1(t_n),5l_2(t_n),\dots,5l_d(t_n))$ $|Q_n^*|=5^d|Q_n|$

(Q_n, Q_n^* 的中心為相同)

$K_{n+1}\overset{def}{=}\{Q\in K_n\mid Q\cap Q_n=\phi\}$

$K_{n+1}'\overset{def}{=}\{Q\in K_n\mid Q\cap Q_n\ne\phi\}$ $\forall Q\in K_{n+1}'$ $Q\subseteq Q_n^*$

我們得 $\{t_n^*\}_{n\geq 1}$ 的遞減序列.

① $\exists N\in\mathbb{N}$ st $t_{N+1}=0 \Rightarrow K_{N+1}=\phi \Rightarrow K=K_1=K_2'\cup K_3'\cup\cdots\cup K_{N+1}'$

$\therefore E\subseteq \bigcup_{Q\in K} Q = \bigcup_{n=1}^{N} \underbrace{\bigcup_{Q\in K_{n+1}'} Q}_{} \subset \bigcup_{n=1}^{N} Q_n^*$

$\therefore |E|_e \leq \sum_{n=1}^{N}|Q_n^*| = 5^d\sum_{n=1}^{N}|Q_n| \quad \therefore \beta=5^{-d}$

② $\exists\delta>0 \quad \inf_{n\geq 1} t_n^*\geq\delta>0 \Rightarrow t_n>\frac{t_n^*}{2}\geq\frac{\delta}{2}>0 \quad \therefore \sum_{n=1}^{N}|Q_n|\nearrow\infty$ (as $N\to\infty$)

$\therefore$ 題明成立.

③ $\lim_n t_n^*=0$. 「$\forall Q\in K_1 \quad Q\subset\bigcup_{n=1}^{\infty}Q_n^*$」

($\because$ 若存在 $\exists Q\in K_1$ st $Q\not\subset\bigcup_{n=1}^{\infty}Q_n^* \Rightarrow Q\not\subset Q_n^*$ (for all n)

$\Rightarrow Q\notin K_{n+1}' \Rightarrow Q\in K_{n+1} \Rightarrow Q(t): t\leq t_n^*$ (for all n)

$\Rightarrow t=0 \quad \therefore$ 矛盾)

故 $E\subseteq\bigcup_{Q\in K_1}Q \subset \bigcup_{n=1}^{\infty}Q_n^* \Rightarrow |E|_e\leq\sum_{n=1}^{\infty}|Q_n^*| = 5^d\sum_{n=1}^{\infty}|Q_n|$.

$\Rightarrow 5^{-d}|E|_e\leq\sum_{n=1}^{\infty}|Q_n|$

取一個 $\beta<5^{-d}$ 和很大的 N 滿足 $\beta|E|_e\leq\sum_{n=1}^{N}|Q_n|$

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$\boxed{7.4}$ $\{E_1, E_2\} \subseteq \mathcal{L}^1(\mathbb{R})$ (Lebesgue可测) $|E_1| > 0$ $|E_2| > 0$

$E_{1+x} \overset{\text{def}}{=} \{x+a \mid a \in E_1\}$

$\int_R |E_{1+x} \cap E_2| \, dx = \int_R dx \int_R I_{E_1+x}(t) \cdot I_{E_2}(t) \, dt$

Tonelli's theorem

$= \int_R \int_R I_{E_1+x}(t) \, I_{E_2}(t) \, dx \, dt$

$= \int_R I_{E_2}(t) \, dt \int_R I_{E_1+x}(t) \, dx$

(注) $I_{E_1+x}(t) = \begin{cases} 1 & t \in E_{1+x} \Leftrightarrow t-x \in E_1 \\ 0 & \text{elsewhere} \end{cases}$

$\therefore = I_{E_1}(t-x)$

$= \int_R I_{E_2}(t) \, dt \int_R I_{E_1}(t-x) \, dx$

$= \int_R I_{E_2}(t) \, dt \int_R I_{E_1}(-x) \, dx$ (6.13)

$= \int_R I_{E_2}(t) \int_R I_{E_1}(x) \, dx$

$= |E_1| \cdot |E_2| > 0$

由此可知, $|\{x \in R \mid |E_{1+x} \cap E_2| > 0\}| > 0$

$\exists x_0$ st $|E_{1+x_0} \cap E_2| > 0$

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$E \overset{\text{def}}{=} E_1+x_0 \cap E_2$

根據 Lemma 3.37. $\exists I \subseteq \{x-y \mid x\in E, y\in E\}$

$\Rightarrow I-x_0 \subseteq \{x-x_0-y \mid x\in E, y\in E\}$

$\subseteq \{x_1-x_2 \mid x_1\in E_1, x_2\in E_2\}$

$I_0 \overset{\text{def}}{=} I-x_0 = \{-x_0+y \mid y\in I\}$

為我們所求的開區間

實分析(I)

HW11 應用數學所(數統)

R05246013 森元俊成

$\boxed{7.5}$

- $\int_a^b \phi df$, $\int_a^b \phi f' dx$, $\int_a^b \phi dh$ 皆存在 (Lebesgue 積分)

$\int_a^b \phi df$ … ϕ: 連續 且 f: 有界變動 $\Rightarrow$ Rieman Stieltjes 積分存在

$\int_a^b \phi dh$ … ϕ: 連續 且 h: 有界變動 $\Rightarrow$ 〃 ($\Rightarrow$ Lebesgue 可積分)

($\because h = f-g$, f, g: 有界變動)

$\int_a^b \phi f' dx$ … $f' = g'$ (a.e) ($\because h' = 0$ (a.e))

g 為絕對連續, 故 g' 於 $[a,b]$ 可積分.

另外 ϕ 於 $[a,b]$ 上連續 $\Rightarrow$ 有界 $\Rightarrow$ $\phi f'$ 可積分

- $\int_a^b \phi df = \int_a^b \phi d(g+h) = \underbrace{\int_a^b \phi dg}_{①} + \underbrace{\int_a^b \phi dh}_{②}$

$① \overset{\text{Theorem 7.32}}{=} \int_a^b \phi g' dx \overset{g'=f' \text{(a.e)}}{=} \int_a^b \phi f' dx$

故 $① + ② = \int_a^b \phi f' dx + \int_a^b \phi dh$

∴ 證明完成

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$\boxed{7.7}$

* 絕對連續的定義

$\forall \varepsilon>0\ \exists \delta>0$ s.t. $\forall \mathbb{I}=\{[s,t] \mid [s,t]\subseteq[a,b]\}$ where $\forall I_1, I_2 \in \mathbb{I}$
I_1, I_2: non-overlapping, $\sum_{[s,t]\in\mathbb{I}} (t-s)<\delta \Rightarrow \sum_{[s,t]\in\mathbb{I}} |f(t)-f(s)|<\varepsilon$

① $\Rightarrow$ 顯然成立.

② $\Leftarrow$ 證明 contraposition. (「$A\Rightarrow B$」$\Leftrightarrow$「not $B \Rightarrow$ not A」)

假設 f 非絕對連續. 故假設:

存在一個正數 $\varepsilon>0$, 對於任意正數 $\forall \delta>0$ (很小的正數)
滿足:「$\sum_{i=1}^{\infty}(b_i-a_i)<\delta$ $\sum_{i=1}^{\infty}|f(b_i)-f(a_i)|\geq\varepsilon$」

$S_{+,N} \stackrel{def}{=} \{i \mid f(b_i)-f(a_i)\geq 0\} \cap \{1,2,3,\dots N\}$
$S_{-,N} \stackrel{def}{=} \{i \mid f(b_i)-f(a_i)<0\} \cap \{1,2,3,\dots N\}$

我們取一個夠大的自然數 N 使得 $\sum_{i=1}^{N}|f(b_i)-f(a_i)|>\frac{\varepsilon}{2}$

那麼滿足 $\sum_{i=1}^{N}(f(b_i)-f(a_i))^+>\frac{\varepsilon}{4}$ or $\sum_{i=1}^{N}(f(b_i)-f(a_i))^->\frac{\varepsilon}{4}$

故 $\sum_{i\in S_{+,N}}(f(b_i)-f(a_i))>\frac{\varepsilon}{4}$ or $-\sum_{i\in S_{-,N}}(f(b_i)-f(a_i))>\frac{\varepsilon}{4}$

(where $\sum_{i\in S_{+,N}}(b_i-a_i)<\delta$ & $\sum_{i\in S_{-,N}}(b_i-a_i)<\delta$)

由此可知, $^{\exists}\varepsilon_0>0$ $^{\forall}\delta>0$ $^{\exists}\mathbb{I}=\{[s,t] \mid [s,t]\subseteq[a,b]\}$: $^{\text{st}}\ulcorner$ $^{\forall}I_1, I_2\in\mathbb{I}$ non-overlapping, $\#\mathbb{I}<\infty$, $\sum_{[s,t]\in\mathbb{I}}(t-s)<\delta$, $\sum_{[s,t]\in\mathbb{I}}|f(t)-f(s)|\geq\varepsilon_0 \lrcorner$

$(\varepsilon_0\to\frac{\varepsilon}{4}$, $\mathbb{I}=\{[a_i,b_i]\}_{i\in S_{+,N}}$ or $\{[a_i,b_i]\}_{i\in S_{-,N}})$

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$\boxed{7.9}$ f: $[a,b]$上的有界變動函數.

① $\int_a^b |f'| \leq V[a,b]$ 的證明

根據 Theorem 7.24, f為有界變動 $\Rightarrow$ $|f'| = V'(x)$ (ae $[a,b]$)

故 $\int_a^b |f'| = \int_a^b V'$

另外 $V(x)$ 為遞增函數, 故 根據 Theorem 7.21,
$0 \leq \int_a^b V' dx \leq V(b-) - V(a+) \leq V(b) - V(a)$ ∎

② 若等號成立, 則 f為 絕對連續
$(\int_a^b V' = V[a,b])$

$\int_a^x |f'| \overset{(ae)}{=} \int_a^x V' \leq V[a,x]$ ...⊛
$\int_x^b |f'| = \int_x^b V' \leq V[x,b]$

$\int_a^x |f'| + \int_x^b |f'| = \int_a^b |f'| = V[a,b] = V[a,x] + V[x,b]$

由此可知, ⊛ 的等號應同時成立

$\int_a^x |f'| = \int_a^x V' = V[a,x] = V(x)$ $(V' = |f'|$ (ae), f 有界變動$)$

再者, 根據 Theorem 7.29, V' 存在 a.e. in $[a,b]$ 且 V'於$[a,b]$上可積, 且 $V(x) = \int_a^x V'$. $\Rightarrow$ V為 絕對連續.

最後由ex8的結論可知，f為絕對連續

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$\boxed{7.10}$ (a)

① $Z \in \mathcal{L}^1$ (Lebesgue可測), $|Z| = 0 \Rightarrow |f(Z)| = 0$

Z的外測度為0，考慮一個開集合G，$Z \subseteq G$

$|Z| \leq |G| < \delta$

存在 non-overlapping 的閉區間 $\{I_n = [a_n, b_n]\}_{n\geq 1}$ s.t

$G = \bigcup_{n\geq 1}^{\infty} I_n$ ∴ $|G| = \sum_{n=1}^{\infty} |I_n| < \delta$

$\sum_{n=1}^{\infty}(b_n - a_n) < \delta$

$f(Z) \subseteq f\left(\bigcup_{n\geq 1}^{\infty}[a_n, b_n]\right) \subseteq \bigcup_{n=1}^{\infty} f([a_n, b_n])$

$\subseteq \bigcup_{n\geq 1}^{\infty} [f(x_n), f(y_n)]$

where $f(x_n)$ 為 $\min\limits_{[a_n,b_n]} f(x)$ $(x_n \in [a_n, b_n])$

$f(y_n)$ 為 $\max\limits_{[a_n,b_n]} f(x)$ $(y_n \in [a_n, b_n])$

∴ $|f(Z)| \leq \sum_{n=1}^{\infty} |f(y_n) - f(x_n)| \leq \sum_{n=1}^{\infty} V[x_n, y_n] = \sum_{n=1}^{\infty} (V(y_n) - V(x_n))$

$\leq \sum_{n=1}^{\infty} (V(b_n) - V(a_n)) < \varepsilon$ ∴ 證明完成

(根據 Theorem 7.31，f為絕對連續 ⇒ V亦為絕對連續)

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② $E \in \mathcal{L}^1$ (Lebesgue 可測) ⇒ $E = H \cup Z$

⇒ $f(E) = f(H) \cup f(Z)$ (H 為 F_σ-set，可寫為 compact 集合的 countable union $\therefore H = \bigcup_{n=1}^{\infty} F_n$)

$f(H) = \bigcup_{n=1}^{\infty} f(F_n)$　　f 為連續，故 $f(F_n)$ 亦為 compact 集合
⇒ $f(H) \in \mathcal{L}^1$ (Lebesgue 可測)

∴ 證明完成

(b) $g(x) \overset{def}{=} x + f_{CL}(x)$ (where f_{CL} 為 Cantor-Lebesgue 函數)
$f(x) \overset{def}{=} g^{-1}(x)$ (g 為嚴格遞增，$g: [0,1] \to [0,2]$ 為 one-to-one & onto)

令 C 為 Cantor 集合 ($|C| = 0$)，$[0,1] \setminus C = \bigcup_{n=1}^{\infty} [a_n, b_n]$

$|f^{-1}([0,1] \setminus C)| = |f^{-1}(\bigcup_{n=1}^{\infty} [a_n, b_n])| = |\bigcup_{n=1}^{\infty} [a_n, b_n]| = |[0,1] \setminus C| = 1$

$|f^{-1}([0,1]) \setminus f^{-1}(C)| = |[0,2]| - |f^{-1}(C)|$　∴ $|f^{-1}(C)| = 1$ ($\neq 0$)

另外，證明 f 為 Lipschitz continuous

「$\exists C > 0$ st $|f(x) - f(y)| \le C|x - y|$
⇔ 「$\exists C > 0$ st $|x - y| \le C|f^{-1}(x) - f^{-1}(y)|$」　(∵ f one-to-one)

$= |x + f_{CL}(x) - y - f_{CL}(y)|$　($x < y$)

$C = 1$ 顯然滿足不等式

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$\boxed{7.12}$ 照提示證明.

若 $\exists j$ s.t. $a_j = 0$，則 不等式顯然成立

以下考慮 $a_j > 0$ (for all $j = 1, 2, \dots N$)

$x_j \overset{\text{def}}{=} p_j \ln a_j \Rightarrow a_j = \exp\left(\frac{x_j}{p_j}\right)$

$\prod_{j=1}^{N} a_j = \prod_{j=1}^{N} \exp\left(\frac{x_j}{p_j}\right) = \exp\left(\sum_{j=1}^{N} \frac{x_j}{p_j}\right)$

$\sum_{j=1}^{N} \frac{a_j^{p_j}}{p_j} = \sum_{j=1}^{N} \frac{1}{p_j} \exp(x_j)$

$c_j \overset{\text{def}}{=} \frac{1}{p_j} \quad \left(c_1 + \dots + c_N = \frac{1}{p_1} + \dots \frac{1}{p_N} = 1\right)$

根據 Theorem 7.35, $\exp\left(\frac{c_1 x_1 + \dots c_N x_N}{c_1 + \dots + c_N}\right) \leq \frac{c_1 \exp(x_1) + \dots c_N \exp(x_N)}{c_1 + \dots c_N}$

$\Leftrightarrow \exp(c_1 x_1 + \dots + c_N x_N) \leq c_1 \exp(x_1) + \dots + c_N \exp(x_N)$

$\Leftrightarrow \exp\left(\sum_{j=1}^{N} \frac{x_j}{p_j}\right) \leq \sum_{j=1}^{N} \frac{1}{p_j} \exp(x_j)$

$\therefore$ 證明完成

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$\boxed{7.14}$ ① 「$\Leftarrow$」的證明…

「claim」: $\phi\left(\frac{\sum_{j=1}^{2^n} x_j}{2^n}\right) \le \sum_{j=1}^{2^n} \frac{1}{2^n}\phi(x_j)$

利用歸納法來證明上述的 claim.

• $n=1$.. 顯然成立 $\left(\because \phi\left(\frac{x_1+x_2}{2}\right) \le \frac{1}{2}\left(\phi(x_1)+\phi(x_2)\right)\right)$

• $n=k$.. 假設成立 $\left(\therefore \phi\left(\frac{\sum_{j=1}^{2^k} x_j}{2^k}\right) \le \sum_{j=1}^{2^k} \frac{1}{2^k}\phi(x_j)\right)$

• $n=k+1$:

$$\phi\left(\frac{\sum_{j=1}^{2^{k+1}} x_j}{2^{k+1}}\right) = \phi\left(\frac{\frac{1}{2^k}\sum_{j=1}^{2^k} x_j + \frac{1}{2^k}\sum_{j=2^k+1}^{2^{k+1}} x_j}{2}\right) \overset{(假設)}{\le} \frac{1}{2}\left(\phi\left(\frac{1}{2^k}\sum_{j=1}^{2^k} x_j\right) + \phi\left(\frac{1}{2^k}\sum_{j=2^k+1}^{2^{k+1}} x_j\right)\right)$$

$$\overset{(n=k \text{ 的 case})}{\le} \frac{1}{2}\left(\sum_{j=1}^{2^k} \frac{1}{2^k}\phi(x_j) + \sum_{j=2^k+1}^{2^{k+1}} \frac{1}{2^k}\phi(x_j)\right) = \frac{1}{2^{k+1}}\sum_{j=1}^{2^{k+1}} \phi(x_j)$$

故 $n=k+1$ 亦成立 $\Rightarrow$ claim 的證明完畢.

• 接下來利用 claim:

$x_1=x_2=x_3=\cdots=x_m=x$ $\quad (m=0,1,\dots,2^n)$ $\quad (x,y)\subseteq(a,b)$

$x_{m+1}=x_{m+2}=\cdots=x_{2^n}=y$

$$\Rightarrow \phi\left(\frac{mx}{2^n}+\frac{2^n-m}{2^n}y\right) \le \frac{m}{2^n}\phi(x)+\frac{2^n-m}{2^n}\phi(y)$$

$A \overset{\text{def}}{=} \left\{\frac{m}{2^n} \,\middle|\, n\in\mathbb{N},\ m\in\{0,1,\dots,2^n\}\right\} \subseteq [0,1]$

$\forall \alpha\in A$. 滿足 $\phi(\alpha x+(1-\alpha)y) \le \alpha\phi(x)+(1-\alpha)\phi(y)$

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A為[0,1]的稠密集合,

故 $\forall \theta \in [0,1]$ $\exists \{a_n\}_{n\geq 1} \subseteq A$ st $a_n \to \theta$

$\therefore$ $\phi(a_n x + (1-a_n)y) \leq a_n \phi(x) + (1-a_n)\phi(y)$

$n \to \infty$ 得 $\phi(\theta x + (1-\theta)y) \leq \theta\phi(x) + (1-\theta)\phi(y)$

($\because \phi$ 連續)

② 「$\Rightarrow$」的證明

$\phi\left(\frac{x+y}{2}\right) \leq \frac{1}{2}\phi(x) + \frac{1}{2}\phi(y)$ 顯然成立.

根據 Theorem 7.40, 連續性成立.

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利用7.14

$\boxed{7.15}$ 證明 $\frac{\phi(x_1)+\phi(x_2)}{2} \geq \phi\left(\frac{x_1+x_2}{2}\right)$

$\Leftrightarrow \phi(x_1)+\phi(x_2) \geq 2\phi\left(\frac{x_1+x_2}{2}\right)$

$\Leftrightarrow \left(\phi(x_2)-\phi\left(\frac{x_1+x_2}{2}\right)\right)-\left(\phi\left(\frac{x_1+x_2}{2}\right)-\phi(x_1)\right) \geq 0$

$\Leftrightarrow \underbrace{\int_{\frac{x_1+x_2}{2}}^{x_2} f(t)\,dt}_{①} - \underbrace{\int_{x_1}^{\frac{x_1+x_2}{2}} f(t)\,dt}_{②} \geq 0$

由於 f 為遞增函數，故 $① \geq \int_{\frac{x_1+x_2}{2}}^{x_2} f\left(\frac{x_1+x_2}{2}\right)dt$

$= \left(\frac{x_2-x_1}{2}\right) f\left(\frac{x_1+x_2}{2}\right)$

同樣，$② \leq \int_{x_1}^{\frac{x_1+x_2}{2}} f\left(\frac{x_1+x_2}{2}\right)dt = \left(\frac{x_2-x_1}{2}\right) f\left(\frac{x_1+x_2}{2}\right)$

$\therefore ①-② \geq 0$

另外，證明 ϕ 為連續函數

$\forall x \in (a,b) \ldots \quad |\phi(x+h)-\phi(x)| = \left|\int_x^{x+h} f(t)\,dt\right|$

$\leq \int_x^{x+h} |f(t)|\,dt$

取一個 $\varepsilon > 0$ st $[x, x+h] \subseteq (a, b-\varepsilon) \subseteq (a,b)$

$\phi(x)$ 於 (a,b) 上為 well-defined，故 $f(t)$ 應於 $(a, b-\varepsilon)$ 上可積分

$(|f|)$

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夠小的 h

$\forall h>0$: $|f(t)|\cdot \mathbb{I}_{[x,x+h]}(t) \leq |f(t)|\cdot \mathbb{I}_{(a,b-\varepsilon)}(t) \in L^1((a,b-\varepsilon))$

利用 Lebesgue 控制收斂定理

$$\lim_{h\to 0}|\phi(x+h)-\phi(x)| \leq \lim_{h\to 0}\int_x^{x+h}|f(t)|\,dt = \lim_{h\to 0}\int_a^{b-\varepsilon}|f(t)|\cdot \mathbb{I}_{[x,x+h]}(t)\,dt$$

$$= \int_a^{b-\varepsilon}\lim_{h\to 0}|f(t)|\cdot \mathbb{I}_{[x,x+h]}(t)\,dt$$

$$= \int_a^{b-\varepsilon}|f(x)|\cdot \underbrace{\mathbb{I}_{\{x\}}(t)}_{0\ (ae)}\,dt$$

$$=0$$

同理 $\lim_{h\to 0}|\phi(x)-\phi(x-h)|=0$

故 ϕ 為連續函數

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$\boxed{7.16}$

- $f \overset{\text{def}}{=} CL(x) \cdot \mathbb{I}_{[0,1]}(x)$ $\quad CL(x)$: Cantor Lebesgue函數
- $g \overset{\text{def}}{=} \exp\left(\frac{1}{x^2-1}\right) \cdot \mathbb{I}_{(-1,1)}(x)$ $\quad \mathrm{supp}(g) = \overline{\{x \in \mathbb{R} \mid g \neq 0\}} = [-1,1]$

g於$(-1,1)$無窮可微

(右) $f' = 0$ (ae) 故 $-\int_{-\infty}^{\infty} f' g \, dx = 0$

(左) $g'(x) = \frac{-2x}{(x^2-1)^2} \cdot \exp\left(\frac{1}{x^2-1}\right)$. 故 $\int_{-\infty}^{\infty} f g' =$

$= \int_0^1 \frac{-2x}{(x^2-1)^2} \cdot \exp\left(\frac{1}{x^2-1}\right) \cdot CL(x) \, \mathbb{I}_{[0,1)}(x) \, dx$

$f'g < 0$ (ae in $[0,1]$)

故 $\int_{-\infty}^{\infty} f g' < 0$

(主) $f \in M((E, \mathcal{L}^d|_E) \to ([0,\infty], \mathbb{B}([0,\infty])))$ $\quad (E \in \mathcal{L}^d)$
(非負可測函數)

「若 $f > 0$ 則 $\int_E f > 0$」

∵ $\int_E f = 0 \Rightarrow f = 0$ (ae)

(證) $\int_E f = 0 \Rightarrow \int_{\{x \in E \mid f > \frac{1}{n}\}} f = 0 \Rightarrow |\{x \in E \mid f > \frac{1}{n}\}| = 0$ $(n \in \mathbb{N})$

$\left|\bigcup_{n=1}^{\infty} \{x \in E \mid f > \frac{1}{n}\}\right| \leq \sum_{n=1}^{\infty} |\{x \in E \mid f > \frac{1}{n}\}| = 0$

$\overset{\parallel}{|\{x \in E \mid f > 0\}|}$ $\quad$ 故 $f = 0$ (ae)

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$\boxed{7.17}$ $\{f_k\}_{k\geq 1} \cup \{f\} \subseteq L^1((0,1))$ $f_k \to f$ (ae)

① 「$\Rightarrow$」 $\lim_{k\to\infty} \int_{(0,1)} |f_k - f| = 0$

我們注意 $\{|f_k|\}_{k\geq 1} \cup \{f\} \subseteq L^1((0,1))$，故 $\int_E |f|$ $\int_E |f_k|$
為絕對連續的集合函數.

由於 $\lim_{k\to\infty} \int_{(0,1)} |f_k - f| = 0$，故存在 k_0 st $\forall k > k_0$
$\int_{(0,1)} |f_k - f| < \frac{\varepsilon}{2}$ （ε為任意正數）

另外，存在 $\{\delta_0, \delta_1, \cdots \delta_{k_0}\} \subseteq (0,\infty)$ 使得：

- $\forall E \in \mathcal{L}^1((0,1))$: $|E| < \delta_0$ $\int_E |f| < \frac{\varepsilon}{2}$
- $\forall E \in \mathcal{L}^1((0,1))$: $|E| < \delta_j$ $\int_E |f_j| < \frac{\varepsilon}{2}$ $(j = 1, 2, \cdots k_0)$

取 $\delta < \min\{\delta_0, \delta_1, \cdots \delta_{k_0}\}$ 以及 $E \in \mathcal{L}^1((0,1))$ $|E| < \delta$

(i) $1 \leq k \leq k_0$

$$\left|\int_E f_k\right| \leq \int_E |f_k| < \frac{\varepsilon}{2}$$

$(\because |E| < \delta = \min\{\delta_0, \cdots \delta_{k_0}\} \leq \delta_k)$

(ii) $k > k_0$

$$\left|\int_E f_k\right| \leq \int_E |f_k| \leq \underbrace{\int_E |f_k - f|}_{<\frac{\varepsilon}{2}} + \underbrace{\int_E |f|}_{<\frac{\varepsilon}{2}} < \varepsilon$$

$(\because |E| < \delta \leq \delta_0)$

$\therefore$ 證明完成

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② 「$\Leftarrow$」: $\Phi_k(E)=\int_E f_k$ 為均勻絕對連續

我們注意 $\int_E |f_k|$ 亦為均勻絕對連續

$\because \int_E |f_k| = \int_{\{x\in E \mid f_k\ge 0\}} f_k - \int_{\{x\in E\mid f_k<0\}} f_k$

$= \left|\int_{\{x\in E\mid f_k\ge 0\}} f_k\right| + \left|\int_{\{x\in E\mid f_k<0\}} f_k\right|$

$= \left|\Phi_k(\{x\in E\mid f_k\ge 0\})\right| + \left|\Phi_k(\{x\in E\mid f_k<0\})\right|$

我們取 E where $|E|<\delta \Rightarrow |\{x\in E\mid f_k\ge 0\}|, |\{x\in E\mid f_k<0\}| \le \delta$

$(0,1)$ 為有限測度，故 $f_n\to f$ (a.e.) $\Rightarrow f_n\to f$ (in measure)

$\forall \varepsilon>0\ \exists k_0$ s.t. $k\ge k_0$ $|\underbrace{\{x\in(0,1) \mid |f_k-f|\ge\varepsilon\}}_{E_k}|<\delta$

故 $k\ge k_0 \Rightarrow \int_{(0,1)}|f_k-f| \le \int_{E_k}|f_k-f| + \int_{(0,1)\setminus E_k}|f_k-f|$

$\le \int_{E_k}|f_k| + \int_{E_k}|f| < 2\varepsilon$

$\le \int_{(0,1)\setminus E_k}\varepsilon$

$\le \varepsilon\,|(0,1)| = \varepsilon$

($\because$ $\int_E |f|$.. 絕對連續

$\int_E |f_k|$.. 均勻絕對連續

所以 $|E|<\delta \Rightarrow \int_E |f|<\varepsilon,\ \int_E |f_k|<\varepsilon$ for all k)

$\therefore$ 證明完成

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7.20

(a) $f \overset{def}{=} \mathbb{I}_{\mathbb{R}^d \setminus \mathbb{Q}^d}(x)$

f為局部可積函數。根據 Lebesgue 微分定理，

$\lim\limits_{Q \downarrow \{x\}} \frac{1}{|Q|} \int_Q f(y)dy = f(x)$ (a.e.)

我們注意 $f(y)=1$ a.e. in Q (for any Q)

$\therefore \frac{1}{|Q|} \int_Q f(y)dy = \frac{1}{|Q|} |Q| = 1$

極限隨時存在且為1.

另外，$\lim\limits_{Q \downarrow \{x\}} \frac{1}{|Q|} \int_Q |f(y)-f(x)|dy$

① $x \in \mathbb{Q}^d$ ... $\lim\limits_{Q \downarrow \{x\}} \frac{1}{|Q|} \int_Q |\underset{\downarrow 1.(a.e)}{f(y)}-0|dy = \lim\limits_{Q \downarrow \{x\}} \frac{1}{|Q|} |Q| = 1$

② $x \in \mathbb{R}^d \setminus \mathbb{Q}^d$ ... $\lim\limits_{Q \downarrow \{x\}} \frac{1}{|Q|} \int_Q \underbrace{|f(y)-1|}_{= 0 \ (a.e)} dy = \lim\limits_{Q \downarrow \{x\}} \frac{0}{|Q|} = 0$

故 $\mathbb{R}^d \setminus \mathbb{Q}^d$ 為 f 的 Lebesgue-Set

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$\boxed{7.25}$

① $x^p \ln(1+x)$ …

$$f(x) = x^p \ln(1+x)$$

$$f'(x) = p x^{p-1} \ln(1+x) + \frac{x^p}{1+x}$$

$$f''(x) = p(p-1) x^{p-2} \ln(1+x) + \frac{p x^{p-1}}{1+x} + \frac{p x^{p-1}}{(1+x)} - \frac{x^p}{(1+x)^2}$$

$$= p(p-1) x^{p-2} \ln(1+x) + \frac{2p x^{p-1}(1+x) - x^p}{(1+x)^2}$$

$$= \frac{p(p-1) x^{p-2} (1+x)^2 \ln(1+x) + 2p x^{p-1}(1+x) - x^p}{(1+x)^2}$$

$$= \underbrace{\frac{x^{p-2}}{(1+x)^2}}_{>0} \left\{ p(p-1)(1+x)^2 \ln(1+x) + 2px(1+x) - x^2 \right\}$$

$$h(x) \overset{\text{def}}{=} p(p-1)(1+x)^2 \ln(1+x) + 2px(1+x) - x^2$$

$$= p(p-1)\underbrace{(1+x)^2}_{>0} \underbrace{\ln(1+x)}_{>0} + \underbrace{(2p-1)}_{>0} x^2 + \underbrace{2px}_{>0}$$

$\therefore h(x) > 0 \quad (x > 0)$ 　　故 $f''(x) > 0 \Rightarrow f'(x)$：遞増函数

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② $x^p(1+\ln^+x) = \begin{cases} x^p & (x<1) \\ x^p(1+\ln x) & (x\geq 1) \end{cases}$

- $\frac{d}{dx}(x^p) = px^{p-1}$ … 遞增函數 $(0<x<1)$
- $\frac{d}{dx}(x^p(1+\ln x)) = px^{p-1}(1+\ln x) + x^{p-1}$ … 遞增函數 $(x>1)$

$\because \frac{d^2}{dx^2}(x^p\cdot(1+\ln x)) = p(p-1)\underbrace{x^{p-2}}_{>0}\underbrace{(1+\ln x)}_{>0} + \underbrace{p}_{>0}x^{p-2} + \underbrace{(p-1)}_{>0}\underbrace{x^{p-2}}_{>0} > 0$

於 $(0,1)\cup(1,\infty)$, $\frac{d}{dx}x^p(1+\ln^+x)$ 為遞增函數（且有限）

且 $x^p(1+\ln^+x)$ 於 $(1,\infty)$ 連續

根據 Exercise 7.24, $x^p(1+\ln^+x)$ 為 convex 函數

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㊟ Extra Exercises

$\boxed{5.28}$

取 ln … ㊧ $\lim_{p\downarrow 0} \frac{\ln \int_E f^p - \ln|E|}{p}$ （ln為連續函數，lim & ln 可替換）

$\boxed{\text{claim}}$ $\lim_{p\downarrow 0} \int_E f^p = |E|$

$$\int_E f^p = \int_{\{x\in E \mid 0<f\le 1\}} f^p + \int_{\{x\in E \mid f>1\}} f^p$$

$p\downarrow 0 \Rightarrow f^p \nearrow 1$ 　 $0\le f^p \le f \in L^1(E)$

MCT (or LDCT) 　 ∴ LDCT

$$\therefore \lim_{p\downarrow 0} \int_E f^p = \int_E \lim_{p\downarrow 0} f^p = \int_E 1 = |E|$$

接下來考慮，$\lim_{p\downarrow 0} \frac{\ln \int_E f^p - \ln|E|}{p}$ 　 $\left(g(p) \overset{\text{def}}{=} \int_E f^p\right)$

利用 L'hôpital's rule: $\lim_{p\downarrow 0} \frac{g(p)-g(0)}{p} = g'(0)$

$$\therefore g'(0) = \lim_{h\downarrow 0} \int_E \frac{f^h - 1}{h}$$

$$= \lim_{h\downarrow 0} \int_{\{x\in E \mid 0<f\le 1\}} \frac{f^h-1}{h} + \int_{\{x\in E \mid f>1\}} \frac{f^h-1}{h}$$

MCT 　 LDCT