105年 台灣大學 應用數學科學研究所 高等統計推論Ⅰ （作業1－6）

Advanced Statistical Inference I

Homework 1: Probability Theory

Due Date: September 22nd, 2016

1. (Review change of variable and integration) Given a set (region),

$$A = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}.$$

Make the transformation $u = x + y$, $v = x$.

   (a) Sketch this region or image of the transformation.

   (b) Find the following integration

$$\int\int_{A \cap D} dxdy$$

   on the set $A \cap D$, where

$$D = \{(x, y) : x + y \leq 1/4\}.$$

2. (Review college level probability and statistics) Given pdf of x,

$$f(x) = \begin{cases} \lambda \exp(-\lambda x) & \text{for } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

where λ is a positive real number. Find the pdf

$$g(x) = \int_{-\infty}^{\infty} f(y) f(x - y) dy$$

for all $x \in R$.
Hint: The above formula is called convolution of $X = Y + Z$, where $Z = X - Y$ and Y and Z are independent random variables with the same distribution f. Be careful with the region of integration.

3. (Review some combinatory) Lay aside m black balls and n read balls in a jug. Suppose $1 \leq r \leq k \leq n$. Each time one draws a ball from the jug at random.

   (a) If each time one draws a ball without return, what is the probability that in the k-th time of drawing one obtains exactly the r-th red ball?

   (b) If each time one draws a ball with return, what is the probability that in the k-th time of drawing one obtained totally an odd number of red ball?

4. (Review uncorrelated and independent) A coin is tossed 10 times. (The implicit assumptions are that the tosses are independent and the chance of heads is $1/2$.) Let V be the number of heads among the last 9 tosses. Let X be $+1$ if the first toss is heads; else, $X = -1$. Set $U = XV$. Show that $Cov(U, V) = 0$ and the random variables U and V are not independent.

5. (Review concept involved conditioning) A coin is tossed 10 times. (The implicit assumptions are that the tosses are independent and the chance of heads is $1/2$.) Let X be the number of heads on the first 5 tosses, and Y the total number of heads. Show that $E(Y|X = x) = x+2.5$ and $var(Y|X = x) = 5/4$ where the possible values of x are $0, 1, \ldots, 5$.

6. (Review of MLE) Let X_i and Y_i, $1 \leq i \leq n$, be independent. Moreover, both X_i and Y_i are normally distributed with mean α_i and varaince σ^2 for $i = 1, \ldots, n$.

   (a) Show that the MLE for α_i is $\hat{\alpha}_i = (X_i + Y_i)/2$ and the MLE for σ^2 is $\hat{\sigma}^2 = n^{-1} \sum_{i=1}^n s_i^2$, where $s_i^2 = (X_i - Y_i)^2/4$.

   (b) Show that $\hat{\sigma}^2$ is not consistent by identifying its mean and variance as n goes to infinity. Review the proof of large numbers and the definition of convergence in probability.

7. Suppose that an experiment is conducted to measure a constant θ. Independent unbiased measurement y of θ can be made with either of two instruments, both of which measure with normal errors: for $i = 1, 2$, instrument i produces independent errors with a $N(0, \sigma_i^2)$ distribution. The two error variances σ_1^2 and σ_2^2 are known. When a measurement y is made, a record is kept of the instrument used so that after n measurements the data is $(a_1, y_1), \ldots, (a_n, y_n)$, where $a_m = i$ if y_m is obtained using instrument i. The choice between instruments is made independently for each observation in such a way that

$$P(a_m = 1) = P(a_m = 2) = 0.5, \quad 1 \leq m \leq n.$$

   Let x denote the entire set of data available to the statistician, in this case $(a_1, y_1), \ldots, (a_n, y_n)$, and let $\ell_\theta(x)$ denote the corresponding log likelihood function for θ. Let $a = \sum_{m=1}^n (2 - a_m)$. Show that the maximum likelihood estimate of θ is given by

$$\hat{\theta} = \left(\sum_{m=1}^n 1/\sigma_{a_m}^2 \right)^{-1} \left(\sum_{m=1}^n y_m/\sigma_{a_m}^2 \right).$$

8. (Review optimization of non-differentiable function) Suppose $x_1, \ldots, x_n$ are real numbers. Suppose n is odd and the x_i are all distinct. There is a unique median $\tilde{x}$: the middle number when the x's are arranged in increasing order. Let c be a real number.

   (a) Show that $f(c) = \sum_{i=1}^n |x_i - c|$, as a function of c, is minimized when $c = \tilde{x}$.
   Hints. You cannot do this by calculus, because f is not differentiable. Instead, show that $f(c)$ is (i) continuous, (ii) strictly increasing as c increases for $c > \tilde{x}$, i.e., $\tilde{x} < c_1 < c_2$ implies $f(c_1) < f(c_2)$, and (iii) strictly decreasing as c increases for $c < \tilde{x}$. It is easier to think about claims (ii) and (iii) when c differs from all the x's. You may as well assume that the x_i are increasing with i. If you pursue this line of reasoning far enough, you will find that f is linear between the x's, with corners at the x's. Moreover, f is convex, i.e., $f((x + y)/2) \leq [f(x) + f(y)]/2$.

   (b) Suppos X_i are independent for $i = 1, \ldots, n$, with common density $0.5 \exp(-|x - \theta|)$, where θ is a parameter, x is real, and n is odd. The MLE for θ is defined as the maximizer of $\prod_{i=1}^n 0.5 \exp(|x_i - \theta|)$. Find MLE of θ.

9. (Review optimization of differentiable function) Suppose that, conditional on the covariates $\mathbf{x} \in R^p$, the Y's are independent $0 - 1$ variables, with logit $P(Y_i = 1|\mathbf{X}_i = \mathbf{x}_i) = \mathbf{x}_i \boldsymbol{\beta}$, i.e., the logit model holds. Here $\mathbf{x}$ is a $p \times 1$ vector. Its log likelihood function can be written as

$$L_n(\boldsymbol{\beta}) = -\left(\sum_{i=1}^n \log[1 + \exp(\mathbf{x}_i \boldsymbol{\beta})] \right) + \left(\sum_{i=1}^n \mathbf{x}_i Y_i \right) \boldsymbol{\beta}.$$

   (a) Show that $L_n(\boldsymbol{\beta})$ is a concave function of $\boldsymbol{\beta}$.

- **(b)** Show that $L_n(\beta)$ is strictly concave if $\mathbf{X}$ has full rank. Here $\mathbf{X}$ is a $p \times n$ matrix and $\mathbf{X} = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$.
- **(c)** Comment on the existence of the maximizer of $L_n(\beta)$ when $\mathbf{X}$ is full rank.

Hint. Let the parameter vector β be $p \times 1$ and $\mathbf{c}$ be a $p \times 1$ vector with $\|\mathbf{c}\| > 0$. You need to show $\mathbf{c}^T L_n''(\beta)\mathbf{c} \leq 0$, with strict inequality if $\mathbf{X}$ has full rank. Check that $\mathbf{c}^T \mathbf{x}_i^T \mathbf{x}_i \mathbf{c} \geq 0$ and $\psi''(\mathbf{x}_i\beta) \leq m < 0$ for all $i = 1, \ldots, n$, where m is a real number that depends on β. Note that $\psi(x) = -\log(1 + e^x)$.

10. (Detect mixture distribution) Two pennies, one with $P(\text{head}) = u$ and one with $P(\text{head}) = w$, are to be tossed together independently. Define $p_0 = P(0 \text{ heads occur})$, $p_1 = P(1 \text{ heads occur})$, and $p_2 = P(2 \text{ heads occur})$. Can u and w be chosen such that $p_0 = p_1 = p_2$? Prove your answer.

11. (Countable additivity and Kolmogorov's Axiom)

    - **(a)** Show that the Axiom of Countable Additivity implies Finite Additivity.
    - **(b)** Prove that the Axiom of Continuity and the Axiom of Finite Additivity imply Countable Addivity. Note that Axiom of Continuity

12. (Information and Conditioning) An employer is about to hire one new employee from a group of N candidates, whose future potential can be rated on a scale from 1 to N. The employer proceeds according to the following rule.

    (Rule 1) Each candidate is seen in successsion (in random order) and a decision is made whether to hire the candidate.

    (Rule 2) Having rejected $m-1$ candidates ($m > 1$), the employer can hire the mth candidate only if the m th candidate is better than the previous $m-1$.

    Suppose a candidate is hired on the ith trial. What is the probability that the best candidate was hired?

13. (Is there a cheating?)

    - **(a)** In a draft lottery containing the 366 days of the year (including February 29), what is the probability that the first 10 days drawn (without replacement) are evenly distributed among the 12 months?
    - **(b)** What is the probability that the first 30 days drawn contain none from September ?

14. (Sampling and central limit theorem)

    - **(a)** A way of approximating the large factorials is through the use of *Stirling's Formula*:

    $$n! \approx \sqrt{2\pi} n^{n+1/2} \exp(-n),$$

    a complete derivation of which is difficult. Instead, prove the easier fact,

    $$\lim_{n\to\infty} \frac{n!}{n^{n+1/2}\exp(-n)} = \text{a constant}.$$

(b) Suppose that we are going to calculate all possible averages of four numbers selected from $\{2, 4, 9, 12\}$ where we draw the numbers with replacement. Prove that the average of $\{2, 4, 9, 12\}$, 29/4, has the highest probability.

(c) Prove that. in general, if we sample with replacement from the set $\{x_1, x_2, \cdots, x_n\}$, the outcome with average $(x_1+x_2+\cdots+x_n)/n$ is the most likely, having probability $n!/n^n$.

(d) Use Stirling's formula to show that $n!/n^n \approx \sqrt{2n\pi}e^{-n}$.

(e) Show that the probability that a particular x_i is missing from an outcome is $(1-1/n)^n \to \exp(-1)$ as $n \to \infty$.

15. (Model time series and etc) Derive the autocovariance function of the following autoregressive process:

$$X_t = 0.5X_{t-1} + Z_t;$$

where $E(Z_t) = 0$, $var(Z_t) = 1$, and $Z_1, Z_2, \ldots$ are independent. For a time series $\{X_t\}$, its autocovariance function is defined as

$$\gamma_X(t+h, t) = Cov(X_{t+h}, X_t) = E[(X_{t+h} - E(X_{t+h}))(X_t - E(X_t))].$$

16. (What is the meaning of random assignment? Think of lottery.)

(a) My telephone rings 12 times each week, the calls being randomly distributed among the seven days. What is the probability that I get at least one call each day? (Answer: 0.2285.)

(b) What is the probability distribution of X_1?

(c) If someone does this experiment once, you are asked to guess the outcome of X_1 defined in Exercise 1.46. Let Y_j denote the rule that someone will make a guess j. If the guess matches the outcome of X_1, a prize of 100 dollars will be given. Otherwise, there is no reward. Determine the expected return of rule Y_j and decide which Y_j gives the highest return.

17. (Comparison of two random variables)

(a) A cdf F_X is *stochastically* larger than a cdf F_Y if $F_X(t) \leq F_Y(t)$ for all t and $F_X(t) < F_Y(t)$ for some t. Prove that if $X \sim F_X$ and $Y \sim F_Y$, then

$$P(X > t) \geq P(Y > t) \quad \text{for every } t$$

and

$$P(X > t) \geq P(Y > t) \quad \text{for some } t,$$

that is, X tends to be bigger than Y.

(b) Let X and Y be two random variables defined as in Example 1.5.4 with p_X and p_Y, respectively. Here p_X and p_Y refer to the probability of a head on any given toss for those two coins, respectively. When $p_X > p_Y$, is F_Y is stochastically greater than F_X? Give reason to support your conclusion.

18. The paper "Methods of Studying Coincidences" by Diaconis and Mosteller (1989, JASA) shows how the "The Law of Truly Large Numbers" might explain what we think of as *rare* events. That is, when we truly enumerate the sample space and the event, things are not so remarkable as they might once seem.
The *Birthday Problem* is an example of this, as is the *Double Lottery Winner*. To the average person, it seems that the odds are astronomical that someone could win the lottery twice, but that is not so! Lets do some calculations for the Florida Lotto.

    (a) The Florida Lottery states *Select six numbers from* 1 *through* 53 *in one panel on your FLORIDA LOTTO playslip*, and gives the odds of winning as $1 : 22,957,480$. Verify that this number is correct. (You might check out http://www.flalottery.com)

    (b) There was a front page story in The New York Times that reported a *1 in 17 trillion* long shot of a woman who won the New Jersey lottery twice. The 1 in 17 trillion number is the correct answer to a not very relevant question. It is the probability that YOU will win the lottery twice. Calculate this number for Florida Lotto

    (c) The important question is, What is the chance that some person, out of all the millions and millions of people who buy lottery tickets in the United States, hits a lottery twice in a lifetime? (We must remember that many people buy multiple tickets on each of many lotteries, but we will ignore that fact here.) The population of Florida is $15,982,378$ people. What is the probability of a double lottery winner if (i) one out of every 10 people play Lotto and (ii) one out of every 5 people play

    (d) Finally, realize that we should not only consider one lottery, but take into account the fact that the lottery is run every week. Redo the calculations in part (b) assuming that (i) the lottery is played every week for one year (ii) the lottery is played every week for five year

    (e) After the The New York Times article appeared, Steve Samuels and George McCabe, both Professors in the Department of Statistics at Purdue University, did some calculations and called the event "practically a sure thing," calculating that it is better than even odds to have a double winner in seven years someplace in the United States. It is better than 1 in 30 that there is a double winner in a four month period - the time between the winnings of the New Jersey woman.
    For the Florida Lotto, give a scenario in which it would be *better than even odds* that there will be a double lottery winner.

The following item was reported in the February 14, 1986 edition of The New York Times: *A New Jersey woman wins the New Jersey State Lottery twice within a span of four months.* She won the jackpot for the first time on October 23, 1985 in the Lotto 6/39. Then she won the jackpot in the new Lotto 6/42 on February 13, 1986. Lottery officials declare that the probability of winning the jackpot twice in one lifetime is approximately one in 17.1 trillion. What do you think of this statement?
Solution. The claim made in this statement is easily challenged. The officials' calculation proves correct only in the extremely farfetched case scenario of *a given person entering a six-number sequence for Lotto 6/39 and a six-number sequence for Lotto 6/42 just one time in his/her life.* In this case, the probability of getting all six numbers right, both times, is equal to

$$\frac{1}{C(39,6)} \times \frac{1}{C(42,6)} = 1.71 \times 10^{13}.$$

But this result is far from miraculous when you begin with an *extremely large number* of people who have been playing the lottery for *a long period of time*, each of whom submit more than one entry for each weekly draw. (Refer to *American Statistician* (1992): 197-202.) For example, if every week 50 million people randomly submit five six-number sequences to one of the (many) Lottos 6/42, then the probability of one of them winning the jackpot twice in the coming four years is approximately equal to 63%. The calculation of this probability is based on the Poisson distribution, and goes as follows. The probability of your winning the jackpot in any given week by submitting five six-number sequences is

$$\frac{5}{C(42,6)} = 9.531 \times 10^{-7}.$$

The number of times that a given player will win a jackpot in the next 200 drawings of a Lotto 6/42, then, is Poisson distributed with expected value

$$\lambda_0 = 200 \times \frac{5}{C(42,6)} = 1.983 \times 10^{-4}.$$

For the next 200 drawings, this means that

$P(\text{any given player wins the jackpot two or more times}) = 1-\exp(-\lambda_0)-\lambda_0\exp(-\lambda_0) = 1.965\times10^{-8}$.

Subsequently, we can conclude that the number of people under the 50 million mark, who win the jackpot two or more times in the coming four years, is Poisson distributed with expected value

$$\lambda = 50,000,000 \times (1.965 \times 10^{-8}) = 0.9825.$$

The probability in question, that at some point in the coming four years at least one of the 50 million players will win the jackpot two or more times, can be given as $1 - \exp(-\lambda) = 0.626$. A few simplifying assumptions are used to make this calculation, such as the players choose their six-number sequences randomly. This does not influence the conclusion that it may be expected once in a while, within a relatively short period of time, that someone will win the jackpot two times.

R0524 6013

森元俊成（應數所・1年級）

No.
Date ① A

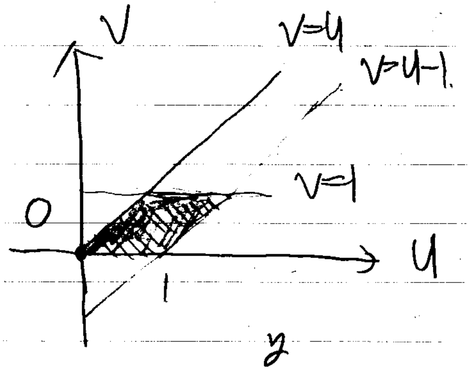
$\boxed{1}$

可以寫成

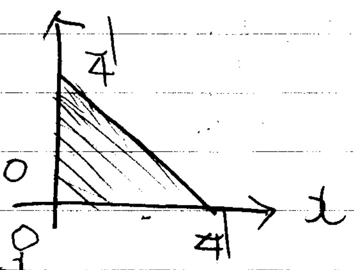
(a) $x=v$, $y=u-v$ と書けるので

$$0 \leqq \underset{x}{\underbrace{v}} \leqq 1, \quad 0 \leqq \underset{y}{\underbrace{u-v}} \leqq 1$$

$$\Leftrightarrow \underbrace{0 \leqq v \leqq 1, \quad u-1 \leqq v \leqq u}$$



(b) $A \cap D$ ：



$$\int_{x=0}^{x=\frac{1}{4}} \int_{y=0}^{y=\frac{1}{4}-x} dy\,dx$$

$$= \int_0^{\frac{1}{4}} \left(\frac{1}{4}-x\right) dx = \left[\frac{x}{4}-\frac{x^2}{2}\right]_0^{\frac{1}{4}} = \frac{1}{16}-\frac{1}{32} = \frac{1}{32}$$

or

又は，Lebesgue 測度空間 $(\mathbb{R}^2, \mathcal{L}_2, \mu)$ μ：Lebesgue 測度

㊟ $\chi_E = \begin{cases} 1 & (x \in E) \\ 0 & (x \notin E) \end{cases}$

$(A \cap D)^c$ … open set（開集合） ∴ $A \cap D \in \mathcal{L}_2$

$$\iint_{A \cap D} dx\,dy = \int_{\mathbb{R}^2} \underbrace{\chi_{A \cap D}(x)}\, d\mu = \mu(A \cap D) = A \cap D\text{の面積} = \frac{1}{32}$$

No.

Date ① B

2 因為 $x-y \geq 0, y \geq 0$

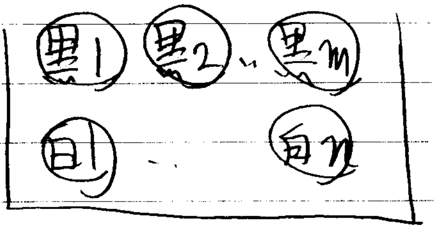
$$\therefore \int_{y=0}^{y=x} \lambda \cdot \exp(-\lambda y) \cdot \lambda \exp(-\lambda(x-y))\, dy$$

$$\int_0^x \lambda^2 \cdot \exp(-\lambda x)\, dy = \lambda^2 \exp(-\lambda x) \int_0^x dy$$

$$= \lambda^2 \exp(-\lambda x) \cdot x \quad (x \geq 0)$$

$$\therefore f_z(x) = \begin{cases} \lambda^2 \cdot x \exp(-\lambda x) & (x \geq 0) \\ 0 & (\text{else}) \end{cases}$$

3



把red 改為 white

(a) 非復元抽出で k個の球を取り出すとき

我們區分所有的球。從 m+n 個球抽出 k 個球時，

有 ${}_{m+n}C_k \left(= \binom{m+n}{k}\right)$ 個組合.

其中包含 r 個白球的情況有 ${}_nC_r \cdot {}_mC_{k-r}$ 個組合

由於區分所有的球，每個組合出現的機率為相同

$$\therefore \frac{{}_nC_r \cdot {}_mC_{k-r}}{{}_{m+n}C_k} \quad (\text{超幾何分布})$$

(b) 復元抽出でk回取り出す時、令 $f(k)$ 爲抽k次時出現奇數次白球的機率。 $f(k+2)$ 和 $f(k)$ 的關係遞回式…

$$f(k+2) = f(k)\cdot\left\{\left(\frac{n}{m+n}\right)^2+\left(\frac{m}{m+n}\right)^2\right\} + (1-f(k))\cdot\frac{2mn}{(m+n)^2}$$

$$= f(k)\cdot\left\{\left(\frac{m-n}{m+n}\right)^2\right\} + \frac{2mn}{(m+n)^2}$$ 把此式改爲 →

$$\left(f(k+2)-\alpha\right) = \left(\frac{m-n}{m+n}\right)^2\left(f(k)-\alpha\right)$$ 的形式。

$$\alpha\left(1-\left(\frac{m-n}{m+n}\right)^2\right) = \frac{2mn}{(m+n)^2}$$

$$\therefore \alpha = \frac{\frac{2mn}{(m+n)^2}}{1-\left(\frac{m-n}{m+n}\right)^2} = \frac{2mn}{(m+n)^2-(m-n)^2} = \frac{2mn}{4mn} = \frac{1}{2}$$

令 $g(k)=f(k)-\frac{1}{2}$

$$\therefore \underbrace{\left(f(k+2)-\frac{1}{2}\right)}_{g(k+2)} = \underbrace{\left(\frac{m-n}{m+n}\right)^2\left(f(k)-\frac{1}{2}\right)}_{\left(\frac{m-n}{m+n}\right)^2 g(k)}$$

$$g(k+2) = \left(\frac{m-n}{m+n}\right)^2 g(k)$$ 以下考慮 k=奇數的場合 和 偶數的場合

ここで、偶数と奇数の場合で分ける。

①

$$g(0) = f(0)-\frac{1}{2} = \frac{-1}{2}$$

$$g(2k) = \left(\frac{m-n}{m+n}\right)^{2k}\cdot\left(\frac{-1}{2}\right)$$

$$\therefore f(2k) = \frac{1}{2}-\frac{1}{2}\left(\frac{m-n}{m+n}\right)^{2k}$$

No.

Date ② B

接着考虑奇数的情况

③ kが奇数の場合……

$k=1$… $f(1)=\frac{n}{m+n}$ であるから $g(1)=f(1)-\frac{1}{2}=\frac{n}{m+n}-\frac{1}{2}$

$$=\frac{2n-(m+n)}{2(m+n)}=\frac{n-m}{2(m+n)}$$

$$\therefore g(2k-1)=\left(\frac{m-n}{m+n}\right)^{2(k-1)}\cdot\frac{n-m}{2(m+n)}=\frac{-(m-n)^{2k-1}}{2(m+n)^{2k-1}}$$

$$=-\frac{1}{2}\cdot\left(\frac{m-n}{m+n}\right)^{2k-1}=f(2k-1)-\frac{1}{2}$$

$$\therefore f(2k-1)=\frac{1}{2}-\frac{1}{2}\cdot\left(\frac{m-n}{m+n}\right)^{2k-1}$$

より 偶数のとき、$f(2k)=\frac{1}{2}-\frac{1}{2}\cdot\left(\frac{m-n}{m+n}\right)^{2k}$

奇数のとき $f(2k-1)=\frac{1}{2}-\frac{1}{2}\cdot\left(\frac{m-n}{m+n}\right)^{2k-1}$

以上の結論は $f(k)=\frac{1}{2}-\frac{1}{2}\cdot\left(\frac{m-n}{m+n}\right)^{k}$ と書ける

無論偶數還是奇數都是 $f(k)=\frac{1}{2}-\frac{1}{2}\left(\frac{m-n}{m+n}\right)^{k}$

No.

Date ③ A

但

令 $I_j = \begin{cases} 1 & (若第j次出現 正面) \\ 0 & (〃 反面) \end{cases}$ $I_j \sim b(\frac{1}{2})$

④ $\begin{cases} V \cdots I_2 + I_3 + \cdots I_{10} & （2回目\sim10回目, トスでの表） \\ X = (-1)^{I_1+1} \end{cases}$

$I_1 \sim I_{10} \sim b(\frac{1}{2})$ （独立同態）

① $\mathrm{cov}[U, V] = E[UV] - E[U]E[V]$

$= E[XV^2] - E[XV]E[V]$

XとVは独立であるから （因為X與V獨立）

$= E[X]E[V^2] - E(X)E[V]^2 = E[X] \cdot \mathrm{Var}[V] = 0$

$\because E[X] = 0$ である。 $Pr(X=1) = \frac{1}{2}$ $Pr(X=-1) = \frac{1}{2}$

$E(X) = \frac{1}{2} + \frac{1}{2}(-1) = 0$

② UとVが独立でないことの証明 （在此證明U與V並非獨立）

$\exists u, v$ $Pr[U=u]\,Pr[V=v] \neq Pr[U=u, V=v]$ を言えば良い。

$Pr[U=9]\,Pr[V=0] = Pr[X=1, V=9] \cdot Pr[V=0]$

$= Pr[X=1] \cdot Pr[V=9]\,Pr[V=0] = \frac{1}{2} \cdot \left(\frac{1}{2}\right)^9 \cdot \left(\frac{1}{2}\right)^9 = \left(\frac{1}{2}\right)^{19}$

但 $Pr[U=9, V=0] = Pr[X=1, \underbrace{V=9, V=0}_{不可能同時滿足 V=0 \cap V=9.}] = 0.$

③13

No.

Date

令 $I_j \begin{cases} 1 & (\text{若第}j\text{次出現正面}) \\ 0 & (\quad〃\quad\text{反面}) \end{cases}$ $\quad I_j \sim b(\frac{1}{2})$

$\downarrow$

$\boxed{5}$ $X = I_1 + I_2 + \dots I_5$

$Y = I_1 + I_2 + \dots I_{10}$

まず $Y|X=x$ の分布を求める (求 $Y|X=x$ 的分布)

$$\Pr[Y=y \mid X=x] = \frac{\Pr(X=x, Y=y)}{\Pr(X=x)} = \frac{\Pr(X=x, Y-X=y-x)}{\Pr(X=x)}$$

$$= \frac{\Pr(X=x)\cdot\Pr(Y-X=y-x)}{\Pr(X=x)}$$ ($Y-X$ と X は独立) ($Y-X$ vs X 獨立)

$$= \Pr(Y-X = y-x) = \Pr(I_6+I_7+\dots I_{10} = y-x)$$

であるので $Y|X$ に関して $Y-x \mid X=x \sim Bin(5, \frac{1}{2})$

$E[Y-x \mid X=x] = \frac{5}{2}$ $\quad\therefore \Pr[Y \mid X=x] = x + \frac{5}{2}$

$Var[Y-x \mid X=x] = 5 \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{5}{4}$

$\parallel$

$Var[Y \mid X=x)$ である $\quad\therefore$ 示せた

(完成證明)

最尤推定量, 問題

No.
Date ⊕ 1A

$\boxed{6}$ $X_i, Y_i \overset{\text{iid.}}{\sim} N(\alpha_i, \sigma^2)$

(a) $(X_1, Y_1, X_2, Y_2, \cdots X_n, Y_n) = (x_1, y_1, \cdots x_n, y_n)$ であるとき

確率密度関数を求める $f(x_1, y_1) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-1}{2\sigma^2}(x_1-\alpha_1)^2\right)$
↳(求 $X_1 \sim X_n, Y_1 \sim Y_n$ 的 聯合機率密度函數)

$\cdot \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-1}{2\sigma^2}(y_1-\alpha_1)^2\right) = \left(\frac{1}{2\pi\sigma^2}\right) \exp\left\{\frac{-1}{2\sigma^2}\left((x_1-\alpha_1)^2+(y_1-\alpha_1)^2\right)\right\}$

$\therefore f(x_1, x_2 \cdots x_n, y_1, \cdots y_n) = \left(\frac{1}{2\pi\sigma^2}\right)^n \cdot \exp\left(\sum_{i=1}^{n} \frac{-1}{2\sigma^2}\left((x_i-\alpha_i)^2+(y_i-\alpha_i)^2\right)\right)$

$\log f(x_1 \sim x_n, y_1 \sim y_n) = -n\log(2\pi\sigma^2) - \sum_{i=1}^{n} \frac{1}{2\sigma^2}\left((x_i-\alpha_i)^2+(y_i-\alpha_i)^2\right)$

$\frac{\partial \log f}{\partial \alpha_i^2} = \frac{1}{\sigma^2}(x_i+y_i-2\alpha_i) = 0 \cdots \hat{\alpha}_i = \frac{x_i+y_i}{2}$ である

$\frac{\partial \log f}{\partial \sigma^2} = \frac{-n}{\sigma^2} + \sum_{i=1}^{n} \frac{1}{2\sigma^4}\left((x_i-\alpha_i)^2+(y_i-\alpha_i)^2\right) = 0$

$\therefore \sigma^2 = \frac{1}{2n} \sum_{i=1}^{n}\left((x_i-\alpha_i)^2+(y_i-\alpha_i)^2\right)$

$\alpha_i = \frac{x_i+y_i}{2}$ を代入.

$\hat{\sigma}^2 = \frac{1}{2n} \sum_{i=1}^{n} \frac{(x_i-y_i)^2}{2} = \frac{1}{4n} \sum_{i=1}^{n} (x_i-y_i)^2$

No.

Date ④ B

(b) $\hat{\sigma}^2$は一致推定量では無い事を証明する

(證明 $\hat{\sigma}^2$ 並非 σ^2 的一致估計量)

$X_i, Y_i \sim N(\alpha_i, \sigma^2)$

$X_i - Y_i \sim N(0, 2\sigma^2)$

$\frac{1}{\sqrt{2}} \frac{X_i - Y_i}{\sigma} \sim N(0,1)$

∴ $\frac{(X_i - Y_i)^2}{2\sigma^2} \sim \chi^2_1$ (自由度1のχ^2分布) (自由度1的卡方分布)

$T = \sum_{i=1}^{n} \frac{(X_i - Y_i)^2}{2\sigma^2} \sim \chi^2_n$ (自由度nのχ^2分布) (自由度n的卡方分布)

$\Gamma(\frac{n}{2}, \frac{1}{2})$

$\frac{\sigma^2 T}{2n} = \hat{\sigma}^2_{MLE}$ である。 ここで $\hat{\sigma}^2_{MLE}$のモーメント母関数 $M(\theta)$を求める

(求 $\hat{\sigma}^2_{MLE}$ 的 moment generating function.) $M(\theta)$

$$M(\theta) = E\left[\exp\left(\frac{\theta \cdot \sigma^2 T}{2n}\right)\right] = E\left[\exp\left(T \cdot \left(\frac{\theta \sigma^2}{2n}\right)\right)\right]$$

である。 また $\Gamma(\alpha, \beta)$のモーメント母関数 (先求$\Gamma(\alpha,\beta)$的mgf)

$$\int_0^\infty \frac{x^{\alpha-1}}{\Gamma(\alpha)\beta^\alpha} \exp\left(\frac{1}{\beta}x\right) \cdot \exp(\theta x)\, dx$$

$$= \int_0^\infty \frac{x^{\alpha-1}}{\Gamma(\alpha)\beta^\alpha} \exp\left(-\left(\frac{1}{\beta}-\theta\right) x\right) dx \quad \frac{1}{\beta} - \theta = r \quad rx = t$$

$$\frac{dt}{dx} = r \quad \int_0^\infty \frac{\left(\frac{t}{r}\right)^{\alpha-1}}{\Gamma(\alpha)\beta^\alpha} \exp(-t) \frac{dt}{r}$$

$$= \int_0^\infty \frac{t^{\alpha-1}}{\Gamma(\alpha)(\beta r)^\alpha} \exp(-t)\, dt = \frac{1}{(\beta r)^\alpha} = \frac{1}{(1-\beta\theta)^\alpha}$$

(略 $\frac{n}{2}$ を^{2}代入 α, β)

ここで $\alpha = \frac{n}{2}$ $\beta = 2$ とすれば $\frac{1}{(1-2\theta)^{\frac{n}{2}}}$.

ここで $\theta \to \frac{\theta\sigma^2}{2n}$ に更新すると $\frac{1}{(1-\frac{\theta\sigma^2}{n})^{\frac{n}{2}}} = M(\theta)$ ($\hat{\sigma}^2_{MLE}$ の mgf)
(θ 改為 $\frac{\theta\sigma^2}{2n}$)

∴これは $\hat{\sigma}^2_{MLE}$ のモーメント母関数である。

$n \to \infty$ のとき $(1-\frac{\theta\sigma^2}{n})^{-\frac{n}{2}} = \left\{(1-\frac{\theta\sigma^2}{n})^{\frac{-n}{\theta\sigma^2}}\right\}^{\frac{\theta\sigma^2}{2}}$

$= e^{\frac{\theta\sigma^2}{2}}$ となるので、これは $\frac{\sigma^2}{2}$ に退化することを表わす (計算 $\lim_{n\to\infty} M(\theta) = \frac{\sigma^2}{2}$ 記表示 $\hat{\sigma}^2_{MLE} \xrightarrow{P} \frac{\sigma^2}{2}$ (退化))

よって $\hat{\sigma}^2_{MLE} \xrightarrow{P} \frac{\sigma^2}{2}$ となる。よって σ^2 の一致推定量では無い
(∴ $\hat{\sigma}^2_{MLE}$ 並非 σ^2 之一致估計量)

定理 大数の弱法則 $X_n (n=1,2,\ldots)$ $E[X_n] = \mu$, $V[X_n] = \sigma^2$
(弱大數法則) $< +\infty$ $< +\infty$

$\frac{X_1+\cdots+X_n}{n} \xrightarrow{P} \mu$ となること

確率収束 定義 $^\forall\varepsilon > 0$ に対して、$^\exists N \in \mathbb{N}$
(機率收斂)

$n \geq N \Rightarrow$ $Pr(|X_n - \mu| > \varepsilon) = 0 \Rightarrow X_n \xrightarrow{P} \mu$ と言う

No.

Date ⑤ B

$\boxed{7}$ $I_j = \begin{cases} 0 \cdots & 若 Y_j 由儀器1來測量 \\ 1 & 若 Y_j 由儀器2來測量 \end{cases}$

($I_j = 0 \to a_j = 1$ $\quad I_j = 1 \to a_j = 2$

$\therefore a_j = I_j + 1$)

給定 $I_j = \delta_j$ (0 or 1)時. Y_j 服從 $N(\theta, \sigma_1^{2(1-I_j)} \cdot \sigma_2^{2I_j})$

$(I_j \sim b(\frac{1}{2}))$

$$\therefore f(Y_j = y_j \mid I_j = \delta_j) = \frac{1}{\sqrt{2\pi \sigma_1^{2(1-\delta_j)} \cdot \sigma_2^{2\delta_j}}} \exp\left(\frac{-(y_j-\theta)^2}{2\sigma_1^{2(1-\delta_j)} \sigma_2^{2\delta_j}}\right)$$

$$\therefore f(Y_j = y_j, I_j = \delta_j) = \frac{\sqrt{\frac{1}{2}}}{\sqrt{\pi \sigma_1^{2(1-\delta_j)} \sigma_2^{2\delta_j}}} \exp\left(\frac{-(y_j-\theta)^2}{2\sigma_1^{2(1-\delta_j)} \sigma_2^{2\delta_j}}\right)$$

接著考慮 $(Y_1, I_1, Y_2, I_2 \cdots Y_n, I_n)$ 之聯合

機率密度函數 =

$$f(Y_1 = y_1, I_1 = \delta_1, \cdots Y_n = y_n, I_n = \delta_n)$$

$$= f(Y_1 = y_1, I_1 = \delta_1) \cdot f(Y_2 = y_2, I_2 = \delta_2) \cdot \cdots$$

$$\cdot f(Y_n = y_n, I_n = \delta_n)$$

$$\left(\frac{2}{\pi}\right)^{\frac{n}{2}} \cdot \frac{1}{(\sigma_1^2)^{\frac{n-(\delta_1+\cdots+\delta_n)}{2}} (\sigma_2^2)^{\left(\frac{\delta_1+\cdots+\delta_n}{2}\right)}} \cdot \exp\left(\sum_{i=1}^{n} \frac{-(y_i-\theta)^2}{2\sigma_1^{2(1-\delta_i)}\sigma_2^{2\delta_i}}\right)$$

$$\log f = \frac{n}{2}\log\left(\frac{2}{\pi}\right) - \frac{\delta_1+\cdots+\delta_n}{2}\log(\sigma_2^2) - \frac{n-(\delta_1+\cdots+\delta_n)}{2}\log(\sigma_1^2)$$

$$+ \sum_{i=1}^{n} \frac{-(y_i-\theta)^2}{2\sigma_1^{2(1-\delta_i)}\sigma_2^{2(\delta_i)}},$$

$$\frac{\partial \log f}{\partial\theta} = \sum_{i=1}^{n} \frac{y_i-\theta}{\sigma_1^{2(1-\delta_i)}\sigma_2^{2\delta_i}} = 0$$

$$\sum_{i=1}^{n} \frac{y_i}{\sigma_1^{2(1-\delta_i)}\sigma_2^{\delta_i}} = \left(\sum_{i=1}^{n} \frac{1}{\sigma_1^{2(1-\delta_i)}\sigma_2^{\delta_i}}\right)\cdot\theta$$

$$\therefore \hat{\theta}_{MLE} = \left(\sum_{i=1}^{n} \frac{1}{\sigma_1^{2(1-\delta_i)}\sigma_2^{2\delta_i}}\right)^{-1} \cdot \left(\sum_{i=1}^{n} \frac{y_i}{\sigma_1^{2(1-\delta_i)}\sigma_2^{2\delta_i}}\right) \perp$$

$$\because \delta_i = a_i - 1, \quad 1-a_i = -\delta_i$$

$$\hat{\theta}_{MLE} = \left(\sum_{i=1}^{n} \frac{1}{\sigma_1^{2(2-a_i)}\sigma_2^{2(a_i-1)}}\right)^{-1}\left(\sum_{i=1}^{n} \frac{y_i}{\sigma_1^{2(2-a_i)}\sigma_2^{2(a_i-1)}}\right) \perp$$

關於 $\sigma_1^{2(a_i-1)} \cdot \sigma_2^{2(2-a_i)}$ 的部分,

$a_i = 1 \cdots \sigma_1^2$
$a_i = 2 \cdots \sigma_2^2$

$$\therefore \sigma_1^{2(a_i-1)}\sigma_2^{2(2-a_i)} = \sigma_{a_i}^2$$

$$\therefore \hat{\theta}_{MLE} = \left(\sum_{i=1}^{n} \frac{1}{\sigma_{a_i}^2}\right)^{-1}\left(\sum_{i=1}^{n} \frac{y_i}{\sigma_{a_i}^2}\right)$$ である

∴ [illegible]

No.

Date ⑥,B

$f(c)=\sum|x_i-c| = \sum|x_{(i)}-c|$

↑

$\boxed{8}$ $x_1 \sim x_n$ 依大小排下来∴ $x_{(1)}<x_{(2)}<\cdots<x_{(n)}$

(a) $f(c)=\sum_{i=1}^{n}|x_i-c|$ に関して まず $x_1,x_2,\ldots,x_n$

を $x_{(1)}<x_{(2)}<\cdots<x_{(n)}$ と順番に並べ変えることを考える。

$(n=2m-1)$

$f(c)=\sum_{i=1}^{n}|x_{(i)}-c|$ とすると

$$\frac{d}{dc}|x-c| = \begin{cases} 1 & (c>x) \\ -1 & (c<x) \end{cases} \quad (\text{else } 0)$$

定

ここで $\mathrm{sgn}(c) = \begin{cases} 1 & c>0 \\ -1 & c<0 \end{cases} \quad (\text{else } 0)$

$\frac{d}{dc}f(c)=\sum_{i=1}^{n}\mathrm{sgn}(c-x_{(i)})$ である ∴ $\hat{x}=\mathrm{med}(x_1 \ldots x_n)=x_{(m)}$.

いま $c=x_{(m)}$ であれば
$$\begin{cases} \mathrm{sgn}(c-x_{(n)}) = \mathrm{sgn}(c-x_{(n-1)}) = \cdots \mathrm{sgn}(c-x_{(m+1)}) = -1 \\ \mathrm{sgn}(c-x_{(m)}) = 0 \\ \mathrm{sgn}(c-x_{(m-1)}) = \mathrm{sgn}(c-x_{(m-2)}) = \cdots \mathrm{sgn}(c-x_{(1)}) = 1 \end{cases}$$

より $\left.\frac{df(c)}{dc}\right|_{x_{(m)}} = 0$.

$c<x_{(m)}$ …… $\mathrm{sgn}(c-x_{(i)})$ $(1 \leq i \leq m)$ のうち -1 になるものが 1 になるものより 大きくなるので $\frac{df(c)}{dc}<0$

$c>x_{(m)}$ … $\frac{df(c)}{dc}>0$ ⇒

以上の事から
(因此:)

c		$X_{(m)}$	
$f'(c)$	$-$	0	$+$
$f(c)$	↘	min	↗

∴ より $c = X_{(m)} = \hat{x}$ の時に最小となる

(b) これは Laplace分布と呼ばれる　$X_1 \sim X_n$ の順序統計量を $X_{(1)} \leqq \cdots \leqq X_{(n)}$ とする

$$f(x) = \frac{1}{2^n} \exp\left(-\sum_{i=1}^{n} |x_i - \theta|\right) \quad (n = 2m-1) \quad (X_{(1)} \leqq \cdots \leqq X_{(n)})$$

は(a)より　$\theta = X_{(m)}$ のときに $\sum_{i=1}^{n} |x_i - \theta|$ … 最小

$\theta = X_{(m)}$ のときに $\sum_{i=1}^{n} -|x_i - \theta|$ … 最大

$\theta = X_{(m)}$ のときに 同時確率密度関数 $f(x)$ も最大

⑦注.

ロジスティック関数 $F(x) = \dfrac{1}{1+e^{-x}}$ の形. $\frac{\partial}{\partial x}$ と $\frac{\partial}{\partial \beta}$ で微分するべきかも

(a)

$\boxed{9}$ 求Hessian矩陣. 為了簡單, 先考慮 $\underline{n=1}$. β 改為 $\beta = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_p \end{pmatrix}$

$$L_0(\beta) = -\log(1+\exp \overset{t}{x}\beta) + Y\overset{t}{x}\beta \text{ と得.}$$

$$= -\log\left(1+\exp(x_1 \cdots x_p)\begin{pmatrix} \beta_1 \\ \vdots \\ \beta_p \end{pmatrix}\right) + Y(x_1 \cdots x_p)\begin{pmatrix} \beta_1 \\ \vdots \\ \beta_p \end{pmatrix}$$

①

$$\frac{\partial L_0}{\partial \beta_j} = \frac{-x_j \exp \overset{t}{x}\beta}{1+\exp \overset{t}{x}\beta} + Yx_j \text{ である.}$$

②

$$\frac{\partial^2 L_0(\beta)}{\partial \beta_k \partial \beta_j} \quad \frac{-x_k x_j \exp \overset{t}{x}\beta}{1+\exp \overset{t}{x}\beta} + \frac{x_j \exp \overset{t}{x}\beta \cdot \exp \overset{t}{x}\beta \cdot x_k}{(1+\exp \overset{t}{x}\beta)^2}$$

$$\frac{x_k x_j \exp \overset{t}{x}\beta}{(1+\exp \overset{t}{x}\beta)}\left\{-1+\frac{\exp \overset{t}{x}\beta}{1+\exp \overset{t}{x}\beta}\right\}$$

$$\frac{-x_k x_j \exp \overset{t}{x}\beta}{(1+\exp \overset{t}{x}\beta)^2} \text{ である.}$$

③ $$\text{Hessian} = H_1 = x\overset{t}{x}\left(\underbrace{\frac{\exp \overset{t}{x}\beta}{(1+\exp \overset{t}{x}\beta)^2}}_{a>0}\right) \times (-1)$$

$(n=1 \quad x=x_1)$

証明 H_1 為 Semi-negative definite-matrix

故に H_1 が x によらず 半負定値であることを示せば良い

注 $y = \begin{pmatrix} y_1 \\ \vdots \\ y_p \end{pmatrix}$

$$\overset{t}{y}H_1 y = -(y_1\ y_2 \cdots y_p)\underbrace{\begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix}}_{x}\underbrace{(x_1 \cdots x_p)}_{\overset{t}{x}}\begin{pmatrix} y_1 \\ \vdots \\ y_p \end{pmatrix} \cdot a$$

[illegible]

No.

Date ⑧ A

$$= -\underbrace{(x_1 y_1 + \dots x_p y_p)^2}_{\geq 0} \cdot C_1 \leq 0 \quad (C_1 > 0)$$

$n \geq 2$ の時も同様 $y^t(H_1 + \dots H_n)y = \sum_{j=1}^{n} \underbrace{y^t H_j y}_{(\leq 0)} \leq 0$

(b) $X = (x_1, \dots x_n)$ の full rank

$$C_j = \frac{\exp x^t \beta}{(1+\exp x^t \beta)^2} > 0$$

$X_0 = (\sqrt{C_1} x_1 \dots \sqrt{C_n} x_n)$ .. full rank （因為矩陣の基本操作）

$$y^t \underbrace{(H_1 + H_2 + \dots H_n)}_{H \ (\text{Hessian})} y = -y^t X_0 X_0^t y = -(X_0^t y)^2 = 0$$

X_0^t .. full rank $\quad X_0^t y = 0 \Rightarrow y = \begin{pmatrix} y_1 \\ \vdots \\ y_p \end{pmatrix} = 0$

∴ Hessian .. negative definite

∴ 似然の凹函数
(strictly concave)

(c) 因為 Hessian 是 負定矩陣 $L_n(\beta)$ 有最大值.

Chuyu culture

No.

Date ③ p

$\boxed{10}$ $p(\overset{正面}{表}|\theta_1)=u$ $\quad Pr(\overset{正面}{表}|\theta_2)=w$

$$\begin{cases}(1-u)(1-w)=p_0=\frac{1}{3}\\(1-u)w+u(1-w)=p_1=\frac{1}{3}\\uw=p_2=\frac{1}{3}\end{cases}$$

考慮 u,v 是否有實數解，

まず p_0, p_2 に着目し $uw=\frac{1}{3}$, $1-(u+w)+uw=\frac{1}{3}$

∴ $u+w=1$ （由 $P_0=\frac{1}{3}$ $P_2=\frac{1}{3}$ 得 $uw=\frac{1}{3}$ $u+w=1$）

ここで u,w を解とする 2次方程式は $(t-u)(t-w)=0$

$t^2-t+\frac{1}{3}=0$. $(t-\frac{1}{2})^2-\frac{1}{4}+\frac{1}{3}=(t-\frac{1}{2})^2+\frac{1}{12}>0$

tは実数解を持たない。∴ 存在しない。

考慮2次方程式 $(t-u)(t-w)=t^2-(u+w)t+(uw)=0$

這沒有實數解. ∴ u,w 不可能滿足 $P_0=P_1=P_2=\frac{1}{3}$

F 具有「$A_n \in F \Rightarrow \bigcup_{n=1}^{m} A_n \in F$」的性質
「$A_n = \phi$ $(n \geqq m)$

$\boxed{1}$

(a) σ-加法性を持つとき、これは有限加法性を持つ事を示す

可測空間 (Ω, F) を考える $A_n \in F$ $(n=1,2..)$

ここで $A_n = \phi$ $(n \geqq m+1)$ とすると $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{m} A_n \in F$

であるから有限加法性についても言える

機率空間 (Ω, F, P)
具有①~④的性質時，
證明亦具⑤的性質

(b) 確率空間 (Ω, F, P) において

① $\forall A \in F$ $P(A) \geqq 0$
② $P(\Omega) = 1$
③ $A_1, A_2 \in F$ $A_1 \cap A_2 = \phi$ $P(A_1) + P(A_2) = P(A_1 \cup A_2)$
④ $A_n \in F$ $A_{n+1} \subset A_n$ $A_n \to \phi \Rightarrow \lim_n P(A_n) = 0$

これらの公理から ⑤ $P(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P(A_n)$ $A_n \in F$, $A_i \cap A_j = \phi$ $(i \neq j)$

を導く。 (i) まず $A_n \subseteq A_{n+1}$ $(A_n \in F)$ とする

$A = \bigcup_{n=1}^{\infty} A_n$ $A^c \subseteq A_{n+1}^c \subseteq A_n^c$ このとき $A_n^c - A^c$ は

④ ∴ $A_{n+1}^c - A^c \subseteq A_n^c - A^c$, $A_n^c - A^c \to \phi$ ∴ $1 - P(A) = P(A^c) + \overset{\to 0}{\lim_{n\to\infty} P(A_n^c - A^c)}$

$= \lim_{n\to\infty} P(A_n^c) = \lim_n (1 - P(A_n))$ 「∴ $\lim_{n\to\infty} P(A_n) = P(A) = P(\bigcup_{n=1}^{\infty} A_n)$」 ⊛

(ii) $\tilde{A}_n$: disjoint $A_n = \bigcup_{k=1}^{n} \tilde{A}_k$ ∴ $A = \bigcup_{n=1}^{\infty} \tilde{A}_n$ → 代入 ⊛

→ 示せる

No.

Date 9(B)

$\boxed{2}$ 這個題目的意思很不清楚。比解題，解讀題意更困難。①"future potential can be rated 1 to N"

→ 不知道 N個人的評分是否能重複。(還是都不一樣)
(若不重複的話，best 的解釋又不一樣)

②「Rule 2」→ 不知道 m指的是固定的數字，還是自己代入 m=2,3,4……。③"那麼第一個被審查的人是否被錄取該怎麼決定?" ④"previous m-1"指的是 {·1~m-1 所有人 or / ·只有 m-1 的人}

在網路上查到答案是 $\frac{1}{N-(i-1)}$，所以從答案猜測題意。i=1。$\frac{1}{N-(i-1)}=\frac{1}{N}$ 但從題意看不出來。

在怎樣的情況下，才會錄取第一個人。

但題意可能是每個人的評分(rate)不相同。

$\frac{1}{N}$ 的機率代表，第一個人的評分=N

(若第一個人的評分=N 的話，他就是最好的人，所以不用看後面)

但求的機率是 Pr(1st is best | 1st is hired)。

「第一個人=最好 ⟺ 錄取第一個人」

No.
Date ⑩ A

所以條件機率一定是 1，而不是 $\frac{1}{N}$.

($\sum_{i=1}^{N} \frac{1}{N-i+1} \neq 1$ 所以求的機率應該是條件機率.)

所以先不論評分是否重複.

接下來 $i=N$ 時 $\frac{1}{N-i+1}=1$

所以到N個人才決定錄取者的話，他一定是 the Best Candidate.

其實可能是「previous m-1 包含第1個～第m-1個」

接下來
$i=N-1$ 時 $\frac{1}{N-i+1}=\frac{1}{2}$

其實可能是、第N-1個人是到目前為止唯一超過前面所有的人. 所以第N-1個人被錄取.
但其實第N個人可能比第N-1個人好，所以機率是 $\frac{1}{2}$.

由此可見 $\frac{1}{N-i+1}$ 的意思是、ith～Nth 的 N-i+1 個人裡面第i個是最好的機率.

題意…… • 每個人的評分可以重複、
• 第一個人是否被錄取、→ 這個題目不應該考慮這點.
若看到第N個人時，仍不出現比第一個更好的人
那麼應該錄取第一個人，但是在這種情況下
$Pr($第一個$=$Best $|$ 第一個被錄取$) = 1 \neq \frac{1}{N}$
$(1 \geq 2)$

No.
Date ⑩ B.

13 (a) 在下面

(b) 9月は1日～30日まで存在。(9月有30天)

すべての並べ方は366!通りである。そのうち1～30日まで

9月1日から選ばれない $\dfrac{336\times335\times\cdots307\times336!}{366!}$

第一天～第三十天
不出現、相的排列法

所有的排列

$= \left(\dfrac{336}{366}\times\dfrac{335}{365}\times\cdots\dfrac{307}{337}\right)$

つまり $\prod_{j=1}^{30}\left(\dfrac{306+j}{336+j}\right)$ である

(a) 同じ月のものが重複しないという意味として解釈する

※ 10days $\xrightarrow{改成}$ 12days.

1月	2月	3月	4月	5月	6月	7月	8月	9月	10月	11月	12月

$$\frac{31\times29\times31\times30\times31\times30\times31\times31\times30\times31\times30\times31}{{}_{366}C_{12}}$$

從各月抽一天的組合

所有的組合

$\boxed{14}$ $f_{X_n}(x) = \frac{x^{n-1}}{\Gamma(n)} \exp(-x)$

(a) $X_n \sim \Gamma(n, 1)$ とする。このとき、X_n のモーメント母関数 $M_{X_n}(\theta)$ は $\nearrow X_n$ の mgf

$E[e^{\theta X_n}] = \frac{1}{(1-\theta)^n}$ である。次に、$\frac{X_n - n}{\sqrt{n}} = Y_n$ とし、

Y_n のモーメント母関数は $E\left[e^{\theta\left(\frac{X_n-n}{\sqrt{n}}\right)}\right] = E\left[e^{\frac{\theta X_n}{\sqrt{n}} - \sqrt{n}\theta}\right]$ (mgf)

$= \frac{1}{\left(1-\frac{\theta}{\sqrt{n}}\right)^n} \exp(-\sqrt{n}\theta) = M_{Y_n}(\theta)$ となる

$\log M_{Y_n}(\theta) = -\sqrt{n}\theta - n\log\left(1-\frac{\theta}{\sqrt{n}}\right) \overset{\text{taylor展開}}{=} -\sqrt{n}\theta + n\left(\frac{\theta}{\sqrt{n}} + \frac{1}{2}\frac{\theta^2}{n} + \frac{1}{3}\frac{\theta^3}{n\sqrt{n}} \cdots\right.$

$\left(\because \log(1+x) = -\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots\right) \quad (|x|<1)\right)$

$n \to \infty$ で $\left|\frac{\theta}{\sqrt{n}}\right| \ll 1$ $\log M_{Y_n}(\theta) \to \frac{\theta^2}{2}$ となる

$M_{Y_n}(\theta) \to \frac{\theta^2}{2}$ であるので $n \to \infty$ $Y_n \xrightarrow{d} N(0,1)$ (Levyの連続性定理)

よって n が十分大きいとき $F_{Y_n}(y) \approx \Phi(y)$ (cdf) (n 很大時)

$\therefore F'_{Y_n}(y) \approx \phi(y) \left(= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}y^2\right)\right)$ (注 ϕ N(0,1) pdf, Φ N(0,1) cdf)

ここで Y_n の確率密度関数 $F'_{Y_n}(y) = \frac{(\sqrt{n}y+n)^{n-1}}{\Gamma(n)} \cdot \exp(-\sqrt{n}y - n) \cdot \sqrt{n}$ (Y_n の pdf)

$F'_{Y_n}(0) = \frac{n^{n-\frac{1}{2}}}{\Gamma(n)} \exp(-n) = \frac{n^{n+\frac{1}{2}}}{n!} \exp(-n)$

$\phi(0) = \frac{1}{\sqrt{2\pi}}$ $\therefore \lim_{n\to\infty} \frac{n^{n+\frac{1}{2}}}{n!} \exp(-n) \to \frac{1}{\sqrt{2\pi}}$

$\therefore \lim_{n\to\infty} \frac{n!}{n^{n+\frac{1}{2}} \exp(-n)} \to \sqrt{2\pi}$

No.
Date ⑪ B

14 (b) 復元抽出して {2,4,9,12} から4回抽出する

全ての $4^4 = 256$ 通りの出現仕方

確率

- 4つの種類の目が出る… (出現四種數字) {2,4,9,12} sum 和27 / $\frac{24}{256}$
- 3つの種類の目が出る… (只出現三種數字)
  - {2,4,9}…
    - {2,2,4,9} 和17 $\frac{12}{256}$
    - {2,4,4,9} 和19 $\frac{12}{256}$
    - {2,4,9,9} 和24 $\frac{12}{256}$
  - {2,4,12}…
    - {2,2,4,12} 和20 $\frac{12}{256}$
    - {2,4,4,12} 和22 $\frac{12}{256}$
    - {2,4,12,12} 和30 $\frac{12}{256}$
  - {2,9,12}
    - {2,2,9,12} 和25 $\frac{12}{256}$
    - {2,9,9,12} 和32 $\frac{12}{256}$
    - {2,9,12,12} 和35 $\frac{12}{256}$
  - {4,9,12}
    - {4,4,9,12} 和29 $\frac{12}{256}$
    - {4,9,9,12} 和34 $\frac{12}{256}$
    - {4,9,12,12} 和37 $\frac{12}{256}$
- 2つの種類の目が出る (只出現二種數字)
  - {2,4}…
    - {2,2,2,4} 和10 $\frac{4}{256}$
    - {2,2,4,4} 和12 $\frac{6}{256}$
    - {2,4,4,4} 和14 $\frac{4}{256}$

No.

Date 四 A

		Sum	機率
{2,9}	{2,2,2,9}	和 15	4/256
	{2,2,9,9}	和 22	6/256
	{2,9,9,9}	和 29	4/256
{2,12}	{2,2,2,12}	和 18	4/256
	{2,2,12,12}	和 28	6/256
	{2,12,12,12}	和 38	4/256
{4,9}	{4,4,4,9}	和 21	4/256
	{4,4,9,9}	和 26	6/256
	{4,9,9,9}	和 31	4/256
{4,12}	{4,4,4,12}	和 24	4/256
	{4,4,12,12}	和 32	6/256
	{4,12,12,12}	和 40	4/256
{9,12}	{9,9,9,12}	和 39	4/256
	{9,9,12,12}	和 42	6/256
	{9,12,12,12}	和 45	4/256
-179目. (只出現一種 點數)	{2,2,2,2}	和 8	1/256
	{4,4,4,4}	和 16	1/256
	{9,9,9,9}	和 36	1/256
	{12,12,12,12}	和 48	1/256

∴ 以上可 和=27 的機率最高

(Sum=27 的機率最高 : Avg = $\frac{27}{4}$)

No.

Date ⑫ D　$x_i \sim x_j$　$x_i\, x_i \ldots$

全出現一次

問 (c) $\{x_1 \sim x_n\}$ のうち 1つずつ1回ずつ出現する確率は

$= \frac{n!}{n^n}$ である。一方、重複して出現するものがあると(き)
(若重複出現時)

$x_1 \ldots k_1$個　$x_2 \ldots k_2$個 … $x_n \ldots k_n$個　$(k_1+k_2+\ldots+k_n=n)$

$$\left(\frac{1}{n}\right)^{k_1}\cdot\left(\frac{1}{n}\right)^{k_2}\cdots\left(\frac{1}{n}\right)^{k_n}\cdot\frac{(k_1+\cdots+k_n)!}{k_1!\,k_2!\cdots k_n!}$$

$$= \frac{1}{n^n}\frac{n!}{k_1!\cdots k_n!} < \frac{n!}{n^n} \quad (\because \exists k_j \geq 2)$$

しかし $\frac{k_1x_1+\cdots+k_nx_n}{n} = \frac{l_1x_1+\cdots+l_nx_n}{n}$ $(k_1, k_2 \ldots k_n) \neq (l_1, l_2 \ldots l_n)$

の場合もあるので、この証明は完全ではない。

(由於可能 $\exists (k_1,\ldots,k_n) \neq (l_1 \ldots l_n)$ $\frac{k_1x_1+\cdots+k_nx_n}{n} = \frac{l_1x_1+\cdots+l_nx_n}{n}$.
這個證明法不完整)

(d) $n! \approx \sqrt{2\pi}\cdot n^{n+\frac{1}{2}}\cdot \exp(-n)$

$\frac{n!}{n^n} \approx \sqrt{2\pi}\cdot n^{\frac{1}{2}}\cdot \exp(-n)$

(e) x_i が出現しない確率… (不出現)　$X_i \sim Bn\left(\frac{1}{n}, n\right)$　$Pr(X_i=0) = \left(\frac{n-1}{n}\right)^n = \left(1-\frac{1}{n}\right)^n$

$\lim_{t\to 0}(1+t)^{\frac{1}{t}} = e$　$\therefore \lim_{t\to -0}(1+t)^{\frac{1}{t}} = e$　$\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{-n} = e$

$\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n = e^{-1}$

(題目並未提及 X_0 之定義
在此假設 $X_0=0$),

$\boxed{15}$ 問題文中では X_0 についての言及がない. $X_1 = 0.5X_0 + Z_1$

よって $X_0=0$ という仮定を立てる. つまり $X_1 = Z_1$

$X_2 = 0.5X_1 + Z_2 = 0.5Z_1 + Z_2$

$X_3 = 0.5X_2 + Z_3 = 0.25Z_1 + 0.5Z_2 + Z_3$.

以後同様に $X_n = Z_n + \left(\frac{1}{2}\right)Z_{n-1} + \left(\frac{1}{2}\right)^2 \cdot Z_{n-2} + \cdots \left(\frac{1}{2}\right)^{n-1} Z_1$

となる. (注: $Z_1 \sim Z_n \cdots$ iid)

$E[X_n] = E\left[Z_n + \left(\frac{1}{2}\right)Z_{n-1} + \cdots \left(\frac{1}{2}\right)^{n-1} \cdot Z_1\right] = 0$

$E[X_t] = 0$, $E[X_{t+h}] = 0$ である

$$\gamma_X(t+h,t) = E\left[\left(Z_{t+h} + \left(\frac{1}{2}\right)Z_{t+h-1} + \cdots \left(\frac{1}{2}\right)^h \cdot Z_t + \cdots \left(\frac{1}{2}\right)^{t+h-1} Z_1\right) \cdot \left(Z_t + \left(\frac{1}{2}\right)^1 Z_{t-1} + \cdots \left(\frac{1}{2}\right)^{t-1} \cdot Z_1\right)\right]$$

また $E[Z_i Z_j] = 0 \ (i \neq j)$ に注意
$\quad\quad\quad\quad = 1 \ (i = j)$

$$\gamma_X(t+h,t) = E\left[\left(\frac{1}{2}\right)^h \cdot Z_t^2 + \left(\frac{1}{2}\right)^{h+2} Z_{t-1}^2 + \cdots + \left(\frac{1}{2}\right)^{2t+h-2} Z_1^2\right]$$

$$= \left(\frac{1}{2}\right)^h + \left(\frac{1}{2}\right)^{h+2} + \cdots \left(\frac{1}{2}\right)^{2t+h-2} = \frac{\left(\frac{1}{2}\right)^h \cdot \left(1 - \left(\frac{1}{2}\right)^t\right)}{1 - \frac{1}{4}} = \frac{4}{3}\left(\frac{1}{2}\right)^h\left(1 - \left(\frac{1}{2}\right)^t\right)$$

である.

No.
Date 考慮這個方程式

(a) idea1…(這 idea1是跟Answer不同的方法. idea2在下一頁)
↓

$\boxed{16}$ → $x_1+x_2+\dots x_7=12$ の方程式の解について考える

($x_1\geq0\dots x_7\geq0$, $x_1\sim x_7\dots$整数)

の「① ② ③ … ⑫ $\underset{1}{|}\ \underset{2}{|}\ \underset{3}{|}\ \underset{4}{|}\ \underset{5}{|}\ \underset{6}{|}$ 」の並べ方
之排列法

$= \frac{18!}{6!\cdot 12!}$ (組合) (${}_7H_n$)

次に $x_1\geq1$, $x_2\geq1$ $x_3\geq1\dots x_7\geq1$ の条件の

場合は $x_1-1=x_1'$ $x_2-1=x_2'\dots x_7-1=x_7'$ とり

$x_1'+x_2'+\dots x_7'=5$

($x_1'\geq0$, $x_2'\geq0\dots x_7'\geq0$)

これは「① ② ③ ④ ⑤ $\underset{1}{|}\ \underset{2}{|}\ \underset{3}{|}\ \underset{4}{|}\ \underset{5}{|}\ \underset{6}{|}$ 」

の並べ方… $\frac{11!}{5!\,6!}$

よって $\frac{\left(\frac{11!}{5!\,\cancel{6!}}\right)}{\left(\frac{18!}{\cancel{6!}\,12!}\right)} = \frac{11!\,12!}{18!\,5!} = 0.0249 \neq 0.2285$

(これはAnswerと異なる)

idea2在下一頁

⑭ A

(a) idea2… 多項分布を考えるとき、（假設、服從多項分布時…）

$\mathrm{Multi}\left(\frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}; 12\right)$

$\Pr(X_1 \geq 1 \overset{\cap}{,} X_2 \geq 1 \overset{\cap}{,} X_3 \geq 1 \overset{\cap}{,} X_4 \geq 1 \overset{\cap}{,} X_5 \geq 1 \overset{\cap}{,} X_6 \geq 1 \overset{\cap}{,} X_7 \geq 1)$ を計算するため

$\boxed{\text{先計算}}$
$\rightarrow \Pr(X_1=0 \cup X_2=0 \cup X_3=0 \cup X_4=0 \cup X_5=0 \cup X_6=0 \cup X_7=0)$

$= \left(\Pr(X_1=0) + \Pr(X_2=0) + \dots \Pr(X_7=0)\right)^{①\,7\text{個}} +$

$(-1)^1 \left(\Pr(X_1=0, X_2=0) + \dots \quad \Pr(X_6=0, X_7=0)\right)^{②\,{}_7C_2=21\text{個}}$

$+ \left(\Pr(X_1=0, X_2=0, X_3=0) \dots \quad \Pr(X_5=0, X_6=0, X_7=0)\right)^{③\,{}_7C_3=35\text{個}},$

$(-1)^3 \left(\Pr(X_1=0, X_2=0, X_3=0, X_4=0) \dots\dots \Pr(X_4=0, X_5=0, X_6=0, X_7=0)\right)^{④\,{}_7C_4=35\text{個}}$

$+ \left(\Pr(X_1=0, X_2=0\, X_3=0\ \ X_4=0\ X_5=0) + \dots + \Pr(X_3 \sim X_7=0)\right)^{⑤\,{}_7C_5=21\text{個}}$

$(-1)^5 \left(\Pr(X_1 \sim X_6=0) + \dots \quad \Pr(X_2 \sim X_7=0)\right)^{⑥\,7\text{個}},$

$+ \Pr(X_1 \sim X_7=0)^{⑦\,1\text{個}}$

① $\Pr(X_1=0) = \left(\frac{1}{7}\right)^0 \cdot \left(\frac{6}{7}\right)^{12} \cdot$ ①の計 $\left(\frac{6}{7}\right)^{12} \times 7$

② $\Pr(X_1=0, X_2=0) = \left(\frac{5}{7}\right)^{12}$ ②の計 $\left(\frac{5}{7}\right)^{12} \times 21$

③ $\Pr(X_1 \sim X_3=0) = \left(\frac{4}{7}\right)^{12}$ ③の計 $\left(\frac{4}{7}\right)^{12} \times 35$

④ $\Pr(X_1 \sim X_4=0) = \left(\frac{3}{7}\right)^{12}$ ④の計 $\left(\frac{3}{7}\right)^{12} \times 35$

⑤ $Pr(X_1 \cap X_5 = 0) = \left(\frac{2}{7}\right)^{12}$ ⑤の計 $\left(\frac{2}{7}\right)^{12} \times 21$

⑥ $Pr(X_1 \cap X_6 = 0) = \left(\frac{1}{7}\right)^{12}$ ⑥の計 $\left(\frac{1}{7}\right)^{12} \times 7$

⑦ $Pr(X_1 \cap X_7 = 0) = 0$

∴ $Pr(X_1 = 0 \cup X_2 = 0 \ldots \cup X_7 = 0)$

$= \left(\frac{6}{7}\right)^{12} \times 7 - \left(\frac{5}{7}\right)^{12} \times 21 + \left(\frac{4}{7}\right)^{12} \times 35 - \left(\frac{3}{7}\right)^{12} \times 35 + \left(\frac{2}{7}\right)^{12} \times 21 - \left(\frac{1}{7}\right)^{12} \times 7$

$= 0.77154$

∴ $1 - 0.77154 = 0.22846 \ldots$

(b) $X_1 \sim Bn\left(\frac{1}{7}, 12\right)$

(c) $Pr(X_1 = 0) = \left(\frac{6}{7}\right)^{12}$

$Pr(X_1 = 1) = \left(\frac{1}{7}\right)\left(\frac{6}{7}\right)^{11} \cdot {}_{12}C_1$

$Pr(X_1 = 2) = \left(\frac{1}{7}\right)^{2} \left(\frac{6}{7}\right)^{10} {}_{12}C_2$

⇒ 考慮 $\frac{Pr(X_1 = x)}{Pr(X_1 = x-1)}$ を比(例).

$$P_x = \frac{Pr(X_1 = x)}{Pr(X_1 = x-1)} = \frac{\left(\frac{1}{7}\right)^{x} \cdot \left(\frac{6}{7}\right)^{12-x} \cdot {}_{12}C_x}{\left(\frac{1}{7}\right)^{x-1} \cdot \left(\frac{6}{7}\right)^{13-x} \cdot {}_{12}C_{x-1}}$$ を考える

$(x \geq 2)$

$$= \frac{\left(\frac{1}{7}\right)}{\left(\frac{6}{7}\right)} \cdot \frac{\frac{12!}{x!\,(12-x)!}}{\frac{12!}{(x-1)!\,(13-x)!}} = \frac{13-x}{x} \cdot \frac{1}{6}$$

$x=1 \ldots \quad P_x = 2 \qquad \therefore \frac{Pr(X_1=1)}{Pr(X_1=0)} = 2$

$x=2 \ldots \quad P_x = \frac{11}{12} \qquad \frac{Pr(X_1=2)}{Pr(X_1=1)} = \frac{11}{12}$

$x=3 \quad P_x = \frac{10}{18} \qquad \frac{Pr(X_1=3)}{Pr(X_1=2)} = \frac{10}{18}$

$\vdots$

以上の事から $Pr(X_1=0) < \underset{(MAX)}{\underset{\parallel}{Pr(X_1=1)}} > Pr(X_1=2) > \ldots\ldots$
(由此可見)

よって常に「1」と予測すると良い。 (Pr(X₁=1) 的機率最大， 所以 predict X₁=1 就好了。)

$Pr(X_1=1) = \left(\frac{1}{7}\right)\cdot\left(\frac{6}{7}\right)^{11}\cdot {}_{12}C_1 = \frac{6^{11}\cdot 12}{7^{12}} \fallingdotseq 0.315$

$\therefore 100 \times 0.315 = 31.5$ dollars (期望值)

$\boxed{17}$ (a) ①は、F_X is stochastically larger than F_Y

①: $\forall t \quad F_X(t) \leqq F_Y(t)$ であるから

$\Leftrightarrow \quad Pr(X \leqq t) \leqq Pr(Y \leqq t)$

$\Leftrightarrow \quad 1 - Pr(X > t) \leqq 1 - Pr(Y > t)$

$\Leftrightarrow \quad Pr(Y > t) \leqq Pr(X > t)$ となる。

② 次に $\exists t_0, \quad F_X(t_0) < F_Y(t_0)$

$\Leftrightarrow \quad Pr(X \leq t_0) \leqq Pr(Y \leqq t_0)$

$\Leftrightarrow \quad 1 - Pr(X > t_0) \leqq 1 - Pr(Y > t_0)$

$\Leftrightarrow \quad Pr(Y > t_0) \leqq Pr(X > t_0)$ である。

題目中並未提及 Example 1.5.4 為何. 所以參考 Casella-Berger Example 1.6.4

(b) 問題文中にExample 1.5.4に関して何の説明も無かった為 Casella-Berger：Example 1.5.4として見なす

つまり $X \sim Ge(P_x)$ $Y \sim Ge(P_y)$ (X, Y：試行の回数)

幾何分布
$(X, Y =$ trial 的次數)

$$\begin{cases} Pr(X=t) = P_x(1-P_x)^{t-1} \\ Pr(Y=t) = P_y(1-P_y)^{t-1} \end{cases}$$

No.

Date (1b) A

$$\Pr(X \leq t) = F_X = \frac{p_x(1-(1-p_x)^t)}{1-(1-p_x)} = 1-(1-p_x)^t.$$

$$\Pr(Y \leq t) = F_Y = 1-(1-p_y)^t$$

$F_Y(t) - F_X(t) = (1-p_y)^t - (1-p_x)^t$ である

いま $p_x > p_y \Leftrightarrow 1-p_y > 1-p_x.$

$\therefore (1-p_y)^t > (1-p_x)^t \quad (t=1,2,3\dots)$

$\therefore F_Y(t) - F_X(t) > 0 \quad (t=1,2,3\dots)$

よし F_X is stochastically larger than F_Y

從{1〜53}選擇6個數字

$\boxed{18}$ (a) 1〜53の中から 6つの数字をえらぶ… ${}_{53}C_6 = \frac{53!}{6!\,47!} = 22957480$

(b) $\frac{1}{22957480^2} = \frac{1}{52,7045,8879,5040}$

(c)(d) 題意不太清楚，所以假設：

① Florida人口:1600,0000 裡面，$\frac{1}{10}$(160万人)的人

每周購買 Florida 樂透

② 每周開獎一次 (一年50次)

(1) 求 160,0000人裡面有人在人生中中獎2次(以上)的機率

(假設購買60年)

某人中獎次數 $X \sim Bn\left(3000, \overset{\to p}{\frac{1}{22957480}}\right)$

$Pr(X \geqq 2) = 1 - P(X=1 \cup X=0)$

$= 1 - p(1-p)^{2999} \cdot 3000 - (1-p)^{3000} = 8.5 \cdot 10^{-7}$

$Pr(X \leqq 1) = 1 - 8.5 \cdot 10^{-7}$;

$Pr(X_1 \leqq 1, \sim X_{100,000} \leqq 1) = \left((1 - 8.5 \cdot 10^{-7})^{-\left(\frac{10^7}{8.5}\right)}\right)^{-1.36} \fallingdotseq e^{-1.36} \fallingdotseq 0.257$

$= Pr(X_1 \geqq 2 \cup \cdots X_{100,000} \geqq 2) = 1 - 0.257 = 0.743$ (很高!)

$10^{17} \div 2^{5}$

No.

Date ⑰ A

(2) 求5年內出現中獎2次以上的人的機率

某人中獎次數 $X \sim Bin\left(250, \frac{1}{22957480}\right)$

$Pr(X \geq 2) = 1 - Pr(X=1 \cup X=0) =$

$= 1 - p(1-p)^{249} \cdot 250 - (1-p)^{250}$

$\fallingdotseq 5.83 \cdot 10^{-9}$

$Pr(X_1 \leq 1, X_2 \leq 1, \ldots X_{160000} \leq 1) = (1 - 5.83 \cdot 10^{-9})^{160,000}$

$Pr = \left\{ (1 - 5.83 \cdot 10^{-9})^{-\frac{10^9}{5.83}} \right\}^{-\frac{1}{1071}} \fallingdotseq e^{-\frac{1}{1071}} \fallingdotseq 0.9977$

$\therefore Pr(X_1 \geq 2 \ldots X_{160000} \geq 2) \approx \underline{0.0023}$ //

Advanced Statistical Inference I

Homework 2: Transformations and Expectations
Due Date: October 6th

1. Let Ω be a sample space and let $A_1, A_2, \cdots$ be events. Define $B_n = \cup_{i=n}^{\infty} A_i$ and $C_n = \cup_{i=n}^{\infty} A_i$.
   (a) Show that $B_1 \supset B_2 \cdots$ and that $C_1 \subset C_2 \subset \cdots$.
   (b) Show that $w \in \cap_{i=1}^{\infty} B_n$ if and only if w belongs to an infinite number of the events $A_1, A_2, \cdots$.

2. Let $X \sim Uniform(0, 1)$. Let $0 < a < b < 1$. Let $Y = 1$ when $0 < x < b$. Otherwise $Y = 0$ Let $Z = 1$ when $a < x < 1$. Otherwise, $Z = 0$.
   (a) Are Y and Z independent? Why/Why not?
   (b) Find $E(Y|Z)$. Hint: What values z can Z take?

3. Let X have mean 0. We say that X is sub-Gaussian if there exists $\sigma > 0$ such that $\log(E[\exp(tX)]) \leq t^2\sigma^2/2$ for all t.
   (a) Show that X is sub-Gaussian if and only if $-X$ is sub-Gaussian.
   (b) Let X have mean μ. Suppose that $X - \mu$ is sub-Gaussian. Show that $P(|X - \mu| \geq t) \leq 2\exp(-t^2/(2\sigma^2))$.
   Remark: When people say "X is sub-Gaussian they often mean that "$X - \mu$ is sub-Gaussian.
   (c) Suppose that X is sub-Gaussian. Show that, for any $p > 0$,
   $$E[|X|^p] \leq p2^{p/2}\sigma^p\Gamma(p/2).$$

4. Let$X_1, \cdots, X_n$ be iid, with mean μ, $Var(X_i) = \sigma^2$ and $|X_i| \leq c$. Bernsteins inequality says that
   $$P(|\bar{X}_n - \mu| > t) \leq 2\exp\left(-\frac{nt^2}{2\sigma^2 + 2ct/3}\right).$$
   Suppose that $\sigma^2 = O(1/n)$. Use Bernsteins inequality to show that $\bar{X}_n - \mu = O_P(1/n)$.

5. An urn contains b black balls and r red balls. One of the balls was drawn at random, and putted back in the urn with a additional balls of the same color. Now suppose that the second ball drawn at random is red. What is the probability that the first ball drawn was black?

6. Let X_1 and X_2 be iid Uniform(0, 3). Find the density of $Y = X_1/X_2$.

7. Let $X_1, \cdots, X_n \sim Uniform(a, b)$ where $a < b$. Let $Y_n = \max\{X_1, \cdots, X_n\}$. Find the density of Y_n.

8. Consider the random variable $X \sim U[-1, 1]$. Derive the CDF and (for continuous case) the density function for the following random variables.

   (a) $Y = \begin{cases} 0 & \text{if } X \in [-1/2, 1/2] \\ X & \text{otherwise} \end{cases}$.

(b) $Z = F(Y)$, where $F(Y)$ is the CDF of Y as defined in (a).

(c) Find $E(Y)$, $E(Z)$, $Var(Y)$, and $Var(Z)$.

9. Let Y be a random variable following the exponential distribution, i.e.,$f_Y(y) = \exp(-y)I(y \geq 0)$. Conditional on $Y = y$, X is a random variable following a normal distribution with mean y and variance y.

   (a) Compute $E[X]$ and $Var(X)$.

   (b) Find the distribution of $(X - Y)^2$.

10. Suppose that X has a continuous distribution with p.d.f. $f_X(x) = 2x$ on the interval $(0, 1)$, and $f_X(x) = 0$ elsewhere. Suppose that Y is a continuous random variable such that the conditional distribution of Y given $X = x$ is uniform on the interval $(0, x)$. Find the mean and variance of Y in two different approaches.

    (a) Determine the unconditional (marginal) distribution of Y. Use it to compute $E[Y]$ and $Var(Y)$.

    (b) Use the the relationships $E(Y) = E[E(Y|X)]$ and $Var(Y) = E[Var(Y|X)]+Var(E(Y|X))$.

11. (Median) (a) Suppose continuous random variable X has the exponential distribution $X \sim Exp(\lambda)$ with pdf $f(x) = \lambda \exp(-\lambda x) 1_{(0,\infty)})(x)$ for $\lambda > 0$. What is the median for X and find an expression for $Pr(X > s + t|X > s)$.
    (b) When the median of a random variable X (or its distribution) is any value m such that $P(X \geq m) \geq 1/2$ and $P(X \leq m) \geq 1/2$, show that the set of medians is a closed interval $[m_0, m_1]$.

12. (Data summary) For any set of numbers $x_1, \ldots, x_n$ and a monotone function $h(\cdot)$, show that the value of a that minimizes $\sum_{i=1}^n [h(x_i) - h(a)]^2$ is given by $a = h^{-1}(\sum_{i=1}^n h(x_i)/n)$.
    Find functions h that will yield the arithmetic, geometric, and harmonic means as minimizes. Recall that the geometric mean of non-negative numbers is $(\prod_{i=1}^n x_i)^{1/n}$ and the harmonic mean is $[n^{-1} \sum_{i=1}^n (1/x_i)]^{-1}$.

高等統計推論 作業(2)

$\boxed{1}$ (a) $B_n = B_{n+1} \cup A_n$ 顯然 $B_{n+1} \subset B_n$.

(可能題目錯誤： $C_n = \bigcup_{j=n}^{\infty} A_j \to \bigcap_{j=n}^{\infty} A_j$)

$C_{n+1} \cap A_n = C_n$. 顯然 $C_{n+1} \supset C_n$.

(b) $w \in \bigcap_{n=1}^{\infty} B_n \Rightarrow \forall n \in N,\ w \in B_n$

$w \in \bigcup_{j=n}^{\infty} A_j$. 若非無限個 $A_1, A_2 \dots$ 包含 w, 則有

最大的自然數 N_0 $w \in A_{N_0}$, $w \notin A_n$ $(n = N_0+1, N_0+2 \dots)$

但 $\forall n \in N$, $w \in \bigcup_{j=n}^{\infty} A_j$ | $n = N_0+1$.

$w \in \bigcup_{j=N_0+1}^{\infty} A_j$ $A_{N_0+1}, A_{N_0+2}, \dots$ 裡至少有一個集合包含 w.

$\Rightarrow$ 矛盾. 存在無限個 A_n $(n \in N)$ 包含 w. $\perp$

接下來, 證明 inverse. $(\Leftarrow)$ $\forall N$ (自然數) ($\because A_1, A_2 \dots A_n \dots$ 中存在無限個集合包含 w)

$n \geq N$, $\{A_N, A_{N+1}, A_{N+2}, \dots\}$ 裡至少有個 $A_n \ni w$,

$\therefore \forall n$ $\bigcup_{j=n}^{\infty} A_j \ni w$ $\therefore$ 所有的 B_n 皆包含 w

$\therefore \bigcap_{n=1}^{\infty} B_n \ni w$. $\therefore$ 證明完成.

$\boxed{2}$ $X \sim U(0,1)$

(1) $Pr(Y=0, Z=0) = Pr(b \le X < 1, 0 < X \le a) = 0$

$(\because a<b)$

$Pr(Y=0) = Pr(b \le X < 1) = 1-b > 0$

$Pr(Z=0) = Pr(0 < X \le a) = a > 0$

$Pr(Y=0)\,Pr(Z=0) = a(1-b) \neq Pr(Y=0, Z=0)$

$\therefore$ 並非獨立

(2) $E[Y|Z] = \sum_{y=0,1} y \cdot Pr(Y=y|Z) = Pr(Y=1|Z)$

① $Z=0$ 時… $Pr(Y=1|Z=0) = \frac{Pr(Y=1, Z=0)}{Pr(Z=0)}$

$Pr(Y=1, Z=0) = Pr(0<X<b, 0<X\le a) = Pr(0<X\le a)$
$= a$

$\therefore E[Y|Z=0] = 1$

② $Z=1$ 時… $Pr(Y=1|Z=1) = \frac{Pr(Y=1, Z=1)}{Pr(Z=1)}$

$Pr(Z=1) = 1-a$. $Pr(Y=1, Z=1) = Pr(0<X<b, a<a<1)$

$= Pr(a<X<b) = b-a$

$E[Y|Z=1] = \frac{b-a}{1-a}$

(1) (2) より $E[Y|Z=z] = \left(\frac{b-a}{1-a}\right)^z \quad (z=0,1)$

$\boxed{3}$

(1) ① X: sub-Gaussian $\Rightarrow \log E[e^{tX}] \le \frac{\sigma^2 t^2}{2}$

$\therefore \log E[e^{(-t)(-X)}] \le \frac{\sigma^2(-t)^2}{2}$ $\qquad M(-X)(\theta)$

$-t \to \theta \qquad \log E[e^{\theta(-X)}] \le \frac{\sigma^2\theta^2}{2}$

$\therefore$ $(-X)$ is also sub-Gaussian.

② $-X$: sub-Gaussian $\Rightarrow \log E[e^{t(-X)}] \le \frac{\sigma^2 t^2}{2}$

$t \to -\theta$ とすると $\log[e^{(-\theta)(-X)}] \le \frac{\sigma^2(-\theta)^2}{2}$

$\therefore \log E[e^{\theta X}] \le \frac{\sigma^2\theta^2}{2}$

$\therefore$ X is also sub-Gaussian.

(2) $\{w \mid |X(w)| \ge t\} = \{w \mid X(w) \ge t\} \cup \{w \mid X(w) \le -t\}$

$$P(|X-\mu| \ge t) = \underbrace{P(\{w \mid X \ge t\})}_{①} + \underbrace{P(\{w \mid -X \ge t\})}_{②}$$

$① = \int_{\{w \mid X \ge t\}} dP = e^{-\theta t} \cdot \int_{\{w \mid X \ge t\}} e^{\theta t} dP$

$\le e^{-\theta t}\int_{\{w \mid X \ge t\}} e^{\theta X} dP = e^{-\theta t}\int_{\Omega} e^{\theta X} \cdot \chi_{\{w \mid X \ge t\}} dP$

Cauchy-Shwartzの不等式を用いて

$$\leqq e^{-\theta t}\cdot\left(\int_\Omega e^{2\theta X}dP\cdot\int_\Omega \chi\{w|X\geqq t\}^2 dP\right)^{\frac{1}{2}}$$

$$= e^{-\theta t}\cdot M_X(2\theta)^{\frac{1}{2}}\cdot P(\{w|X\geqq t\})^{\frac{1}{2}}$$

$$\therefore P(\{w|X\geqq t\})^{\frac{1}{2}}\leqq e^{-\theta t}\cdot M_X(2\theta)^{\frac{1}{2}}$$

$$\therefore P(\{w|X\geqq t\})\leqq e^{-2\theta t}\cdot M_X(2\theta)\leqq \exp(-2\theta t+2\sigma^2\theta^2)$$

$\therefore\ \theta=\frac{t}{2\sigma^2}$ を代入.. $\exp\left(-\frac{t^2}{\sigma^2}+\frac{t^2}{2\sigma^2}\right)=\exp\left(\frac{-t^2}{2\sigma^2}\right)$

同様に、$P(\{w|-X\geqq t\})\leqq\exp\left(\frac{-t^2}{2\sigma^2}\right)$

$\therefore P(\{w|\,|X|\geqq t\})$ (mean of $X=0$) $\leqq 2\exp\left(\frac{-t^2}{2\sigma^2}\right)$

(3) 一般に $X\geqq 0$ のとき、$E[X]=\int_0^\infty xf(x)dx=\int_0^\infty\int_0^x dy\,f(x)dx$

$$=\int_{y=0}^{y=\infty}\int_{x=y}^{x=\infty}f(x)\,dx\,dy=\int_{y=0}^{y=\infty}(1-F(y))dy=\int_{t=0}^{t=\infty}Pr(X\geqq t)dt$$

このことから、$E[|X|^p]=\int_{t=0}^{t=\infty}Pr(\{w|\,|X(w)|^p\geqq t\})dt=\int_{t=0}^{t=\infty}P(\{w|\,|X|\geqq t^{\frac{1}{p}}\})dt$

$=\int_{t=0}^{t=\infty}2\exp\left(\frac{-1}{2\sigma^2}t^{\frac{2}{p}}\right)dt\qquad \frac{1}{2\sigma^2}t^{\frac{2}{p}}=u\qquad t=(2\sigma^2u)^{\frac{p}{2}}$

$$\frac{dt}{du}=\left(\frac{p}{2}\right)\cdot(2\sigma^2u)^{\frac{p}{2}-1}\cdot 2\sigma^2=(2\sigma^2)^{\frac{p}{2}}\cdot\frac{p}{2}\cdot u^{\frac{p}{2}-1}$$

$$\therefore\int_{u=0}^{u=\infty}2\exp(-u)\cdot(2\sigma^2)^{\frac{p}{2}}\cdot\frac{p}{2}u^{\frac{p}{2}-1}du=\int_{u=0}^{u=\infty}p(2\sigma^2)^{\frac{p}{2}}\cdot u^{\frac{p}{2}-1}\exp(-u)\,du$$

$= p\cdot(2\sigma^2)^{\frac{p}{2}}\Gamma\left(\frac{p}{2}\right)$ $\quad\therefore$ 示された

$\boxed{4}$ Bernstein's inequality

$$\Pr\left[|\bar{X}-\mu|>t\right] \leq 2\exp\left(\frac{-nt^2}{2\sigma^2+\frac{2ct}{3}}\right)$$

$$1-2\exp\left(\frac{-nt^2}{2\sigma^2+\frac{2ct}{3}}\right) \leq \Pr\left[|\bar{X}-\mu|\leq t\right]$$

$\sigma^2=\frac{a}{n}$, $t=n^p$ を代入.

$$1-2\exp\left(\frac{-n^{2p+1}}{2\frac{a}{n}+\frac{2cn^p}{3}}\right)$$

- $p=-0.9$ 時…… $1-2\exp\left(\frac{-n^{0.2}}{\frac{2a}{n}+\frac{2c}{3n^{0.9}}}\right)$

$$1-2\exp\left(\frac{-n^{1.2}}{2a+\frac{2}{3}cn^{0.1}}\right)\xrightarrow{n\to\infty}1$$

- $p=-0.99$ 時…… $1-2\exp\left(\frac{-n^{0.02}}{\frac{2a}{n}+\frac{2c}{3}\cdot n^{-0.99}}\right)$

$$1-2\exp\left(\frac{-n^{1.02}}{2a+\frac{2}{3}cn^{0.01}}\right)\xrightarrow{n\to\infty}1$$

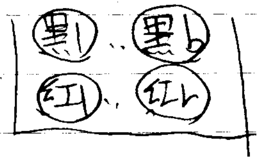
- $p=-0.999$ 時… $1-2\exp\left(\frac{n^{0.002}}{2a+\frac{2}{3}cn^{0.001}}\right)\xrightarrow{n\to\infty}1$

$$\left(\Pr\left[|\bar{X}-\mu|\leq n^{-0.999}\right]\xrightarrow{n\to\infty}1 \quad\cdots\quad \bar{X}-\mu=o_p\left(\frac{1}{n^{0.999}}\right)\right)$$

以上より $\inf\left\{-p \,\middle|\, 1-2\exp\left(\frac{-n^{2p+1}}{\frac{2\sigma}{n}+\frac{2cn^{p}}{3}}\right) \xrightarrow{n\to\infty} 1\right\} = -1.$

$\therefore\ \bar{X}-\mu = O_p\left(\frac{1}{n}\right),$

$\boxed{5}$



$$\Pr(\text{第1個}=\text{黑球} \mid \text{第2個}=\text{紅球})$$

$$= \frac{\Pr(\text{第1個}=\text{黑球}, \text{第2個}=\text{紅球})}{\Pr(\text{第2個}=\text{紅球})}$$

$$= \frac{\Pr(\text{1st}=\text{黑}, \text{2nd}=\text{紅})}{\Pr(\text{1st}=\text{黑}, \text{2nd}=\text{紅})+\Pr(\text{1st}=\text{紅}, \text{2nd}=\text{紅})}$$

① $\Pr(\text{1st}=\text{黑}, \text{2nd}=\text{紅}) = \underbrace{\left(\frac{b}{b+r}\right)}_{\Pr(\text{1st}=\text{黑})} \times \underbrace{\left(\frac{r}{b+r+1}\right)}_{\Pr(\text{2nd}=\text{紅} \mid \text{1st}=\text{黑})} = \frac{br}{(b+r)(b+r+1)}$

② $\Pr(\text{1st}=\text{紅}, \text{2nd}=\text{紅}) = \underbrace{\left(\frac{r}{b+r}\right)}_{\Pr(\text{1st}=\text{紅})} \times \underbrace{\left(\frac{r+1}{b+r+1}\right)}_{\cdot\,\Pr(\text{2nd}=\text{紅} \mid \text{1st}=\text{紅})} = \frac{r(r+1)}{(b+r)(b+r+1)}$

$$\therefore \frac{①}{①+②} = \frac{\left\{\frac{br}{(b+r)(b+r+1)}\right\}}{\left\{\frac{br}{\cancel{(b+r)}(b+r+1)} + \frac{r(r+1)}{\cancel{(b+r)}(b+r+1)}\right\}} = \frac{br}{br+r(r+1)}$$

$$= \frac{br}{r(b+r+1)} = \frac{b}{b+r+1}.$$

$\boxed{6}$ $\begin{cases} y = \frac{X_1}{X_2} \\ z = X_2 \end{cases}$ $\quad X_1 = yz$, $X_2 = z$

$\frac{\partial x_1}{\partial y} = z$, $\frac{\partial x_1}{\partial z} = y$ $\quad \therefore J = z\ (\geq 0)$

$\frac{\partial x_1}{\partial y} = 0$ $\quad \frac{\partial x_2}{\partial z} = 1$

$dx_1 dx_2 = z\,dy\,dz$

$\underbrace{0 \leq yz \leq 3}_{X_1}$, $\underbrace{0 \leq z \leq 3}_{X_2}$ より $0 \leq z \leq 3$, $0 \leq z \leq \frac{3}{y}$

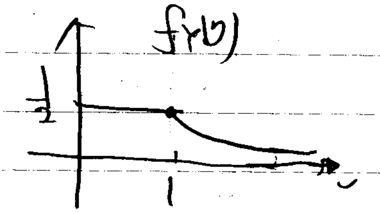
より $0 \leq z \leq \min\{\frac{3}{y}, 3\}$

X_1, X_2 の pdf $\quad 1 = \iint_{X_1, X_2} \frac{1}{9}\, dx_1 dx_2$

$= \iint_{Y,Z} \frac{1}{9} \cdot z\, dy\, dz \qquad f_Y(y) = \int_z \frac{1}{9} z\, dz$

$= \left[\frac{z^2}{18}\right]_0^{\min\{\frac{3}{y}, 3\}} = \frac{\left(\min\{\frac{3}{y}, 3\}\right)^2}{18}$

$\begin{cases} 0 < y \leq 1 \quad f_Y(y) = \frac{1}{2} \\ 1 < y < \infty \quad f_Y(y) = \frac{1}{2} \cdot \frac{1}{y^2} \end{cases}$



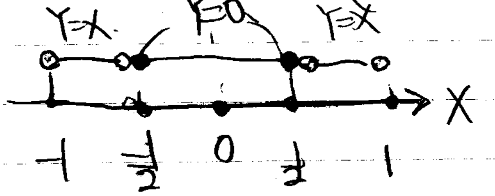
$\boxed{7}$ $Pr(Y_n \leq y) = Pr(\{X_1, X_2, .. X_n\} \leq y)$

$= Pr(X_1 \leq y \cap X_2 \leq y \cap .. X_n \leq y) \qquad (a<y<b)$

$= \left(\frac{y-a}{b-a}\right)^n = \frac{(y-a)^n}{(b-a)^n}$

$\frac{d Pr(Y_n \leq y)}{dy} = \frac{n(y-a)^{n-1}}{(b-a)^n} \cdots f_{Y_n}(y) \quad (Y_n$ の pdf$)$

$\boxed{8}$ (a) $Y = \begin{cases} 0 & x \in [-\frac{1}{2}, \frac{1}{2}] \\ X & x \notin [-\frac{1}{2}, \frac{1}{2}] \end{cases}$ $\qquad X \sim U(-1,1)$ $\quad \therefore f_X(x) = \frac{1}{2}$



$Pr(Y \leq y)$ について

$\quad \|$

① $-1 \leq y < -\frac{1}{2} \cdots$ $Pr(Y \leq y) = Pr(X \leq y, X \notin [-\frac{1}{2}, \frac{1}{2}])$

$\qquad + \underbrace{Pr(Y \leq y, X \in [-\frac{1}{2}, \frac{1}{2}])}_{0}$

$= Pr(X \leq y) = F_X(y) = \underline{\frac{y+1}{2}}$

② $-\frac{1}{2} \leq y < 0 \cdots$ $Pr(Y \leq y) = Pr(\underbrace{Y \leq y}_{Y=X}, X \notin [-\frac{1}{2}, \frac{1}{2}])$

$\qquad + Pr(\underbrace{Y \leq y}_{Y=0}, X \in [-\frac{1}{2}, \frac{1}{2}])$

$= Pr(X < -\frac{1}{2}) = \underline{\frac{1}{4}}$

③ $0 \leq y < \frac{1}{2}$ ∵ $Pr(Y \leq y) = Pr(Y \leq y, X \notin [-\frac{1}{2}, \frac{1}{2}]) +$
$Pr(Y \leq y, X \in [-\frac{1}{2}, \frac{1}{2}])$

$= Pr(X < -\frac{1}{2}) + Pr(X \in [-\frac{1}{2}, \frac{1}{2}]) = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$

④ $\frac{1}{2} \leq y \leq 1$ ∵ $Pr(Y \leq y) = Pr(Y \leq y, X \notin [-\frac{1}{2}, \frac{1}{2}]) +$
$Pr(Y \leq y\ X \in [-\frac{1}{2}, \frac{1}{2}])$

$= \frac{1}{4} + \frac{y - \frac{1}{2}}{2} + \frac{1}{2} = \frac{y+1}{2}$

∴ $F(y) = \begin{cases} \frac{y+1}{2} & (-1 \leq y < -\frac{1}{2},\ \frac{1}{2} \leq y \leq 1) \\ \frac{1}{4} & (-\frac{1}{2} \leq y < 0) \\ \frac{3}{4} & (0 \leq y < \frac{1}{2}) \end{cases}$

(b) $Y: -1 \to 1$ $Z: 0 \to 1$ $\int_{-1}^{1} \left(\frac{dF(y)}{dy}\right) dy = \int_{0}^{1} dz$

∴ $Z \sim U(0,1)$

(c) $E[Z] = \int_{0}^{1} z\,dz = \frac{1}{2}$ $E[Z^2] = \int_{0}^{1} z\,dz = \frac{1}{3}$

$V(Z) = E(Z^2) - E(Z)^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$

$E[Y] = E[E[Y|X]]$, $E[Y^2] = E[E[Y^2|X]]$

- $x \in [-\frac{1}{2}, \frac{1}{2}]$ ∵ $E[Y|X=x] = 0$ $E[Y^2|X=x] = 0$
- $x \notin [-\frac{1}{2}, \frac{1}{2}]$ ∵ $E[Y|X=x] = x$ $E[Y^2|X=x] = x^2$

∴ $E[E[Y|X=x]] = \int_{x \notin [-\frac{1}{2}, \frac{1}{2}]} x f_X(x)\,dx = 0 = E[Y]$

$E[E[Y^2|X=x]] = \int_{x \notin [-\frac{1}{2}, \frac{1}{2}]} x^2 f_X(x)\,dx = \left[\frac{x^3}{6}\right]_{\frac{1}{2}}^{1} + \left[\frac{x^3}{6}\right]_{-1}^{-\frac{1}{2}} = \frac{7}{24}$

∴ $V[Y] = \frac{7}{24}$ $E[Y] = 0$

$\boxed{9}$ Y.. 隨機變數　$Y \sim \exp(1)$

$X | Y=y \sim N(y, y)$

(a) $E[X] = E[\underbrace{E[X|Y]}_{\overset{\|}{y}}] = E[Y] = 1.$

$V[X] = E[X^2) - \underbrace{E[X]^2}_{1}$ である.

$E[X^2) = E[\underbrace{E[X^2|Y]}_{}] = E[Y^2+Y)$

$\left(\begin{array}{l} E[X^2|Y) - \underbrace{E(X|Y)^2}_{(y)^2} = V[X|Y) = y \\ \therefore E[X^2|Y) = Y^2+Y \end{array} \right.$

$E[X^2] = E[Y^2+Y) = 3$

$\therefore V[X) = \underbrace{E[X^2)}_{3} - \underbrace{E[X]^2}_{1} = 2.$

(b) $X-Y | Y=y \sim N(0, y)$

$\frac{X-Y}{\sqrt{Y}} | Y=y \sim N(0,1)$

$\therefore \frac{(X-Y)^2}{Y} | Y=y \sim \chi^2(1) = \Gamma(\frac{1}{2}, 2)$

$Z = \underbrace{(X-Y)^2}_{\overset{\|}{Z}} | Y=y \sim \Gamma(\frac{1}{2}, 2y)$

$$f_{Z|Y}(z|y) = \frac{z^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})(2y)^{\frac{1}{2}}} \exp\left(\frac{-z}{2y}\right)$$

$$f_{Z,Y}(z,y) = \frac{z^{-\frac{1}{2}}}{\sqrt{2\pi y}} \cdot \exp\left(\frac{-z}{2y}\right) \cdot \exp(-y)$$

$$= \frac{z^{-\frac{1}{2}}}{\sqrt{2\pi y}} \exp\left(\frac{-z}{2y} - y\right)$$

$$\therefore f_Z(z) = \int_{y=0}^{y=\infty} \frac{z^{-\frac{1}{2}}}{\sqrt{2\pi y}} \exp\left(\frac{-z}{2y} - y\right) dy$$

（無法再簡化）

No

Date

$\boxed{10}$ $Y|X=x \sim U(0,x)$ $X \sim U(0,1)$

(a) $f_{Y|X}(y|x) = \frac{1}{x}$ $(0<y<x)$

$\therefore f_{X,Y}(x,y) = f_{Y|X}(y|x) \cdot f_X(x) = \frac{1}{x}$

$\therefore \iint_{0<y<x<1} \frac{1}{x} dx dy = 1.$

$\int_{\cancel{x=y\to1}} \frac{1}{x} dx = [\log x]_y^1 = -\log y$

$\therefore f_Y(y) = -\log y$ $(0<y\leq 1)$

① $E[Y] = \int_0^1 y(-\log y)dy$

$= \left[-\frac{y^2}{2}\log y + \frac{y^2}{4}\right]_0^1 = \frac{1}{4}$

② $E[Y^2] = \int_0^1 y^2(-\log y)dy$

$= \left[\frac{y^3}{3}(-\log y) + \frac{y^3}{9}\right]_0^1 = \frac{1}{9}$

$\therefore V[Y] = \frac{1}{9} - \frac{1}{16} = \frac{7}{144}$

(b) $E[Y|X] = \frac{x}{2}$ $E[Y] = E[\frac{X}{2}] = \int_0^1 \frac{x}{2} dx = \left[\frac{x^2}{4}\right]_0^1 = \frac{1}{4}$

$E[Y^2|X] = \frac{x^2}{3}$ $E[Y^2] = E[\frac{X^2}{3}] = \int_0^1 \frac{x^2}{3} dx = \left[\frac{x^3}{9}\right]_0^1 = \frac{1}{9}$

$E[Y^2] - E[Y]^2 = \frac{1}{9} - \left(\frac{1}{4}\right)^2 = \frac{7}{144}$

⑮

$\boxed{\text{II}}$

(a) $X \sim e(\lambda)$ $\quad f_X(x) = \lambda\exp(-\lambda x)$

$\int_0^m \lambda\exp(-\lambda x)\,dx = \frac{1}{2}$ $\quad m \cdots$ median

$[\exp(-\lambda x)]_0^m = \frac{1}{2}$

$1 - \exp(-\lambda m) = \frac{1}{2}$

$\therefore \frac{1}{2} = \exp(-\lambda m)$

$\therefore -\ln 2 = -\lambda m$ $\quad \therefore m = \frac{1}{\lambda}\ln 2$

$$\frac{Pr(X>s+t, X>s)}{Pr(X>S)} = \frac{\int_{s+t}^{\infty} \lambda\exp(-\lambda x)\,dx}{\int_{S}^{\infty} \lambda\exp(-\lambda x)\,dx}$$

$$= \frac{[-\exp(-\lambda x)]_{s+t}^{\infty}}{[-\exp(-\lambda x)]_{s}^{\infty}} = \frac{\exp(-\lambda(s+t))}{\exp(-\lambda s)} = \exp(-\lambda t)$$

(與s無關)

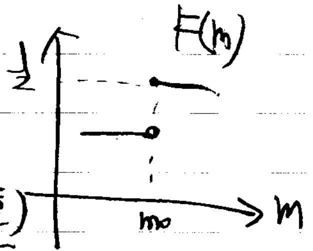
→

(b) $\{m \mid \underset{A}{P(X \geqq m) \geqq \frac{1}{2}},\ \underset{B}{P(X \leqq m) \geqq \frac{1}{2}}\} = [m_0, m_1]$ の証明

$= \{m \mid P(X \geqq m) \geqq \frac{1}{2}\} \cap \{m \mid \underbrace{P(X \leqq m)}_{F(x)} \geqq \frac{1}{2}\}$

「$\sup A \in A$, $\inf B \in B$ を証明する。」

① $\underset{B}{\{m \mid P(X \leqq m) \geqq \frac{1}{2}\}} = \{m \mid F_X(m) \geqq \frac{1}{2}\}$



$F_X(m)$…広増函数, $\lim_{m \to m_0+0} F_X(m) = F(m_0)$ (右連續)

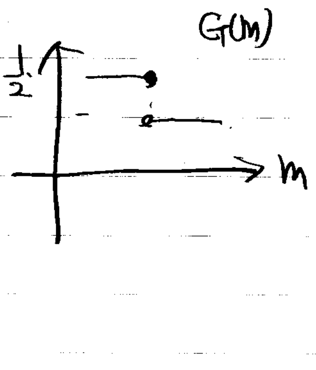
∴ $\inf\{m \mid F(m) \geqq \frac{1}{2}\} = \inf\{m \mid F(m) = \frac{1}{2}\} \ni \{m \mid F(m) \geqq \frac{1}{2}\}$

② $\underset{A}{\{m \mid \underbrace{P(X \geqq m)}_{G(m)} \geqq \frac{1}{2}\}}$ (P…測度, X…可測函数)

$\lim_{m \to m_1 - 0} G(m) = \lim_{n \to \infty} G(m_1 - \frac{1}{n}) = \lim_{n \to \infty} P(X^{-1}([m_1 - \frac{1}{n}, \infty)))$

$= P(\bigcap_{n=1}^{\infty} X^{-1}([m_1 - \frac{1}{n}, \infty))) = P(X^{-1}([m_1, \infty)) = G(m_1)$

∴ G(m)…左連續, 広減函数



∴ $\sup\{m \mid G(m) \geqq \frac{1}{2}\} = \sup\{m \mid G(m) = \frac{1}{2}\}$

$\in \{m \mid P(X \geqq m) \geqq \frac{1}{2}\}$

∴ ①, ②より 証明出来た

$\boxed{12}$ - h: monotone   or

$x<y \quad h(x) \leq h(y) \quad / \quad h(x) \geq h(y)$

$$Q(a) = \sum_{j=1}^{n} \{h(x_i) - h(a)\}^2$$

$$\frac{\partial Q}{\partial a} = -\sum_{j=1}^{n} 2(h(X_i) - h(a)) \cdot h'(a) = 0$$

$$\Rightarrow \sum_{j=1}^{n} (h(X_i) - h(a)) = 0$$

$$\sum_{j=1}^{n} h(X_i) = n \cdot h(a)$$

$$\therefore \frac{1}{n} \sum_{j=1}^{n} h(X_i) = h(a)$$

$$a = h^{-1}\left(\frac{1}{n} \sum_{j=1}^{n} h(X_i)\right)$$

(h: monotone function)

x		$h^{-1}\left(\frac{\sum_{j=1}^{n} h(X_i)}{n}\right)$	
Q'	$-$	0	$+$
Q	↘	min	↗

(㊟ if h: increasing $h'(a) \geqq 0$
h: decreasing $h'(a) \leqq 0$

①

arithmetic mean

$$\frac{X_1+\dots+X_n}{n} = h^{-1}\left(\sum_{i=1}^{n}\frac{h(X_i)}{n}\right)$$

$$h\left(\frac{X_1+\dots+X_n}{n}\right) = \sum_{i=1}^{n}\frac{h(X_i)}{n}$$ … Linear Function $h(x)=ax$ $(a\in R)$

② $(x_1x_2\cdots x_n)^{\frac{1}{n}} = h^{-1}\left(\sum_{i=1}^{n}\frac{h(X_i)}{n}\right)$

$$h\left((x_1\cdots x_n)^{\frac{1}{n}}\right) = \sum_{i=1}^{n}\frac{h(X_i)}{n}$$ $h(x)=a\log x$

③ $\dfrac{1}{n\left(\frac{1}{x_1}+\frac{1}{x_2}+\dots\frac{1}{x_n}\right)} = h^{-1}\left(\sum_{i=1}^{n}\frac{h(X_i)}{n}\right)$

$$h\left(\frac{n}{\left(\frac{1}{x_1}+\dots\frac{1}{x_n}\right)}\right) = \sum_{i=1}^{n}\frac{h(X_i)}{n}$$ Fn $h(x)=\frac{a}{x}$

Advanced Statistical Inference I
Homework 3: Transformations and Expectations
Due Date: October 24th

1. Let $X \sim N(0, 1)$ be the standard normal distribution. Show that
   (a) The moment generating function $M_X(t) = \exp(t^2/2)$ for all $t \in R$.
   (b) $M_{X^2}(t) = (1 - 2t)^{-1/2}$ for $-\infty < t < 1/2$.

2. The Weibull cumulative distribution function is

$$F(x) = 1 - \exp\left[-\left(\frac{x}{\alpha}\right)^{\beta}\right], \quad x \geq 0, \alpha > 0, \beta > 0.$$

   (a) Find the density function.
   (b) Show that if W follows a Weibull distribution, then $X = (W/\alpha)^{\beta}$ follows an exponential distribution.
   (c) How could Weibull random variables be generated from a uniform random number generator?

3. Let X have two-sided exponential distribution with density

$$f(x) = \frac{1}{2}\exp(-|x|), \quad \text{for } x \in R.$$

   (a) Find the moment generating function of X.
   (b) Use your answer in (a) to find $E(X)$, $E(X^n)$ and $Var(X)$.
   (c) For any $\mu \in R$, and $\sigma > 0$, what is the density of $Y = \sigma X + \mu$?

4. Let Y have the binomial(n, p) distribution and let X have the beta(α, β) distribution.
   (a) Show that $P(Y \leq y) = P(X \leq 1 - p)$ if $\alpha = n - y$ and $\beta = y + 1$.
   (b) Use the relationship in part (a) to find the median of a beta$(5, 3)$ random variable.

5. For any random variable X, be it continuous, discrete, or whatever, a uth-quantile of X is defined to be a number x such that the following holds.

$$P[X \leq x] \geq u \quad \text{and} \quad P[X \geq x] \geq 1 - u.$$

   The standard exponential distribution has density

$$f(x) = \begin{cases} \exp(-x), & \text{for } x \geq 0, \\ 0, & \text{for } x < 0. \end{cases}$$

   What are its quantiles?

6. (Refer to the previous question on the definition of quantile.) Suppose Z is a standard normal random variable, with density $\phi(z) = \exp(-z^2/2)/\sqrt{2\pi}$ for $-\infty < z < \infty$.
   (a) Show that $P[Z \geq z] = (1 + o(1))\phi(z)/z$ as z goes to ∞; here $o(1)$ denotes a quantity that tends to 0 as z goes to 1. [Hint: integration by parts.]

(b) Let q_α be the $(1-\alpha)$th-quantile of Z. Show that

$$q_\alpha = \sqrt{2\log(1/\alpha) - \log(\log(1/\alpha)) - \log(4\pi) + o(1)}$$

as $\alpha \to 0$; here $o(1)$ denotes a quantity that tends to 0 as $\alpha \to 0$.

7. Suppose Y is a standard Cauchy random variable.

   (a) What are the first and third quartiles of Y?

   (b) Show that $P[Y \geq y] \approx 1/(\pi y)$ as $y \to \infty$.

8. Adapted from *A Simple Population Estimate Based on Simulation for Capture-Recapture and Capture-Resight Data* by Minta and Mangel, Ecology 1989.

   > In the fall of 1984, North American badgers (Taxidea taxus) were snowtracked in a 15 km^2 area on the National Elk Refuge, Jackson, Wyoming. The size and shape of the target area were dictated by topographic and plant community features that created a relatively isolated area of high badger density. Fifteen of the badgers were radiotagged and known to be occupying or overlapping the area. During the 2-month tracking period there was no death or emigration of radiotagged badgers, and radiotagged badgers outside the target area did not immigrate. One badger emigrated near the end of the sampling period. During daylight and under suitable weather conditions, the target was searched for badger snowtracks. A total of 24 tracks could be followed to a terminal hole, where the badger would be inactive in an underground burrow. All telemetry frequencies were then scanned to determine whether the badger was *marked* or *unmarked*. Radiotelemetry revealed that 11 of the tracks were generated by marked badgers.

   (a) Let N be the (unknown) total badger population size. Explain why the hypergeometric distribution can be used to model this experiment. Identify the values of the parameters M and K.

   (b) For the values of the parameters given above, what is your best guess (estimate) of N.

   (c) For the value of N from part (b), draw the hypergeometric distribution. How likely is the observed value of x?

   (d) For values of N near that of part (b), evaluate the probability of the observed value of x. What might you conclude about the population size?

9. Let $X_1, \cdots, X_n$ be independent random variables, taking values from $[0,1]$ and $S_n = \sum_{i=1}^n X_i$. Show that, for any $t \geq E(S_n)$,

   $$P(S_n \geq t) \leq \left(\frac{E(S_n)}{t}\right)\left(\frac{n-E(S_n)}{n-t}\right)^{n-t}.$$

   Hint. Use Chernoffs bounding method.

10. Let $Y_1, Y_2, Y_3, \cdots$ be a sequence of i.i.d. random variables with mean $E(Y_i) = \mu$, and finite variance $Var(Y_i) = \sigma^2$. Define the sequence $\{X_n, n = 2, 3, \cdots\}$ as

    $$X_n = \frac{Y_1Y_2 + Y_2Y_3 + \cdots + Y_{n-1}Y_n + Y_nY_1}{n}, \quad \text{for } n = 2, 3, \cdots.$$

    Show that X_n converges to μ^2 in probability.

11. Let $Y_1, Y_2, Y_3, \cdots$ be a sequence of positive i.i.d. random variables with $0 < E[\ln Y_i] = \gamma < \infty$. Define the sequence $\{X_n, n = 1, 2, 3, \cdots\}$ as

$$X_n = (Y_1 Y_2 Y_3 \cdots Y_{n-1} Y_n)^{1/n}, \qquad \text{for} \quad n = 1, 2, 3, \cdots.$$

    Show that X_n converge to $\exp(\gamma)$ in probability.

12. Let X_n be uniform on the points $\{1/n, 2/n, \cdots, n/n = 1\}$. As n goes to the infinity, show that $Eh(X_n)$ converges to $Eh(X)$ where X is a uniform random variable on the interval $[0, 1]$. (Think of the convergence of a Riemann sum to a Riemann integral.)

13. Consider the following sequence of random variables:

$$X_n = \begin{cases} n & \text{with probability } 1/n \\ 0 & \text{with probability } 1 - 1/n. \end{cases}$$

    Find $E(X_n)$, $Var(X_n)$, and show that X_n converges to 0 in probability.

①

高等統計推論(Ⅰ) Homework3 森元俊成

1

(a) $M_X(t) = E[e^{tX}] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right) \cdot e^{tx} dx$

$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-1}{2}(x-t)^2 + \frac{1}{2}t^2\right) dx$

$= \exp\left(\frac{t^2}{2}\right) \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-1}{2}(x-t)^2\right) dx$

$x - t = y \quad \frac{dy}{dx} = 1$

$= \exp\left(\frac{t^2}{2}\right) \cdot \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-y^2}{2}\right) dy}_{\overset{\|}{\int_{-\infty}^{\infty} \phi(y)\, dy = 1}}$

$\therefore M_X(t) = \frac{t^2}{2} \quad (t \in \mathbb{R})$

(b) $M_{X^2}(t) = E[e^{tX^2}] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right) \cdot \exp(tx^2)\, dt$

$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\left(\frac{1}{2} - t\right)x^2\right) dx$

- case. $t = \frac{1}{2}$ … $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} dx = \frac{1}{\sqrt{2\pi}} \mu(\mathbb{R}) = +\infty$

  (㊟ μ… Lebesgue測度)

  ∴ 非可積分

- case $t > \frac{1}{2}$… $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(\underbrace{\left(t - \frac{1}{2}\right)}_{>0} x^2\right) dx$

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(\left(t-\frac{1}{2}\right)x^2\right)dx \geq \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}dx = \frac{1}{\sqrt{2\pi}}\mu(\mathbb{R}) = +\infty$$

∴ 非可積分

- case $t<\frac{1}{2}$ .. $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\underbrace{\left(\frac{1}{2}-t\right)}_{>0}x^2\right)dx$

$$\frac{1}{2}-t=\beta>0 \quad \sqrt{\beta}x=y \quad \frac{dy}{dx}=\sqrt{\beta}$$

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-y^2\right)\cdot\frac{1}{\sqrt{\beta}}dy$$

$$=\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\beta}} \exp\left(-y^2\right)dy$$

$$=\frac{\sqrt{\pi}}{\sqrt{2\pi\beta}}=\frac{1}{\sqrt{2\beta}}=\frac{1}{\sqrt{(1-2t)}}=(1-2t)^{-\frac{1}{2}} \quad \left(t<\frac{1}{2}\right)$$

$\boxed{2}$ (1) $\frac{dF}{dx}$ = 機率密度函數 $= \frac{d}{dx}\left(1-\exp\left(-\left(\frac{x}{\alpha}\right)^{\beta}\right)\right)$

$= -\exp\left(-\left(\frac{x}{\alpha}\right)^{\beta}\right)\cdot\frac{d}{dx}\left(-\left(\frac{x}{\alpha}\right)^{\beta}\right)$

$= \frac{\beta x^{\beta-1}}{\alpha^{\beta}}\cdot\exp\left(-\left(\frac{x}{\alpha}\right)^{\beta}\right) = \left(\frac{\beta}{\alpha}\right)\cdot\left(\frac{x}{\alpha}\right)^{\beta-1}\cdot\exp\left(-\left(\frac{x}{\alpha}\right)^{\beta}\right)$

(2) $W \geq 0$, $\Pr(X \leq x) = \Pr\left(\left(\frac{W}{\alpha}\right)^{\beta} \leq x\right)$

$= \Pr\left(\left(\frac{W}{\alpha}\right) \leq x^{\frac{1}{\beta}}\right) = \Pr\left(W \leq \alpha x^{\frac{1}{\beta}}\right)$

$= F\left(\alpha x^{\frac{1}{\beta}}\right) - 1-\exp\left(-\frac{\left(\alpha x^{\frac{1}{\beta}}\right)^{\beta}}{\alpha^{\beta}}\right) = 1-\exp\left(-\frac{\alpha^{\beta}\cdot x}{\alpha^{\beta}}\right)$

$= 1-\exp(-x) = G(x) \quad \frac{dG}{dx} = \exp(-x)$

由此可見 $X \sim \exp(1)$ (mean 1)

(3) 討論關於如何利用產生均勻分布的機器來模擬 Weibull 分布.

我們由(2)得知 X 服從 $\exp(1)$ 時,

$\alpha\cdot X^{\frac{1}{\beta}} = W \sim \text{Weibull}(\alpha,\beta)$.

因此, 我們只要考慮如何將均勻分布轉換為指數分佈即可.

我們考慮 Uniform(0,1). ($\because\ \forall a<b$, Uniform(a,b) 可轉換為 Uni(0,1))

$X_1 \sim X_n$: U(0,1) (iid). 我們利用順序統計量以及其分布收斂. $X_{(1)} \overset{def}{:=} \min\{X_1, .. X_n\}$

$$Pr(nX_{(1)} \le \lambda) = Pr(X_{(1)} \le \frac{\lambda}{n}) = 1 - \underbrace{Pr(X_{(1)} > \frac{\lambda}{n})}_{\parallel}$$

$$Pr(X_1 \sim X_n > \frac{\lambda}{n})$$

$$= 1-(1-\frac{\lambda}{n})^n = F_{nX_{(1)}}(\lambda)$$

$$\lim_{n\to\infty} F_{nX_{(1)}}(x) = 1-\{\underbrace{(1-\frac{\lambda}{n})^{\frac{-n}{\lambda}}}_{\to e}\}^{(-\lambda)} = 1-e^{-\lambda}$$

$\therefore$ $nX_{(1)}$ 分布收斂於 e(1) (期望值為1之指數分布)

總之 $X_1 \sim X_n \sim$ Uni(0,1)

$$nX_{(1)} \xrightarrow{d} exp(1)$$

$$\alpha(nX_{(1)})^{\frac{1}{\beta}} \xrightarrow{d} \text{Weibull}(\alpha,\beta) \quad (\because (2))$$

[3]. (a) Lhaplus分布 $f(X) = \frac{1}{2}\exp(-|X|)$ $(\mu=0.\ \sigma=1)$

$$M_X(t) = E[e^{tX}] = \int_{-\infty}^{\infty} e^{tX}\,\frac{1}{2}\exp(-|X|)\,dx$$

$$= \underbrace{\int_{0}^{\infty} e^{tX}\,\frac{1}{2}\exp(-x)\,dx}_{\parallel} + \underbrace{\int_{-\infty}^{0} e^{tX}\,\frac{1}{2}\exp(x)\,dx}_{\downarrow}$$

$$\int_{0}^{\infty} \frac{1}{2}\underset{\parallel}{\exp(-(1-t)x)}\,dx$$

$$\frac{1}{2}\,\frac{1}{1-t}$$

$-x = y$ $\qquad \frac{dy}{dx} = -1$

$x:\ -\infty \to 0$

$y:\ \infty \to 0$

$$\int_{\infty}^{0} -e^{-ty}\cdot\frac{1}{2}\exp(-y)\,dy$$

$$= \int_{0}^{\infty} \frac{1}{2}\exp(-\underset{\parallel}{(1+t)y})\,dy$$

$$\frac{1}{2}\cdot\frac{1}{1+t}$$

$$M_X(t) = \frac{1}{2}\cdot\left(\frac{1}{1-t} + \frac{1}{1+t}\right) \quad (-1<t<1)$$

$$= \frac{1}{1-t^2}$$

(b) $M_X(t)$ $(-1<t<1)$ の Taylor展開:

$$\frac{1}{1-r} = 1+r+r^2+\cdots\cdots \quad (-1<r<1)$$

$$\therefore \frac{1}{1-t^2} = 1+t^2+t^4+t^6+\cdots\cdots$$

(Taylor展開)

$$= \sum_{n=0}^{\infty} \frac{M_X^{(n)}(0)}{n!} t^n$$ （比較 係數得知）

$\Downarrow$

、n: 奇數時… $M_X^{(n)}(0) = 0$

、n: 偶數時 $M_X^{(n)}(0) = n! = E[X^n]$

$\therefore E[X] = 0$ （∵ 指數=1）

$$\therefore E[X^n] = \begin{cases} 0 & (n:\text{奇數}) \\ n! & (n:\text{偶數}) \end{cases}$$

$$V[X] = E[X^2] - E[X]^2 = 2! - 0^2 = 2$$

(c) $\frac{dy}{dx} = \sigma$. $X = \frac{Y-\mu}{\sigma}$ $\quad X: -\infty \to \infty$, $Y: -\infty \to \infty$ （$\because \sigma > 0$）

$$1 = \int_{-\infty}^{\infty} \frac{1}{2} \exp(-|x|)\, dx = \int_{-\infty}^{\infty} \underbrace{\frac{1}{2\sigma} \exp\left(-\frac{|y-\mu|}{\sigma}\right)}_{f_Y(y)} dy$$

$$\therefore f_Y(y) = \frac{1}{2\sigma} \exp\left(\frac{-|y-\mu|}{\sigma}\right).$$

$\boxed{4}$ $Y \sim Bin(n,p)$ $X \sim Beta(\alpha,\beta)$

(a) $Pr(Y \leq y) = \sum_{k=0}^{y} nC_k \cdot p^k (1-p)^{n-k}$

$$Pr(X \leq 1-p) = \int_0^{1-p} \frac{x^{n-y-1}(1-x)^y}{Be(n-y,\ y+1)}\,dx \quad \left(\because \begin{matrix} \alpha = n-y \\ \beta = y+1 \end{matrix} \right)$$

$$= \int_0^{1-p} \frac{\Gamma(n+1)}{\Gamma(n-y)\Gamma(y+1)}\, x^{n-y-1}(1-x)^y\,dx$$

$$= \int_0^{1-p} \frac{n!}{(n-y-1)!\,y!}\, x^{n-y-1}(1-x)^y\,dx$$

㊟ Integral by parts

$$= \left[\frac{n!}{(n-y)!\,y!}\, x^{(n-y)}(1-x)^y \right]_0^{1-p}$$

$$+ \int_0^{1-p} \frac{n!}{(n-y)!(y-1)!}\, x^{(n-y)}(1-x)^{y-1}\,dx$$

$$= \frac{n! \cdot p^y(1-p)^{n-y}}{(n-y)!\,y!} + \left[\frac{n!}{(n-y+1)!(y-1)!}\, x^{(n-y+1)}(1-x)^{y-1} \right]_0^{1-p}$$

$$+ \int_0^{1-p} \frac{n!}{(n-y+1)!(y-2)!}\, x^{(n-y+1)}(1-x)^{y-2}\,dx$$

(反覆 Integral By Parts)

$$= \frac{n!}{(n-y)!\,y!}\, p^y(1-p)^{n-y} + \frac{n!}{(n-y+1)!(y-1)!}\, p^{y-1}(1-p)^{n-y+1} + \cdots$$

$$+ p^0(1-p)^n = \sum_{k=0}^{y} p^k(1-p)^{n-k} \cdot nC_k = Pr(Y \leq y)$$

∴ 證明完成

(b) $X \sim Be(5,3)$ $\quad Pr(X \geqq 1-p)$ $\quad \underset{\sim\sim\sim}{(p=\frac{1}{2})}$

$= Pr(Y \leqq y)$ $\quad \underset{\underset{5}{\|}}{\alpha} = n-y, \quad \underset{\underset{3}{\|}}{\beta} = y+1$

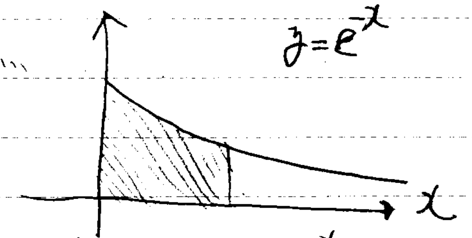
$\therefore y=2, \ n=7 \quad Y \sim Bn(n=7, \ p=\frac{1}{2})$

$$Pr(Y \leqq 2) = \sum_{k=0}^{2} {}_7C_k \cdot \left(\frac{1}{2}\right)^k \cdot \left(\frac{1}{2}\right)^{7-k} = \sum_{k=0}^{2} {}_7C_k \left(\frac{1}{2}\right)^7$$

$$= \frac{{}_7C_0 + {}_7C_1 + {}_7C_2}{128} = \frac{1+7+21}{128} = \frac{29}{128}$$

$\boxed{5}$ the u-th quantile の定義 $Pr[X \leqq x] \geqq u$, $Pr[X \geqq x] \geqq 1-u$

$X \sim \exp(1)$ 時……



- $Pr[X \leqq x] = \int_0^x e^{-t}dt = \left[-e^{-t}\right]_0^x = 1 - e^{-x} \geqq u$

  $1-u \geqq e^{-x}$ $\quad \log(1-u) \geqq -x$ $\quad \therefore\ x \geqq -\log(1-u)$

- $Pr[X \geqq x] = \int_x^\infty e^{-t}dt = \left[-e^{-t}\right]_x^\infty = e^{-x} \geqq 1-u$

  $\therefore\ -x \geqq \log(1-u)$ $\quad \therefore\ x \leqq -\log(1-u)$

$\therefore\ x = -\log(1-u)$

u-th-quantile: $-\log(1-u)$

*. 可能題目有錯誤 ⇒ o(1) denotes a quantity that tends to 0 as z goes to $\underline{\infty}$,（並非1!）

$\boxed{6}$ (a) $Pr[Z \geq z] = \int_z^\infty \phi(t)dt = \int_z^\infty \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-t^2}{2}\right)dt$

$$\int_z^\infty \frac{1}{\sqrt{2\pi}} \cdot \frac{(-t)}{(-t)} \exp\left(\frac{-t^2}{2}\right) dt$$

$$= \left[\frac{1}{\sqrt{2\pi}} \cdot \frac{1}{(-t)} \exp\left(\frac{-t^2}{2}\right)\right]_z^\infty + \int_z^\infty \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{t^2} \exp\left(\frac{-t^2}{2}\right)dt$$

$$= \frac{\exp\left(\frac{-z^2}{2}\right)}{\sqrt{2\pi}z} + \int_z^\infty \frac{1}{\sqrt{2\pi}} \frac{1}{t^2} \exp\left(\frac{-t^2}{2}\right)dt$$

$$= \frac{\phi(z)}{z}\Big(1 + \underbrace{\frac{z}{\phi(z)} \cdot \int_z^\infty \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{t^2} \exp\left(\frac{-t^2}{2}\right) dt}\Big) \rightarrow \underset{\Downarrow}{o(1)}\ ?$$

證明 $z \to \infty$ 時，$\frac{z}{\phi(z)} \int_z^\infty \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{t^2} \exp\left(\frac{-t^2}{2}\right) dt \to 0$

$$= z\exp\left(\frac{z^2}{2}\right) \int_z^\infty \frac{\exp\left(\frac{-t^2}{2}\right)}{t^2} dt$$

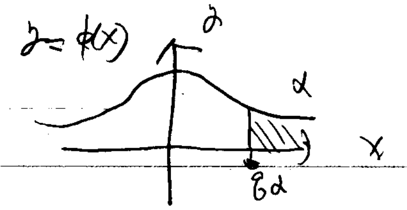
利用 l'hôpital's rule: $\lim_{z\to\infty} \frac{\int_z^\infty \frac{\exp\left(\frac{-t^2}{2}\right)}{t^2}dt}{\left(\frac{1}{z\exp\left(\frac{z^2}{2}\right)}\right)} \xrightarrow{\lim_{z\to\infty}} = \frac{-\frac{\exp\left(\frac{-z^2}{2}\right)}{z^2}}{\left(\frac{-\exp\left(\frac{z^2}{2}\right) - z^2\exp\left(\frac{z^2}{2}\right)}{z^2\exp(z^2)}\right)}$

$$\lim_{z\to\infty} \frac{-\exp\left(\frac{z^2}{2}\right)}{\exp\left(\frac{z^2}{2}\right) + z^2\exp\left(\frac{z^2}{2}\right)} = \lim_{z\to\infty} \frac{-1}{1+z^2} \to 0$$

∴ 證明完成

No.

Date ⑪



$\boxed{6}$ $P(Z \geq z) = (1+o(1))\cdot\frac{\phi(z)}{z}$ ((a)的結果)

(b)

㊟ $\alpha \to 0$ 時 $z_\alpha \to \infty$

$P(Z \geq z_\alpha) = \alpha$ (定義)　　∴ 利用(a)

觀察 $\alpha \to 0 \Rightarrow \frac{\phi(z_\alpha)}{z_\alpha}$ 是否趨近於 α.

計算 $\lim\limits_{\alpha\to+0}\frac{\phi(z_\alpha)}{\alpha z_\alpha} = \lim\limits_{\alpha\to+0}\frac{1}{\sqrt{2\pi}}\cdot\frac{1}{\alpha}\frac{\exp\left(-\frac{1}{2}\left(2\log\frac{1}{\alpha}-\log\log\frac{1}{\alpha}-\log 4\pi\right)\right)}{\sqrt{2\log\left(\frac{1}{\alpha}\right)-\log\log\left(\frac{1}{\alpha}\right)-\log(4\pi)}}$

由於 $\alpha\to+0$ $\log\frac{1}{\alpha}\to\infty$，因此 $\frac{\sqrt{2\log\frac{1}{\alpha}}}{\sqrt{2\log\frac{1}{\alpha}-\log\log\frac{1}{\alpha}-\log 4\pi}}\to 1$.

所以考慮 $\lim\limits_{\alpha\to0^+}\frac{1}{\sqrt{2\pi}}\cdot\frac{1}{\alpha}\cdot\frac{1}{\sqrt{2\log\frac{1}{\alpha}}}\exp\left(-\frac{1}{2}\left(2\log\frac{1}{\alpha}-\log\log\frac{1}{\alpha}-\log 4\pi\right)\right)$

在此 $\log\frac{1}{\alpha}=t$，$\alpha\to0^+$ $t\to\infty$

$$=\lim_{t\to\infty}\frac{1}{\sqrt{2\pi}}e^{t}\frac{1}{\sqrt{2t}}\exp\left(-\frac{1}{2}\left(2t-\log t-\log 4\pi\right)\right)$$

$$=\lim_{t\to\infty}\frac{1}{2\sqrt{\pi}}\frac{1}{\sqrt{t}}\exp\left(\frac{1}{2}\log t+\frac{1}{2}\log 4\pi\right)$$

$$=\lim_{t\to\infty}\frac{1}{2\sqrt{\pi}\sqrt{t}}\exp\left(\log t^{\frac{1}{2}}\right)\cdot\exp\left(\log 2\sqrt{\pi}\right)=1$$ (跟t無關)

總之 $\lim\limits_{\alpha\to+0}\frac{\sqrt{2\log\frac{1}{\alpha}}}{\sqrt{2\log\frac{1}{\alpha}-\log\log\frac{1}{\alpha}-\log 4\pi}}\to 1$ 所以 $\lim\limits_{\alpha\to0}\frac{\phi(z_\alpha)}{\alpha z_\alpha}=1$

∴ $\alpha\to0$ $\frac{\phi(z_\alpha)}{z_\alpha}\approx\alpha$.

$\boxed{7}$ $Y \sim \text{Cauchy}(0,1)$ $\quad f(y) = \frac{1}{\pi}\frac{1}{1+y^2}$

(a) 1st quartile ⇒ 0.25th quantile

3rd quartile ⇒ 0.75th quantile

① $\int_{-\infty}^{x} \frac{1}{\pi}\cdot\frac{1}{1+y^2}dy = \left[\frac{1}{\pi}\arctan y\right]_{-\infty}^{x} = \frac{1}{\pi}\arctan x + \frac{1}{2} = 0.25$

$\arctan x = -\frac{\pi}{4}$ $\quad \therefore \quad x = -1$

② $\int_{-\infty}^{x} \frac{1}{\pi}\cdot\frac{1}{1+y^2}dy = \frac{1}{\pi}\arctan x + \frac{1}{2} = 0.75$

$\therefore x = 1.$

(b) $\int_{x}^{\infty} \frac{1}{\pi}\cdot\frac{1}{1+y^2}dy = \frac{1}{2} - \frac{1}{\pi}\arctan x.$

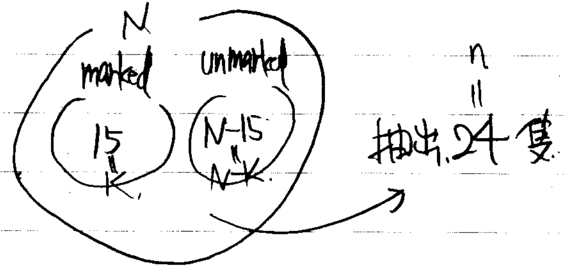
證明 $\lim\limits_{x\to\infty} \frac{\frac{1}{2}-\frac{1}{\pi}\arctan x}{\left(\frac{1}{\pi x}\right)} = 1$

利用 l'hôpital's 定理 $= \lim\limits_{x\to\infty} \frac{-\frac{1}{\pi}\cdot\frac{1}{1+x^2}}{\frac{-1}{\pi x^2}} = \lim\limits_{x\to\infty}\frac{x^2}{1+x^2} = 1$

$\therefore x \to \infty$ 時 $\underset{\substack{\| \\ P(Y \geq x_i)}}{\frac{1}{2}-\frac{1}{\pi}\arctan x} \underset{(\approx)}{≒} \frac{1}{\pi x}$

Date ⑬

8.



$$\underset{n}{\overset{\shortparallel}{}}$$ 抽出24隻 $\begin{cases} \text{marked} \ldots 11 \text{隻} \\ (\text{unmarked} \ldots 13 \text{隻}) \end{cases}$

(a) 若母群體個數很大，可以無視抽出來的樣本數跟母群體個數之比例。（所以可以當成 binominal-trial.）但現在母群體個數並不大（有限），所以我們應該採用超幾何分布之 model。

$\sim HG(N, \underbrace{K=15}_{\text{marked}}, \underbrace{n=24}_{\text{sample number}})$

(b) $\Pr(X=x) = \dfrac{{}_K C_x \cdot {}_{N-K}C_{n-x}}{{}_N C_n} = L(N|x)$ （概似函數），
Liklihood function.

$$\frac{L(N|x)}{L(N-1|x)} = \frac{\frac{\cancel{K!}}{\cancel{x!(K-x)!}}\cdot\frac{(N-K)!}{(N+x-n-K)!\cancel{(n-x)!}}\cdot\frac{(N-1)!}{(N-n-1)!\cancel{n!}}}{\frac{\cancel{K!}}{\cancel{x!(K-x)!}}\cdot\frac{(N-K-1)!}{(N+x-n-K-1)!\cancel{(n-x)!}}\cdot\frac{M!}{(N-n)!\cancel{n!}}}$$

$$= \frac{(N-K)}{(N+x-n-K)}\frac{(N-n)}{N} \leftarrow \begin{cases} n=24 \quad x=11 \\ K=15 \end{cases}$$

(+1)

$$= \frac{(N-15)(N-24)}{(N-28)N}$$

（㊟ unmarked $N-\overset{15}{\overset{\shortparallel}{K}} \geq 13$
$\underset{\sim\sim\sim}{(N \geq 28)}$

$$= \frac{N^2-39N+360}{N^2-28N} = 1 + \frac{-11N+360}{N^2-28N} \left(= \frac{L(N)}{L(N-1)} \right)$$

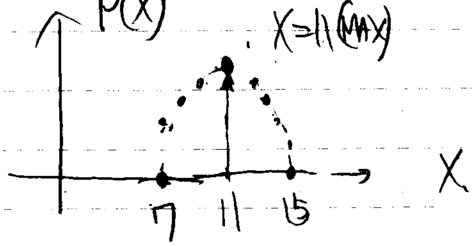
$$1 \leqq \frac{L(29)}{L(28)},\ 1 \leqq \frac{L(30)}{L(29)} \cdots 1 \leqq \frac{L(32)}{L(31)} \quad \frac{L(33)}{L(32)} \leqq 1$$

$$\therefore L(28) \leqq L(29) \leqq \cdots L(32) \geqq L(33) \cdots$$

∴ N 的最大概似估計量 = 32

(c) 在(B)求的N為最大概似估計值，理所當然地，

N=32時、x=11之機率為最大，



(d) 根據(b)之結論，估計N=32時 Pr(x|N)為最大，

無必要重新計算就知道 N=32（以機率來看）為

最佳的估計，

⑮ *題目可能有錯誤：$\left(\frac{E[S_n]}{t}\right)\left(\frac{n-E[S_n]}{n-t}\right)^{n-t}$ 應為 $\underline{\left(\frac{E[S_n]}{t}\right)^{t}}\left(\frac{n-E[S_n]}{n-t}\right)^{n-t}$

$\boxed{9}$ $\mu \overset{def}{=} E[X_j]$ ($j=1\sim n$)

$$E[S_n] = n\mu$$

① Markov 不等式 ($X(\omega) \geqq 0$)

$$E[X] = \int_{\{\omega\in\Omega\}} X(\omega)dP \geqq \int_{\{\omega\in\Omega \mid X(\omega)\geqq\varepsilon\}} X(\omega)dP$$

$$\geqq \varepsilon\int_{\{\omega\in\Omega \mid X(\omega)\geqq\varepsilon)} dP = \varepsilon \cdot P(\{\omega\in\Omega \mid X(\omega) \geqq \varepsilon\})$$

② $X \to e^{\theta S_n}$, $\varepsilon \to e^{\theta t}$ $\underline{(\theta \geqq 0)}$

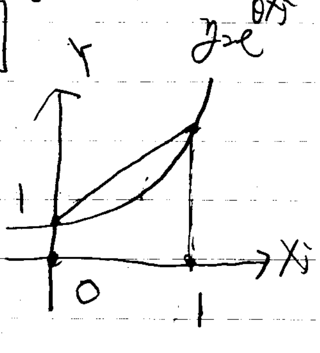
$$\therefore E[e^{\theta S_n}] \geqq e^{\theta t} P(\{\omega\in\Omega \mid S_n(\omega) \geqq t\})$$

③ $E[e^{\theta S_n}] = E[e^{\theta X_1} \cdot e^{\theta X_2} \cdots e^{\theta X_n}] = E[e^{\theta X_1}]^n$

$1 \leqq e^{\theta X_j} \leqq (e^{\theta}-1)X_j + 1$ (∵直線≧曲線)

$E[e^{\theta X_j}] \leq E[(e^{\theta}-1)X_j+1] = \mu(e^{\theta}-1)+1$

$\therefore E[e^{\theta S_n}] \leqq (\mu(e^{\theta}-1)+1)^n = (\mu e^{\theta}+(1-\mu))^n$



$\parallel$

(= 服從 Bin(n,μ) 之隨機變數 的 Moment Generating Function)

已知 $P(\{w\in\Omega \mid S_n(w)\geq t\}) \leq e^{-\theta t}\cdot\left(\mu e^{\theta}+(1-\mu)\right)^{n}$

④ $\overset{\text{代入}}{e^{\theta}} = \dfrac{(1-\mu)\left(\frac{t}{n}\right)}{\mu\left(1-\frac{t}{n}\right)} = \dfrac{(1-\mu)t}{\mu(n-t)}$ (≥ 1) ↓ 下面有證明

$$e^{-\theta t} = \frac{\mu^{t}(n-t)^{t}}{(1-\mu)^{t}t^{t}}$$

$$\mu e^{\theta}+(1-\mu) = \frac{n(1-\mu)}{(n-t)}$$

$$\left(\mu e^{\theta}+(1-\mu)\right)^{n} = \frac{(n-n\mu)^{n}}{(n-t)^{n}}$$

$$e^{-\theta t}\left(\mu e^{\theta}+(1-\mu)\right)^{n} = \left(\frac{n\mu}{t}\right)^{t}\left(\frac{n-n\mu}{n-t}\right)^{n-t}$$ ∴證明完成

(㊟ $e^{\theta}\geq 1$ 之證明 $(\because t\geq n\mu)$

$$e^{\theta}=\frac{(1-\mu)t}{\mu(n-t)} \geq \frac{(1-\mu)(n\mu)}{\mu(n-t)} = \frac{(1-\mu)n}{(n-t)} = \frac{n-n\mu}{n-t} \geq \frac{n-t}{n-t} = 1$$

$\therefore e^{\theta}\geq 1.$ $(\theta\geq 0)$).

$Z_1 = Y_2Y_3$ $Z_3 = Y_3Y_4$ $Z_4 = Y_4Y_5$

No. 17
Date

$\boxed{10}$ $E[Y_i] = \mu$ $V[Y_i] = \sigma^2$

$$X_n = \frac{Y_1Y_2 + Y_1Y_3 + \cdots Y_nY_1}{n}$$

- $E[X_n] = \dfrac{E[Y_1Y_2] + \cdots E[Y_nY_1]}{n} = \dfrac{n\mu^2}{n} = \mu^2$
- $V[X_n]$ ...

令 $Z_1 = Y_1Y_2$ $Z_2 = Y_2Y_3$ .. $Z_n = Y_nY_1$.

$$V[Z_1] = \underbrace{E[(Y_1Y_2)^2]}_{\substack{E[Y_1^2]E[Y_2^2] \\ \| \\ (\mu^2+\sigma^2)(\mu^2+\sigma^2)}} - \underbrace{E[Y_1Y_2]^2}_{(\mu^2)^2}$$

$$= \underbrace{2\mu^2\sigma^2 + \sigma^4}$$

$$\mathrm{Cov}[Z_1, Z_2] = E[Z_1Z_2] - \underbrace{E[Z_1]E[Z_2]}$$

$$E[Z_1Z_2] = E[Y_1Y_2^2Y_3] = \mu^2(\mu^2+\sigma^2) \qquad E[Z_1]E[Z_2] = \mu^2 \cdot \mu^2 = \mu^4$$

$$\therefore \underbrace{\mu^2\sigma^2}$$

$$V[X_n] = V\left[\frac{Z_1+\dots Z_n}{n}\right] = \frac{1}{n^2}V[Z_1+\dots Z_n]$$

$$= \frac{1}{n^2}E\left[\left\{(Z_1-\bar{Z}_1)+\dots+(Z_n-\bar{Z}_n)\right\}^2\right]$$

$$= \frac{1}{n^2}\left\{\sum_{j=1}^{n}V[Z_j] + \sum_{\substack{(i,j)=(1,2),(1,n)\\(2,3)\ (2,1)\\(3,4)\ (3,2)\\ \vdots\\(n,n-1)(n,1)}} \mathrm{cov}[Z_i,Z_j]\right\}$$

(有2n個組合 cov非0)

$$= \frac{1}{n^2}\left\{n(2\mu^2\sigma^2+\sigma^4) + 2n\cdot\underbrace{\mu^2\sigma^2}_{\parallel \atop \mathrm{cov}[Z_1,Z_2]}\right\}$$

$$= \frac{1}{n}(4\mu^2\sigma^2+\sigma^4)$$

$$\left(\because \lim_{n\to\infty}E[X_n]=\mu^2 \quad \lim_{n\to\infty}V[X_n]=0\right)$$

根據 柴比雪夫不等式：

$$Pr\left(|X_n-\mu^2| \geq \underbrace{k\cdot V[X_n]^{\frac{1}{2}}}_{\parallel \atop \varepsilon}\right) \leq \frac{1}{k^2}, \quad k=\frac{\varepsilon}{\sqrt{V[X_n]}}$$

$$\therefore Pr(|X_n-\mu^2| \geq \varepsilon) \leq \frac{V[X_n]}{\varepsilon} = \frac{1}{\varepsilon}\cdot\frac{1}{n}(4\mu^2\sigma^2+\sigma^4)$$

$$n\to\infty \quad \forall\varepsilon>0, \quad Pr(|X_n-\mu^2| \geq \varepsilon)=0$$

$$\therefore X_n \xrightarrow{P} \mu^2$$

No ⑲

Date

$\boxed{11}$ $\log X_n = \dfrac{\log Y_1 + \dots \log Y_n}{n}$

$$E[\log X_n] = E\left[\frac{\log Y_1 + \dots \log Y_n}{n}\right] = r$$

「根據 Khintchine 之弱大數法則，$\{X_i\}$ 為 $\overset{iid}{獨立}$ 同態的隨機變數且 $E[X_i] < \infty$，則

$\overline{X_n} \xrightarrow{P} \mu$，」 $E[\log X_i] < \infty \Rightarrow \dfrac{\log X_1 + \dots \log X_n}{n} \xrightarrow{P} r$

$(\because \log X_i\ i=1\sim n$ 亦為 iid 的隨機變數)

$\therefore \log(X_1 \cdots X_n)^{\frac{1}{n}} \xrightarrow{P} r$

$\therefore (X_1 \cdots X_n)^{\frac{1}{n}} \xrightarrow{P} e^{r}$

(註) Khintchine 之弱大數法則

$\phi_{\bar{X}}(t) = E[e^{it\bar{X}}] = \left(\phi\left(\frac{t}{n}\right)\right)^n = \left[1 + \frac{it\mu}{n} + o\left(\frac{t}{n}\right)\right]^n$

(特徵函數)

$\to e^{it\mu}$ (退化)

根據 Levy 連續定理 $\bar{X} \xrightarrow{P} \mu$ (退化)

Lebesgue可測集合

$\boxed{12}$ 考慮機率空間 (Ω: $[0,1)$, $\mathcal{L}[0,1)$; P = Lebesgue測度)

可測函數: $X_n(\omega) = \frac{1}{n} I[0,\frac{1}{n}) + \frac{2}{n} I[\frac{1}{n},\frac{2}{n}) + \cdots 1 \cdot I[1-\frac{1}{n},1)$

h 應為 Ω 上可積分函數 $\int_\Omega |h| dp < \infty$
(integrable)

根據 Lebesgue's Dominated Convergence Theorem,

若 $h \circ X_n(\omega) \leq |h| \Rightarrow$ 則

$$\underset{n\to\infty}{\lim} \int_\Omega h\circ X_n(\omega)\, dP = \int_\Omega \underset{n\to\infty}{\lim} h\circ X_n(\omega)\, dP \quad (\text{可替換 } \textstyle\int, \underset{n\to\infty}{\lim})$$

$$\overset{\shortparallel}{E[h(X_n)]} \qquad \overset{\shortparallel}{E[\underset{n\to\infty}{\lim} h\circ X_n(\omega)]} = E[h\circ X]$$

函數 $X_n(\omega) \xrightarrow[(n\to\infty)]{a.s} X(\omega) = \omega \quad (\omega\in\Omega)$

$P(\{\omega\in\Omega \mid X(\omega) \leq \lambda\}) =$

$P(\{\omega\in\Omega \mid \omega \leq x\}) = P([0,x]) = x$ (X之cdf)

由此可知 $X(\omega)$ 服從 均勻分布 $U(0,1)$

$$\therefore \underset{n\to\infty}{\lim} E[h(X_n)] \to E[h(X)]$$

$X \sim U(0,1)$

$\boxed{13}$ $E[X_n] = \frac{1}{n}\cdot n + (1-\frac{1}{n})\cdot 0 = 1$

$V[X_n] = \underbrace{E[X_n^2)}_{\frac{1}{n}\cdot n^2} - \underbrace{E[X_n]^2}_{1} = n-1$

$Pr(|X_n - 0| > \varepsilon) = Pr(X_n = n) = \frac{1}{n}$

(㊟ ε為很小的正數)

$\forall \varepsilon > 0, \quad \lim_{n\to\infty} Pr(|X_n - 0| > \varepsilon) = \lim_{n\to\infty} \frac{1}{n} = 0$

$\therefore X_n \xrightarrow{P} 0$

Advanced Statistical Inference I

Homework 4: Common Families of Distributions

Due Date: November 21st

1. Suppose that (Y, X) are random variables where $Y \in \{0, 1\}$ and $X \in R$. Suppose that

$$X|Y = 0 \sim Normal(0, 1)$$

and that

$$X|Y = 1 \sim Normal(2, 1).$$

Suppose that $P(Y = 0) = P(Y = 1) = 1/2$. Find $m(x) = P(Y = 1|X = x)$.

2. Let $X_1, \ldots, X_n \sim Bernoulli(\theta)$ where $0 < \theta < 1$. Let $Y_i = \exp(3X_i)$. Let

$$W_n = \frac{1}{n}\sum_{i=1}^{n} Y_i.$$

(a) Show that there is a number μ such that W_n converges in probability to μ.

(b) Find the limiting distribution of $\sqrt{n}(W_n - \mu)$. d.

(c) Let $Y_n = \sqrt{W_n}$. Show that $\sqrt{n}(Y_n - a) \to N(0, b)$ for some a and b. Fin a and b explicitly.

3. Let X_n be a $Bin(n, 1/2)$ random variable. Set

$$Y_n = \left(1 + \frac{1}{\sqrt{n}}\right)^{X_n}\left(1 - \frac{1}{\sqrt{n}}\right)^{n - X_n}$$

(a) Find $E(Y_n)$.

(b) Find the limiting distribution of $Z_n = \log Y_n$.

4. Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables from an exponential distribution with mean $1/\lambda$ so that their common density function is

$$f(x|\lambda) = \lambda\exp(-\lambda x), \quad x \geq 0.$$

Denote by $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$ the order statistics of $X_1, X_2, \ldots, X_n$. Define, for $i = 1, 2, \ldots, n$,

$$D_i = (n - i + 1)(X_{(i)} - X_{(i-1)})$$

with $X_{(0)} = 0$.

(a) Prove that $D_1, D_2, \ldots, D_n$ are i.i.d. random variables from an exponential distribution with mean $1/\lambda$.

(b) Use the result in (a) to find $E(X_{(n)})$.

(c) For a fixed $K \in \{3, 4, \ldots, n\}$, suppose that you are only able to observe the exact values of $X_{(1)}, X_{(2)}, \ldots, X_{(K)}$ and you only know that each of the $X'_{(j)}s$ for $j > K$ are at least equal to $X_{(K)}$. Define the total-time-on-test (TTOT) statistic

$$T = \sum_{i=1}^{K} X_{(i)} + (n - K)X_{(K)}.$$

Find $E(T)$ and determine the constant c such that $E(cT) = \lambda$. $\frac{1}{\lambda}$

(d) Find an expression for the variance of your estimator in (c).

5. Let X_n be a $Bin(n, 1/2)$ random variable. Set

$$Y_n = \left(1 + \frac{1}{\sqrt{n}}\right)^{X_n} \left(1 - \frac{1}{\sqrt{n}}\right)^{n-X_n}$$

(a) Find $E(Y_n)$.

(b) Find the limiting distribution of $Z_n = \log Y_n$.

6. Note that a random variable X with the following density function

$$f(x|p) = \frac{1}{2^{p/2}}\Gamma(p/2)x^{(p/2)-1}\exp(-x/2), \quad 0 < x < \infty$$

is called a χ^2 random variable with p degrees of freedom. Its moment generating function $M_X(t) = (1-2t)^{-p/2}$ for any $t < 1/2$.

(a) Find the MGF $M_Y(t)$ where $Y = Z^2$. Here Z is a standard normal random variable.

(b) Let U and V be independent and identically distributed χ^2 random variable with 1 degrees of freedom. Find the MGF $M_Y(t)$ of $Y = U + V$. What is the distribution of Y? You can use the above fact on the moment generating function of $M_X(t)$ to answer question (b) - (e).

(c) Find the mean and variance of a χ^2 random variable with 1 degrees of freedom.

(d) If $X_1, \ldots, X_n$ are independently identically distributed χ^2 random variable with 1 degrees of freedom. Find the MGF $M_n(t)$ of the standardized mean,

$$W_n = \frac{\bar{X} - E[X_1]}{\sqrt{Var(X_1)/n}}.$$

(e) What is the limit of $M_n(t)$ as $n \to \infty$? What distribution has this function for its MGF?

4th Homework R05246013.

① 高学統計推論 (I) 森元俊成

$\boxed{1}$ $m(X) = Pr(Y=1 \mid X=\lambda) = \dfrac{f_{XY}(X=\lambda, Y=1)}{f_X(X=\lambda)}$

$$f_{X|Y}(X=\lambda \mid Y=0) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-\lambda^2}{2}\right) = \frac{f(X=\lambda, Y=0)}{Pr(Y=0)}$$

$$f_{X|Y}(X=\lambda \mid Y=1) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-1}{2}(\lambda-2)^2\right) = \frac{f(X=\lambda, Y=1)}{Pr(Y=1)}$$

$$\therefore f_{XY}(X=\lambda, Y=0) = \frac{1}{2} \cdot \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-\lambda^2}{2}\right)$$

$$f_{XY}(X=\lambda, Y=1) = \frac{1}{2} \cdot \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-1}{2}(\lambda-2)^2\right)$$

$$\therefore f_X(X=\lambda) = \frac{1}{2}\frac{1}{\sqrt{2\pi}} \exp\left(\frac{-\lambda^2}{2}\right) + \frac{1}{2} \cdot \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-1}{2}(X-2)^2\right)$$

$$\therefore m(X) = \frac{\frac{1}{\sqrt{2\pi}} \exp\left(\frac{-1}{2}(\lambda-2)^2\right)}{\frac{1}{2}\frac{1}{\sqrt{2\pi}} \exp\left(\frac{-\lambda^2}{2}\right) + \frac{1}{2} \cdot \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-1}{2}(X-2)^2\right)}$$

此題目未明確表示 $\exp(\lambda)$ 為期望值 $\frac{1}{\lambda}$ 還是 λ，在此假設 λ.

$\boxed{2}$ (a) 先求 Y 之期望值以及變異數

$$Y|X \sim \exp(3X)$$

$$E[Y|X] = 3X$$

$$E[Y^2|X] = 18X^2$$

$$E(Y) = E[3X] = 3\theta$$

$$E(Y^2) = E[18X^2] = 18 \cdot E(X^2) = 18\theta$$

$$V[Y] = 18\theta - 9\theta^2 = 9\theta(2-\theta)$$ である.

$$E[W_n] = \frac{1}{n}\sum_{j=1}^{n} E[Y_j] = 3\theta$$

$$V[W_n] = V\left[\frac{Y_1 + \cdots Y_n}{n}\right) = \frac{1}{n} \cdot 9\theta(2-\theta)$$

柴比雪夫不等式...

$$Pr\left[|W_n - 3\theta| > k \cdot \frac{9\theta(2\theta)}{n}\right] \leq \frac{1}{k^2}$$

$$k = \frac{n\varepsilon}{9\theta(2-\theta)}$$

$$Pr\left[|W_n - 3\theta| > \varepsilon\right] \leq \frac{81}{n^2\varepsilon^2}\theta^2(2\theta)^2$$

$$\forall \varepsilon > 0 \quad \lim_{n\to\infty} Pr\left[|W_n - 3\theta| > \varepsilon\right] = 0 \quad \therefore W_n \xrightarrow{p} 3\theta$$

(b) 利用 Moment-Generating-Function,

$$\sqrt{n}(W_n-\mu) = \frac{\sqrt{n}}{n}\left((Y_1-\mu)+(Y_2-\mu)+\cdots(Y_n-\mu)\right)$$

$$E\left[\exp\left(t\sqrt{n}(W_n-\mu)\right)\right] = E\left[\sum_{j=1}^{n}\frac{1}{\sqrt{n}}t(Y_j-\mu)\right]$$

$$=\left\{M_{(Y-\mu)}\left(\frac{t}{\sqrt{n}}\right)\right\}^n \qquad (\mu=3\theta)$$

$$M_{Y-\mu}(0)=1, \quad M'_{Y-\mu}(0)=0 \ (\mu-\mu=0)$$

$$M''_{Y-\mu}(0)=E[(Y-\mu)^2]=V[Y]=9\theta(2-\theta)=\sigma_Y^2$$

$$\therefore \ M_{(Y-\mu)}\left(\frac{t}{\sqrt{n}}\right) = 1+\frac{\sigma_Y^2}{2!}\cdot\left(\frac{t}{\sqrt{n}}\right)^2+o\left(\frac{1}{n\sqrt{n}}\right)$$

$$M_{\sqrt{n}(W_n-\mu)} = \left(1+\frac{\sigma_Y^2t^2}{2n}+o\left(\frac{1}{n\sqrt{n}}\right)\right)^n$$

$$\lim_{n\to\infty} M_{\sqrt{n}(W_n-\mu)} = \left\{\left(1+\frac{\sigma_Y^2t^2}{2n}\right)^{\frac{2n}{\sigma_Y^2t^2}}\right\}^{\frac{\sigma_Y^2t^2}{2}}$$

$$= \exp\left(\frac{1}{2}\sigma_Y^2t^2\right)$$

∴ 由此可知 $\sqrt{n}(W_n-\mu)\xrightarrow{d}N(0,\sigma_Y^2)=N(0,9\theta(2-\theta))$

(Levy 連續性定理)

(c) 利用 δ-method.

$g(X) = x^{\frac{1}{2}}$ $\quad$ $g(X) \doteqdot g'(a)(x-a) + g(a)$

$a = 3\theta$ $\quad$ $g(X) \doteqdot \frac{1}{2\sqrt{3\theta}}(X-3\theta) + \sqrt{3\theta}$

$\therefore \sqrt{X} - \sqrt{3\theta} \doteqdot \frac{1}{2\sqrt{3\theta}}(\underset{\sim}{X-3\theta})$

$(X \to W_n)$ $\quad$ $\sqrt{W_n} - \sqrt{3\theta} \doteqdot \frac{1}{2\sqrt{3\theta}}\underset{\substack{\downarrow \\ N\left(0,\ \frac{9\theta(2-\theta)}{n}\right)}}{\underset{\sim}{(W_n-3\theta)}}$

$\frac{9\theta(2-\theta)}{12\theta n}$

$\therefore \underset{\substack{\parallel \\ Y_n}}{\sqrt{W_n}} - \sqrt{3\theta} \sim N\left(0,\ \frac{3(2-\theta)}{4n}\right)$

$\therefore \sqrt{n}(Y_n - \underset{\substack{\sim \\ a}}{\sqrt{3\theta}}) \sim N\left(0,\ \frac{3}{4}(2-\theta)\right)$

$$\left(\begin{array}{l} a = \sqrt{3\theta} \\ b = \frac{3}{4}(2-\theta) \end{array}\right.$$

Date chapter4

3

(a) $E[Y_n] = E\left[\left(1+\frac{1}{\sqrt{n}}\right)^{X_n}\cdot\left(1-\frac{1}{\sqrt{n}}\right)^{n-X_n}\right]$

$= \sum_{x=0}^{n}\left(1+\frac{1}{\sqrt{n}}\right)^{x}\left(1-\frac{1}{\sqrt{n}}\right)^{n-x}\cdot P(X=x)$

$= \sum_{x=0}^{n}\left(1+\frac{1}{\sqrt{n}}\right)^{x}\left(1-\frac{1}{\sqrt{n}}\right)^{n-x}\cdot\frac{n!}{x!\,(n-x)!}\left(\frac{1}{2}\right)^{x}\cdot\left(\frac{1}{2}\right)^{n-x}$

$= \sum_{x=0}^{n}\left(\frac{1}{2}\right)^{n}\cdot\left(1+\frac{1}{\sqrt{n}}\right)^{x}\left(1-\frac{1}{\sqrt{n}}\right)^{n-x}\cdot {}_nC_x$

$= \left(\frac{1}{2}\right)^{n}\underbrace{\sum_{x=0}^{n}\left(1+\frac{1}{\sqrt{n}}\right)^{x}\left(1-\frac{1}{\sqrt{n}}\right)^{n-x}\cdot {}_nC_x}_{\left(\left(1+\frac{1}{\sqrt{n}}\right)+\left(1-\frac{1}{\sqrt{n}}\right)\right)^{n}}$

$= \left(\frac{1}{2}\right)^{n}\cdot 2^{n} = 1$

(b) 利用 Moment-Generating Function

$E[\exp(tZ_n)] = E[\exp(t\log Y_n)]$

$= E[\exp(\log Y_n^{t})] = E[Y_n^{t}]$

$= E\left[\left(1+\frac{1}{\sqrt{n}}\right)^{tX_n}\cdot\left(1-\frac{1}{\sqrt{n}}\right)^{t(n-X_n)}\right]$

$$= \sum_{x=0}^{n} \left(1+\frac{1}{\sqrt{n}}\right)^{tx}\left(1-\frac{1}{\sqrt{n}}\right)^{tn-tx} \cdot Pr(X=x)$$

$$= \sum_{x=0}^{n} \left(1+\frac{1}{\sqrt{n}}\right)^{tx}\left(1-\frac{1}{\sqrt{n}}\right)^{tn-tx} \cdot {}_nC_x \cdot \left(\frac{1}{2}\right)^{n}$$

$$\left(1+\frac{1}{\sqrt{n}}\right)^{t} = \alpha \quad \left(1-\frac{1}{\sqrt{n}}\right)^{t} = \beta$$

$$= \sum_{x=0}^{n} \alpha^{x}\beta^{n-x} \cdot {}_nC_x \cdot \left(\frac{1}{2}\right)^{n}$$

$$= \left(\frac{1}{2}\right)^{n}(\alpha+\beta)^{n} = \left(\frac{1}{2}\right)^{n}\left(\underbrace{\left(1+\frac{1}{\sqrt{n}}\right)^{t}+\left(1-\frac{1}{\sqrt{n}}\right)^{t}}_{}\right)^{n}$$

$$= M_{Z_n}(t) \quad \perp$$

考慮利用Taylor展開來表示成多項式

- $f(x) = (1+x)^{t} \quad (|x| \ll 1)$

$$f'(x) = t(1+x)^{t-1} \qquad f'(0) = t$$

$$f''(x) = t(t-1)(1+x)^{t-2} \qquad f''(0) = t(t-1)$$

$$\left(1+\frac{1}{\sqrt{n}}\right)^{t} = f(0) + f'(0)\frac{1}{\sqrt{n}} + \frac{1}{2!}f''(0)\frac{1}{n} + o\left(\frac{1}{n\sqrt{n}}\right)$$

$$= 1 + \frac{t}{\sqrt{n}} + \frac{t(t-1)}{2n} + o\left(\frac{1}{n\sqrt{n}}\right)$$

$$\left(1+\frac{1}{\sqrt{n}}\right)^{t} \doteq 1 - \frac{t}{\sqrt{n}} + \frac{t(t-1)}{2n} + o\left(\frac{1}{n\sqrt{n}}\right)$$

$$\therefore \left(1+\frac{1}{\sqrt{n}}\right)^{t} + \left(1-\frac{1}{\sqrt{n}}\right)^{t} = 2 + \frac{t(t-1)}{n} + o\left(\frac{1}{n\sqrt{n}}\right)$$

(b) (part 2) $(1+\frac{1}{\sqrt{n}})^t + (1-\frac{1}{\sqrt{n}})^t \approx 2+\frac{t(t-1)}{n} + o(\frac{1}{n\sqrt{n}})$

∴ $(\frac{1}{2})^n \{ (1+\frac{1}{\sqrt{n}})^t + (1-\frac{1}{\sqrt{n}})^t \}^n \approx (1+\frac{t(t-1)}{2n} + o(\frac{1}{n\sqrt{n}}))^n$

$\lim_{n\to\infty} M_{Z(n)}(t) = \lim_{n\to\infty} (1+\frac{t(t-1)}{2n})^n = \lim_{n\to\infty} \{(1+\frac{t(t-1)}{2n})^{\frac{2n}{t(t-1)}}\}^{\frac{t(t-1)}{2}}$

$= \exp(\frac{t(t-1)}{2}) = \exp(-\frac{1}{2}t + \frac{t^2}{2})$

$N(\mu, \sigma^2)$ 의 mgf ~ $\exp(\mu t + \frac{\sigma t^2}{2})$

∴ $Z_{(n)} \xrightarrow{d} N(-\frac{1}{2}, 1)$

$\boxed{4}$ $f(X_{(1)}=x_1, X_{(2)}=x_2 \cdots X_{(n)}=x_n) = n!\,\lambda^n \exp(-\lambda x_1 - \lambda x_2 \cdots - \lambda x_n)$

$(x_1 \leq x_2 \leq \cdots \leq x_n)$

$$\begin{pmatrix} P_1 \\ P_2 \\ \vdots \\ P_n \end{pmatrix} = \underbrace{\begin{pmatrix} n & & & & \\ -(n-1) & (n-1) & & & \\ 0 & -(n-2) & (n-2) & & \\ \vdots & 0 & 0 & \ddots & \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}}_{A} \begin{pmatrix} X_{(1)} \\ X_{(2)} \\ \vdots \\ X_{(n)} \end{pmatrix}$$

$\det A = n!$

利用矩陣的基本運算，可以將對角線上以外的元素變為零 0。

Step 1: $\begin{pmatrix} n & 0 & \cdots & 0 \\ -(n-1) & (n-1) & \cdots & 0 \end{pmatrix} \downarrow \oplus \times \frac{n-1}{n}$

Step 2: $\begin{pmatrix} n & 0 & \cdots & \\ 0 & (n-1) & 0 & \cdots \\ & -(n-2) & (n-2) & \end{pmatrix} \downarrow \oplus \times \frac{n-2}{n-1}$

(以下省略)

$$\begin{pmatrix} \frac{\partial P_1}{\partial X_{(1)}} & \cdots & \frac{\partial P_1}{\partial X_{(n)}} \\ \frac{\partial P_2}{\partial X_{(1)}} & & \vdots \\ \vdots & & \\ \frac{\partial P_n}{\partial X_{(1)}} & & \frac{\partial P_n}{\partial X_{(n)}} \end{pmatrix} = A$$

$dP_1 dP_2 \cdots dP_n = \det(A) \cdot dx_{(1)} \cdots dx_{(n)}$

$\therefore dP_1 dP_2 \cdots dP_n = n!\, dx_{(1)} dx_{(2)} \cdots dx_{(n)}$

$n X_1 - n X_2$

另外、$D_1+D_2+\dots D_n = X_{(1)}+X_{(2)}+\dots X_{(n)}$

$$\int_{0\le X_1\le\dots\le X_n} f(X_{(1)}=x_1 \dots X_{(n)}=x_n)\, dx_1\dots dx_n \quad (=\text{全概率}=1)$$

$$=\int_{\underbrace{0\le x_1\le\dots\le x_n}_{\Downarrow\; D_1\ge 0,\, D_2\ge 0 \dots D_n\ge 0}} n!\, \exp\Big(-\lambda\big(\underbrace{X_1+\dots X_n}_{(D_1+\dots D_n)}\big)\Big)\, \underbrace{dx_1\dots dx_n}_{\frac{dD_1\dots dD_n}{n!}}$$

$$\therefore 1=\int_{\substack{D_1\ge 0\\ \vdots\\ D_n\ge 0}} \exp\big(-\lambda(D_1+D_2+\dots D_n)\big)\, dD_1\dots dD_n$$

由此可見, $D_1 \sim D_n \sim \exp(\lambda)$ (iid)
$(\text{mean}: \frac{1}{\lambda})$

(2) $E[D_1] = E[nX_{(1)}] = \frac{1}{\lambda}$ $\quad E[X_{(1)}] = \frac{1}{n\lambda}$

$E[D_2] = E[(n-1)(X_{(2)}-X_{(1)})] = \frac{1}{\lambda}$ $\quad E[X_{(2)}-X_{(1)}] = \frac{1}{(n-1)\lambda}$

$E[D_3] = E[(n-2)(X_{(3)}-X_{(2)})] = \frac{1}{\lambda}$ $\quad E[X_{(3)}-X_{(2)}] = \frac{1}{(n-2)\lambda}$

$\therefore E[X_{(1)} + (X_{(2)}-X_{(1)}) + \dots (X_{(n)}-X_{(n-1)})]$

$= \frac{1}{\lambda}\left(\frac{1}{n}+\frac{1}{n-1}+\frac{1}{n-2}+\dots 1\right)$ $\quad \therefore \underbrace{E[X_{(n)}] = \sum_{j=1}^{n} \frac{\lambda}{n-j+1}}$

(3) 題目有錯誤。 $X \xrightarrow{應為} \frac{1}{\lambda}$

(3) 將T用Di來表示:

$$D_1 + D_2 + \cdots D_k = n(X_{(1)}) + (n-1)(X_{(2)} - X_{(1)}) + \cdots (n-k+2)(X_{(k-1)} - X_{(k-2)}) + (n-k+1)(X_{(k)} - X_{(k-1)})$$

$$= X_{(1)} + X_{(2)} + \cdots X_{(k-1)} + (n-k+1) X_{(k)}$$

$$= \sum_{i=1}^{k} X_{(i)} + (n-k) X_{(k)} = T$$

$$\therefore E[D_1 + \cdots D_k] = \frac{k}{\lambda} = E[T)$$

$$E\left[\frac{T}{k}\right] = \frac{1}{\lambda}$$

(4) $V[CT] = V\left[\frac{T}{k}\right] = \frac{1}{k^2} V[T] = \frac{1}{k^2} V(D_1 + \cdots D_k)$

$$= \frac{1}{k^2} \sum_{j=1}^{k} V[D_j] = \frac{1}{k^2} \sum_{j=1}^{k} \frac{1}{\lambda^2} = \frac{1}{k\lambda^2}$$

($\because$ $D_1 \sim D_k$: 獨立)

$\boxed{5}$ 第5題跟第3題完全相同.
(請參閱$\boxed{3}$)

$\boxed{6}$ $f(x|p) \sim \chi^2_p \;(= \Gamma(\frac{p}{2}, \frac{1}{2}))$

(a) $Z \sim N(0,1)$ 時, $Z^2 \sim \chi^2_1$.

$\therefore M_X(t) = (1-2t)^{-\frac{p}{2}}$

$\because p=1 \;\therefore M_Y(t) = (1-2t)^{-\frac{1}{2}}$

(b) $E[\exp(tY)] = E[\exp(tU+tV)] = E[\exp(tU)\cdot\exp(tV)]$

$= \underbrace{E[\exp(tU)]}_{(1-2t)^{-\frac{1}{2}}} \perp \underbrace{E[\exp(tV)]}_{(1-2t)^{-\frac{1}{2}}} = (1-2t)^{-1}$

$\therefore U+V \sim \chi^2_{(2)}$

(c) Cumulant Generating Function $C_Y(t) = \log M_Y(t)$

$= -\frac{1}{2}\log(1-2t)$

$\frac{dC_Y(t)}{dt} = -\frac{1}{2}\cdot\frac{-2}{1-2t} = \frac{1}{1-2t} = C'_Y(t)$

$$C''_Y(t) = \frac{2}{(1-2t)^2}$$

$$C'_Y(0) = 1 \qquad C''_Y(0) = 2$$

$\Big($ ∴ 期望值：1
變異數：2 $\Big)$

(d)
(e) $X_1 + \cdots X_n \sim \chi^2(n)$

$$E\left[\exp(t\bar{X})\right] = E\left[\exp\left(\frac{t}{n}(X_1+\cdots+X_n)\right)\right] = \left(1-\frac{2t}{n}\right)^{-\frac{n}{2}}$$

$$E\left[\exp\left(\frac{\sqrt{n}}{\sqrt{2}}t(\bar{X}-1)\right)\right] = E\left[\exp\left(\frac{\sqrt{n}}{\sqrt{2}}t\bar{X} - \frac{\sqrt{n}}{\sqrt{2}}t\right)\right]$$

$$= \left(1-\frac{2}{n}\frac{\sqrt{n}}{\sqrt{2}}t\right)^{-\frac{n}{2}} \cdot \exp\left(\frac{-\sqrt{n}}{\sqrt{2}}t\right)$$

$$= \left(1-\frac{\sqrt{2}t}{\sqrt{n}}\right)^{-\frac{n}{2}} \cdot \exp\left(\frac{-\sqrt{n}}{\sqrt{2}}t\right) = M_{W_n}(t)$$

$$\log M_{W_n}(t) = \frac{-n}{2}\underbrace{\log\left(1-\frac{\sqrt{2}t}{\sqrt{n}}\right)} - \frac{\sqrt{n}}{\sqrt{2}}t$$

$$\fallingdotseq -\left(\frac{\sqrt{2}}{\sqrt{n}}t + \frac{1}{2}\cdot\frac{2t^2}{n} + \frac{1}{3}\cdot\frac{2\sqrt{2}t^3}{n\sqrt{n}} + \cdots\right)$$

$$\left(\because -\log(1-x) \approx x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots \quad (|x|<1)\right)$$

$$\log M_{W_n}(t) = \frac{n}{2}\left(\cancel{\frac{\sqrt{2}}{\sqrt{n}}t} + \frac{t^2}{n} + \frac{2\sqrt{2}}{3}\cdot\frac{t^3}{n\sqrt{n}} + \cdots\right) - \cancel{\frac{\sqrt{n}}{\sqrt{2}}t}$$

$$= \frac{t^2}{2} + \frac{\sqrt{2}}{3}\frac{t^3}{\sqrt{n}} + \cdots O\left(\frac{1}{n\sqrt{n}}|t\right)$$

$$\lim_{n\to\infty} \log M_{W_n}(t) = \frac{t^2}{2}$$

(d) $\lim_{n\to+\infty} \log M_w(t) = \frac{t^2}{2}$

(e)

$\therefore n \to +\infty \ M_w(t) = \exp\left(\frac{t^2}{2}\right)$

Levy、連續定理, $W_n \xrightarrow{d} N(0,1)$

Advanced Statistical Inference I

Homework 5: Estimationiikelihood
Due Date: December 1st

(野[illegible])

1. ✓ Let $X_1, X_2 \sim Uniform(0, \theta)$ where $\theta > 0$.

   (a) Find the distribution of (X_1, X_2) given T where $T = \max\{X_1, X_2\}$.

   (b) Show that $X_1 + X_2$ is not sufficient.

2. ✓ Let $X_1, \ldots, X_n \sim Uniform(-\theta, 2\theta)$ where $0 < \theta$. Find the likelihood function.

3. An unknown number, say N, of animals inhabit a certain region. To obtain some information about the population size, ecologists often perform the following experiment. They first catch a number, m, of these animals and tag or mark them in some manner. The captured animals are then released back into the region. After allowing the tagged animals time to disperse throughout the region, a new catch of size, say n, is made. Let X denote the number of marked animals in the second catch. If we assume that the number of animals in the region remains essentially constant between the times of the two captures and that each time an animal was caught it was equally likely to be any of the remaining uncaught animals. Derive the distribution of X. (Note that it is hypergeometrically distributed.)

4. ✓ The following data shows the heart rate (in beats/minute) of a person measured through the day.

   $$73, 75, 84, 76, 93, 79, 85, 80, 76, 78, 80,$$

   Assume the data are an iid sample from $N(\theta, \sigma^2)$ where σ^2 is known as the observed sample variance s^2. Thus,

   $$p_\theta(x) = (2\pi\sigma^2)^{-1/2} \exp\left(-\frac{1}{2\sigma^2}(x-\theta)^2\right).$$

   For the following cases: (a) only the first value $x_1 = 73$ is reported, (b) only the sample mean $\bar{x}$ is reported, (c) only the sample median $x_{(6)}$ is reported, and (d) only $x_{(11)} = x_{max}$ is reported. (For the data above, $\bar{x} = 879/11$, $x_{(6)} = 79$ and $x_{(11)} = 93$.) Please derive the distributions needed for each of the cases (a), (b), (c), and (d) .

5. ✓ Again in reference to question 4, consider the following two cases: (a) only $x_{(1)}$ and $x_{(11)}$ are reported and (b) only $x_{(1)}$ and $x_{(2)}$ are reported. Using the distributions for the appropriate order statistics, derive the likelihoods for each of these cases.

6. For determining the half-lives of radioactive isotopes, it is important to know what the background radiation is in a given detector over a period of time. The following data were obtained in a ray detection experiment over 98 ten-second intervals.

   ```
   58 50 57 58 64 63 54 64 59 41 43 56 60 50 46 59 54 60 59 60 67 52
   65 63 55 61 68 58 63 36 42 54 58 54 40 60  58 53 51 73 44 50 53 62
   58 47 63 59 59 56 60 59 50 52 62 51 66 51 56 53 59 57
   ```

   Assuming a Poisson model with parameter λ for the data, derive its likelihoof function and determine its maximum likelihood estimate.

✓7. Suppose that $X_1, X_2, \cdots, X_n$ are i.i.d. with density function

$$f(x|\theta) = \begin{cases} \exp(-(x-\theta)), & x \geq \theta \\ 0, & \text{otherwise}, \end{cases}$$

(a) Find the method of moments estimate of θ.

(b) Find the maximum likelihood estimate of θ. (Hint: Be careful, and dont differentiate before thinking. For what values of θ is the likelihood positive?)

✓8. (Measurement Model with Autoregressive Errors) Let $X_I, \cdots, X_n$ be the n determinations of a physical constant μ. Consider the model where $X_i = \mu + e_i$, $i = 1, \cdots, n$ and assume $e_i = \beta e_{i-1} + \epsilon_i$, $i = 1, \cdots, n$, $e_o = 0$. To find the density $p(X_1, \cdots, X_n)$ by finding the density of $e_1, \cdots, e_n$ using conditional probability theory and $e_i = \beta e_{i-1} + \epsilon_i$. Derive the density as follows

(a) Show that $p(e_1, \cdots, e_n) = f(e_1)f(e_2 - \beta e_1) \cdots f(e_n - \beta e_{n-1})$.

(b) Show that the model for $X_1, \cdots, X_n$ is $p(x_1, \cdots, x_n) = f(x_1 - \mu) \prod_{j=2}^{n} f(x_j - \beta x_{j-1} - (1-\beta)\mu)$.

(c) Give the joint density function when f is the $N(0, \sigma^2)$.

9. Consider the random variable $X \sim U[-1, 1]$. Derive the CDF and (for continuous case) the density function for the following random variables.

(a) $Y = \begin{cases} 0 & \text{if } X \in [-1/2, 1/2] \\ X & \text{otherwise} \end{cases}$.

(b) $Z = F(Y)$, where $F(Y)$ is the CDF of Y as defined in (a).

(c) Find $E(Y)$, $E(Z)$, $Var(Y)$, and $Var(Z)$.

✓10. Let X be a random variable with range $\{0, 1, 2, \ldots\}$. Show that if $E(X) < \infty$, then

$$E(X) = \sum_{n=1}^{\infty} P(X \geq n).$$

✓11. Let X be a random variable having a c.d.f. $F(x)$. Show that if $X \geq 0$, then

$$E(X) = \int [1 - F_X(x)]dx;$$

in general, if $E(X)$ exists, then

$$E(X) = \int_0^{\infty} [1 - F_X(x)]dx - \int_{-\infty}^{0} [F_X(x)]dx.$$

✓12. Let X_1 and X_2 be independent random variables having the standard normal distribution. Obtain the joint p.d.f. of (Y_1, Y_2), where $Y_1 = \sqrt{X_1^2 + X_2^2}$ and $Y_2 = X_1/X_2$. Are the Y_i independent?

✓13. Consider n systems with failure times $X_1, \ldots, X_n$ assumed to be independent and identically distributed with gamma, $\Gamma(\theta, \lambda)$ distributions, where θ and λ are both unknown. Find the method of moments estimates of θ and λ.

14. If time is measured in discrete periods a model that is often used for the time X to failure of an item is,

제출

$$P_\theta[X = k] = \theta^{k-1}(1-\theta), \quad k = 1, 2, \ldots$$

where $0 < \theta < 1$. Suppose that we only record the time of failure, if failure occurs on or before time r and otherwise just note that the item has lived at least $(r+1)$ periods. Thus we observe $Y_1, \ldots, Y_n$ which are independent, identically distributed, and have common frequency function,

$$\begin{aligned} f(k,\theta) &= \theta^{k-1}(1-\theta), \quad k = 1, 2, \ldots, r \\ f(r+1,\theta) &= 1 - \sum_{k=1}^{r} \theta^{k-1}(1-\theta) = \theta^r. \end{aligned}$$

Let M = number of indices I such that $Y_i = r+1$. Show that the maximum likelihood estimate of θ based on $Y_1, \ldots, Y_n$ is

$$\hat\theta(\mathbf{Y}) = \frac{\sum_{i=1}^n Y_i - n}{\sum_{i=1}^n Y_i - M}.$$

15. Suppose X has a Hypergeometric distribution with parameters b, N, n. (Refer to A13.6 of BD for further information on this distribution.) Show that the maximum likelihood estimate of b for N and n fixed is given by,

$$\hat b(X) = \left[\frac{X}{n}(N+1)\right] \quad \text{if } \tfrac{X}{n}(N+1) \text{ is not an integer,}$$

and

$$\hat b(X) = \frac{X}{n}(N+1) \quad or \quad \frac{X}{n}(N+1) - 1$$

otherwise, where $[t]$ is the largest integer which is $\le t$.

16. Let $X_1, \ldots, X_n$ be iid according to a Weibull distribution with density

$$f_\theta(x) = \theta x^{\theta-1} \exp(-x^\theta), \quad x > 0, \theta > 0.$$

Show that there is a unique maximum of the likelihood function.

17. Suppose $X_1, \ldots, X_n$ be iid according to $N(\xi, 1)$ with $\xi > 0$. Show that the maximum likelihood estimate is $\bar X$ when $\bar X > 0$ and does not exist when $\bar X \le 0$.

16. $f_\theta(x_1 \sim x_n) = \theta^n (x_1 \cdot x_n)^{\theta-1} \exp(-x_1^\theta - x_2^\theta \cdot -x_n^\theta)$

$\log f = n\log\theta + (\theta-1)\sum_{j=1}^{n}\log x_j - \sum_{j=1}^{n} x_j^\theta$

$\frac{\partial \log f}{\partial\theta} = \sum_{j=1}^{n}\log x_j - \sum_{j=1}^{n} x_j^\theta \log x_j + \frac{n}{\theta} = 0$

$\Leftrightarrow \frac{n}{\theta} = \sum_{j=1}^{n}(x_j^\theta - 1)\log x_j$

θ에 대해 감소함수

① 高等統計推論、作業(5) R05246013 森元俊成

$\boxed{1}$ $X_1, X_2 \sim U(0,\theta)$ $(\theta>0)$

(1) $f_{X_1,X_2|T}(x_1,x_2|t) = \dfrac{f_{X_1,X_2,t}(x_1,X_2,t)}{f_T(t)}$

$$Pr(T\leq t) = Pr(X_1,X_2\leq t) = \left(\frac{t}{\theta}\right)^2 = \frac{t^2}{\theta^2} \quad (0\leq t\leq\theta)$$

$$\frac{d}{dt}Pr(T\leq t) = \frac{2t}{\theta^2} \quad \therefore f_T(t) = \frac{2t}{\theta^2} \quad (0\leq t\leq\theta)$$

$$f_{X_1,X_2}(x_1,X_2) = \frac{1}{\theta^2} \quad (0\leq x_1\leq\theta,\ 0\leq X_2\leq\theta)$$

$$f_{X_1,X_2,T}(x_1,X_2,t) = \frac{1}{\theta^2} \quad \left(\text{但}\ \begin{matrix}0\leq x_1\leq\theta\\ 0\leq X_2\leq\theta\end{matrix},\ t=\max\{X_1,X_2\}\right)$$

$$\therefore \frac{f_{X_1X_2T}(x_1,X_2,t)}{f_T(t)} = \frac{\left(\frac{1}{\theta^2}\right)}{\left(\frac{2t}{\theta^2}\right)} = \frac{1}{2t}$$

$$f_{X_1,X_2|T}(x_1,X_2|t) = \begin{cases}\dfrac{1}{2t} & (x_1=t,\ 0\leq x_2\leq x_1 \text{ or } x_2=t,\ 0\leq X_1\leq X_2)\\ 0 & (\text{else})\end{cases}$$

$f_{X_1,X_2|T}(x_1,x_2|t)$ 與 θ 無關, 由此可知,

$T(\max\{X_1,X_2\})$ 為 θ 之 充分統計量

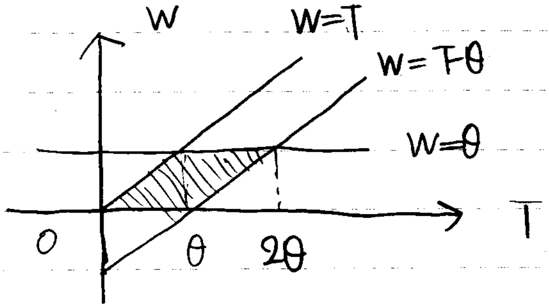
(2) 方法① 證明 $f_{X_1,X_2|T}(x_1,x_2|T=t)$ 跟 θ 有關
$(T=X_1+X_2)$

先求 X_1+X_2 之機率密度函數

$$\begin{cases} T = X_1+X_2 \\ W = X_2 \end{cases} \Leftrightarrow \begin{matrix} X_1 = T-W \\ X_2 = W \end{matrix} \qquad \begin{pmatrix} \frac{\partial x_1}{\partial t} & \frac{\partial x_1}{\partial w} \\ \frac{\partial x_2}{\partial t} & \frac{\partial x_2}{\partial w} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$\therefore |J| = 1 \quad \therefore dx_1 dx_2 = dt\,dw$

$$1 = \iint_{\substack{0 \leq x_1 \leq \theta \\ 0 \leq x_2 \leq \theta}} \frac{1}{\theta^2} dx_1 dx_2 = \iint_{\substack{0 \leq T-W \leq \theta \\ 0 \leq W \leq \theta}} \frac{1}{\theta^2} dt\,dw$$



$\therefore \; 0 \leqq T \leqq \theta \Rightarrow 0 \leqq W \leqq T$
$\quad\; \theta \leqq T \leqq 2\theta \Rightarrow T-\theta \leqq W \leqq \theta$

$\therefore \displaystyle\int_0^t \frac{1}{\theta^2} dw = \frac{t}{\theta^2}$
$\quad \displaystyle\int_{t-\theta}^{\theta} \frac{1}{\theta^2} dw = \frac{2\theta - t}{\theta^2}$

$$\therefore f_T(t) = \begin{cases} \dfrac{t}{\theta^2} & (0 \leq t \leq \theta) \\ \dfrac{2\theta - t}{\theta^2} & (\theta \leq t \leq 2\theta) \end{cases}$$

③

$$f_{X_1X_2|T}(x_1,x_2|T=t)=\frac{f_{X_1,X_2,T}(x_1,x_2,t)}{f_T(t)}$$

$$f_{x_1,x_2,T}(x_1,x_2,t)=\begin{cases}\frac{1}{\theta^2} & (t=x_1+x_2,\ \ 0\leq x_1,x_2\leq\theta)\\ 0 & (\text{else})\end{cases}$$

∴

$$f_{X_1,X_2|T}(x_1,x_2|T=t)=\begin{cases}\frac{1}{t} & (0\leq t\leq\theta,\ \ t=x_1+x_2,\ \ 0\leq x_1,x_2\leq\theta)\\ \frac{1}{2\theta-t} & (\theta<t\leq 2\theta\ \ \ t=x_1+x_2,\ \ 0\leq x_1,x_2\leq\theta)\\ 0 & (\text{else})\end{cases}$$

∴ $t>\theta$ 時 $f_{x_1,x_2|T}(x_1,x_2|t)$ 包含 θ，由此可知

T 並非 θ 之充分統計量，

方法②：找到最小充分統計量

$$\frac{f(y_1,y_2|\theta)}{f(x_1,x_2|\theta)}=\frac{\frac{1}{\theta^2}I(y_1,y_2\leq\theta)}{\frac{1}{\theta^2}I(x_1,x_2\leq\theta)}=\frac{I(\max\{y_1,y_2\}\leq\theta)}{I(\max\{x_1,x_2\}\leq\theta)}$$

跟 θ 無關 $\Rightarrow$ $\max\{x_1,x_2\}=\max\{y_1,y_2\}$

而且 $\max\{x_1,x_2\}=\max\{y_1,y_2\}\Rightarrow\frac{f(y_1,y_2|\theta)}{f(x_1,x_2|\theta)}$ 跟 θ 無關

∴ θ 之最小充分統計量為 $\max\{X_1, X_2\}$

$\nexists$ g (函數) $g(X_1+X_2) = \max\{X_1, X_2\}$.

換言之、無法由 X_1+X_2 得知 $\max\{X_1, X_2\}$ 之值

(觀測 X_1+X_2 之值時，關於 $\max\{X_1, X_2\}$ 之資訊已經消失了)

∴ X_1+X_2 並非充分統計量

$\boxed{2}$ $f(x) = \frac{1}{3\theta} I(-\theta < x < 2\theta)$

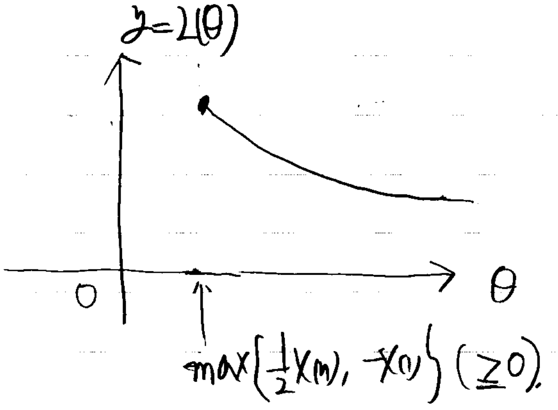
$$f(x_1 \cdots x_n | \theta) = \left(\frac{1}{3\theta}\right)^n I(-\theta < x_1 \sim x_n < 2\theta)$$

$$= \frac{1}{(3\theta)^n} I(-\theta < X_{(1)}, X_{(n)} < 2\theta)$$

$$= \frac{1}{(3\theta)^n} I\left(\frac{1}{2}X_{(n)} < \theta, \; -X_{(1)} < \theta\right)$$

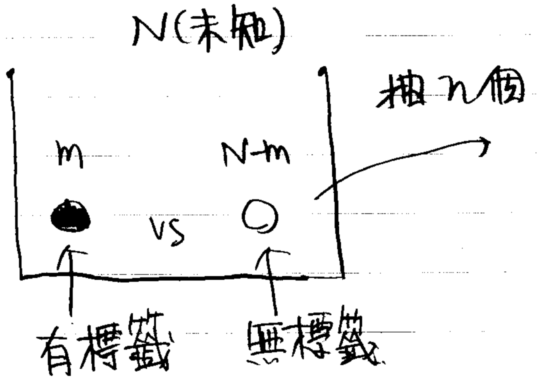
$$= \frac{1}{(3\theta)^n} I\left(\max\left\{\frac{1}{2}X_{(n)}, -X_{(1)}\right\} < \theta\right)$$

$$\therefore L(\theta | x_1 \sim x_n) = \frac{1}{(3\theta)^n} I\left(\max\left\{\frac{1}{2}X_{(n)}, -X_{(1)}\right\} < \theta\right).$$



(由此可知, $\max\{\frac{1}{2}X_{(n)}, -X_{(1)}\}$ 為 θ 之 MLE)

$\boxed{3}$



$X :=$ n個裡面有標籤的動物

區分所有的個體. 抽出來的組合有 ${}_NC_n$ $\left(=\binom{N}{n}\right)$

抽到「x個有標籤的個體」及「n-x個無標籤的個體」之組合為 ${}_mC_x \cdot {}_{N-m}C_{n-x}$.

每個組合出現的機率為相同.

故此 $Pr(X=x)=\dfrac{{}_mC_x \cdot {}_{N-m}C_{n-x}}{{}_NC_n}$ $(\sim HG(N,m,n))$

$\boxed{4}$ $X_1 \sim X_n \sim N(\theta\ \sigma^2)$ $(n=11)$

(a) 只觀測一個樣本 $X_1 \sim N(\theta, \sigma^2)$

$$\therefore f(X_1=x_1) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-1}{2\sigma^2}(x-\theta)^2\right)$$

(b) 觀測樣本平均數 $\bar{X} \sim N(\theta, \frac{\sigma^2}{n})$

$$f(\bar{X}=\bar{x}) = \frac{1}{\sqrt{2\pi(\frac{\sigma^2}{n})}} \exp\left(\frac{-1}{2(\frac{\sigma^2}{n})}(x-\theta)^2\right)$$

(c) 觀測樣本中位數 $X_{(\frac{n+1}{2})}$ 時

為了方便求解，先考慮 $X_1 \sim X_n$ 之 cdf 之值，即 $F(X_1) \cdots F(X_n)$ 之分布. $\underset{\substack{\parallel\\ U_1}}{F(X_1)} \sim \underset{\substack{\parallel\\ U_n}}{F(X_n)} \sim \text{Uni}(0,1)$.

接著考慮 均勻分布 $U(0,1)$ 之順序統計量,

$X_i \leq X_j \Leftrightarrow F(X_i) \leq F(X_j)$ $\therefore$ $F(X_{(i)}) = U_{(i)}$

$U_{(j)} \sim \text{Beta}(j, n-j+1)$ （右頁有證明）→

$$\therefore f(U_{(j)}=u) = \frac{1}{Be(j,n-j+1)} \cdot u^{j-1}(1-u)^{n-j}$$

$$1 = \int_0^1 \frac{1}{Be(j,n-j+1)} u^{j-1}(1-u)^{n-j}\,du \qquad F(X)=U \qquad \frac{du}{dx}=f(x)$$

$$(F^{-1}(U)=X)$$

$$= \int_{-\infty}^{\infty} \frac{1}{Be(j,n-j+1)} F(X)^{j-1}\cdot(1-F(X))^{n-j}\cdot f(x)\,dx$$

$(U: 0\to 1 \quad X: -\infty\to\infty)$

$j=\frac{n+1}{2}$ ∴ $X_{(\frac{n+1}{2})} \sim f = \frac{1}{Be(\frac{n+1}{2},\frac{n+1}{2})} F(x)^{\frac{n-1}{2}} \cdot (1-F(x))^{\frac{n-1}{2}} \cdot f(x)$

$F(x) = Pr(X \leq x) = Pr\left(\frac{X-\theta}{\sigma} \leq \frac{x-\theta}{\sigma}\right) = \Phi\left(\frac{x-\theta}{\sigma}\right)$

∴ $X_{(\frac{n+1}{2})} \sim f = \frac{1}{Be(\frac{n+1}{2},\frac{n+1}{2})} \Phi\left(\frac{x-\theta}{\sigma}\right)^{\frac{n-1}{2}} \cdot \left(1-\Phi\left(\frac{x-\theta}{\sigma}\right)\right)^{\frac{n-1}{2}} \cdot \frac{1}{\sigma}\phi\left(\frac{x-\theta}{\sigma}\right)$

(d) 利用(c) $\underset{\sim\sim}{j=n}$

$$\frac{1}{Be(n,1)} F(x)^{n-1}(1-F(x))^{0} f(x) = n\,F(x)^{n-1} f(x)$$

$$= n \cdot \Phi\left(\frac{x-\theta}{\sigma}\right)^{n-1} \cdot \frac{1}{\sigma} \cdot \phi\left(\frac{x-\theta}{\sigma}\right)$$

㊟ $U(0,1)$ 之順序統計量之分布 $X_{(k)}$ $\quad (X_1 \sim X_n \sim U(0,1))$

$f(x_{(1)} \dots x_{(n)}) = n! \quad (X_{(1)} \leq X_{(2)} \dots \leq X_{(n)})$

全機率 $1 = \int_{x_1 \leq \dots \leq x_n} n!\, dx_1 \dots dx_n$ 求 $X_{(k)}$ 之邊際分布，

① x_1 積分… $\int_0^{x_2} n!\, dx_1 = n!\, x_2$

② x_2 積分 $\int_0^{x_3} n!\, x_2\, dx_2 = n! \frac{x_3^2}{2}$

↓ $\quad\int$

ⓚ⁻¹ x_{k-1} 積分 $\int_0^{x_k} \sim dx_{k-1} = n! \frac{x_k^{(k-1)}}{(k-1)!}$

接著計算 ① $n! \frac{x_k^{(k-1)}}{(k-1)!} \int_{x_{n-1}}^{1} dx_n = n! \cdot \frac{x_k^{(k-1)}}{(k-1)!} \cdot (1-x_{n-1})$

$x_n \to x_{n-1} \to x_{k+1}$ 之積分

② $n! \frac{x_k^{(k-1)}}{(k-1)!} \int_{x_{n-2}}^{1} (1-x_{n-1})\, dx_{n-1} = n! \frac{x_k^{k-1}}{(k-1)!} \frac{(1-x_{n-2})^2}{2!}$

↓

(n-k) $\qquad = n! \frac{x_k^{k-1}}{(k-1)!} \frac{(1-x_k)^{n-k}}{(n-k)!}$

∴ $X_{(k)} \sim Be(k, n-k+1)$

$\boxed{5}$ $n=11$ ϕ: $N(0,1)$之pdf, Φ: $N(0,1)$之cdf

(a) 求 $X_{(1)}, X_{(n)}$ 之聯合機率密度函數.

$$Pr(X_{(1)} \leq x, X_{(n)} \leq y) = Pr(X_{(n)} \leq y) - Pr(x < X_{(1)}, X_{(n)} \leq y)$$

$$= Pr(X_1 \sim X_n \leq y) - Pr(x < X_1 \sim X_n \leq y)$$

$$= Pr\left(\frac{X_j-\theta}{\sigma} \leq \frac{y-\theta}{\sigma}\right)^n - Pr\left(\frac{x-\theta}{\sigma} < \frac{X_j-\theta}{\sigma} \leq \frac{y-\theta}{\sigma}\right)^n$$

$$= \Phi\left(\frac{y-\theta}{\sigma}\right)^n - \left(\Phi\left(\frac{y-\theta}{\sigma}\right) - \Phi\left(\frac{x-\theta}{\sigma}\right)\right)^n$$

$$\frac{\partial^2 Pr(X_{(1)} \leq x, X_{(n)} \leq y)}{\partial x \partial y} = \frac{\partial^2}{\partial x \partial y}\left\{\Phi\left(\frac{y-\theta}{\sigma}\right)^n - \left(\Phi\left(\frac{y-\theta}{\sigma}\right) - \Phi\left(\frac{x-\theta}{\sigma}\right)\right)^n\right\}$$

$$= n(n-1)\left\{\Phi\left(\frac{y-\theta}{\sigma}\right) - \Phi\left(\frac{x-\theta}{\sigma}\right)\right\}^{n-2} \cdot \phi\left(\frac{y-\theta}{\sigma}\right) \cdot \frac{1}{\sigma} \cdot \phi\left(\frac{x-\theta}{\sigma}\right) \frac{1}{\sigma}$$

$$= \frac{n(n-1)}{\sigma^2} \phi\left(\frac{y-\theta}{\sigma}\right) \phi\left(\frac{x-\theta}{\sigma}\right) \left(\Phi\left(\frac{y-\theta}{\sigma}\right) - \Phi\left(\frac{x-\theta}{\sigma}\right)\right)^{n-2}$$

$$(n=11) \quad = f_{X_{(1)},X_{(n)}}(X_{(1)}=x, X_{(n)}=y \mid \theta, \sigma)$$

$$\therefore L(\theta, \sigma \mid X_{(1)}, X_{(n)}) = \frac{n(n-1)}{\sigma^2} \phi\left(\frac{y-\theta}{\sigma}\right) \phi\left(\frac{x-\theta}{\sigma}\right) \left(\Phi\left(\frac{y-\theta}{\sigma}\right) - \Phi\left(\frac{x-\theta}{\sigma}\right)\right)^{n-2} \quad (n=11)$$

$(x \leq y)$

(b) 求 $X_{(1)}, X_{(2)}$ 之聯合機率密度函數.

$X_{(1)} \sim X_{(n)}$ 之聯合機率密度函數為

$$n! \left(\frac{1}{\sigma}\phi\left(\frac{X_{(1)}-\theta}{\sigma}\right)\right)\left(\frac{1}{\sigma}\phi\left(\frac{X_{(2)}-\theta}{\sigma}\right)\right) \cdots \left(\frac{1}{\sigma}\phi\left(\frac{X_{(n)}-\theta}{\sigma}\right)\right)$$

$(X_{(1)} \leq X_{(2)} \cdots \leq X_{(n)})$

全機率 $= 1 = \int_{X_{(1)} \leqq \cdots \leqq X_{(n)}} n! \frac{1}{\sigma} \phi\left(\frac{X_{(1)}-\theta}{\sigma}\right) \cdots \frac{1}{\sigma} \phi\left(\frac{X_{(n)}-\theta}{\sigma}\right) dx_{(1)} \cdots dx_{(n)}$

Step 1 $X_{(n)}$で積分…

$$\int_{X_{(n-1)}}^{\infty} n! \frac{1}{\sigma} \phi\left(\frac{X_{(1)}-\theta}{\sigma}\right) \cdots \frac{1}{\sigma} \phi\left(\frac{X_{(n)}-\theta}{\sigma}\right) dx_{(n)}$$

$$= n! \frac{1}{\sigma} \phi\left(\frac{X_{(1)}-\theta}{\sigma}\right) \cdots \frac{1}{\sigma} \phi\left(\frac{X_{(n-1)}-\theta}{\sigma}\right) \cdot \left[\Phi\left(\frac{X_{(n)}-\theta}{\sigma}\right)\right]_{X_{(n-1)}}^{\infty}$$

$$= n! \frac{1}{\sigma} \phi\left(\frac{X_{(1)}-\theta}{\sigma}\right) \cdots \frac{1}{\sigma} \phi\left(\frac{X_{(n-1)}-\theta}{\sigma}\right) \left(1 - \Phi\left(\frac{X_{(n-1)}-\theta}{\sigma}\right)\right)$$

Step 2 $X_{(n-1)}$で積分

$$\int_{X_{(n-2)}}^{\infty} \sim dX_{(2)} = n! \frac{1}{\sigma} \phi\left(\frac{X_{(1)}-\theta}{\sigma}\right) \cdots \frac{1}{\sigma} \phi\left(\frac{X_{(n-2)}-\theta}{\sigma}\right) \left[-\frac{1}{2}\left(1 - \Phi\left(\frac{X_{(n-1)}-\theta}{\sigma}\right)\right)^2\right]_{X_{(n-2)}}^{\infty}$$

$$= n! \frac{1}{\sigma} \phi\left(\frac{X_{(1)}-\theta}{\sigma}\right) \cdots \frac{1}{2\sigma} \phi\left(\frac{X_{(n-2)}-\theta}{\sigma}\right) \cdot \left(1 - \Phi\left(\frac{X_{(n-2)}-\theta}{\sigma}\right)\right)^2$$

Step j $X_{(n-j+1)}$で積分

$$\frac{1}{\sigma} \phi\left(\frac{X_{(n-j+1)}-\theta}{\sigma}\right)$$

↓

$$\int_{X_{(n-j+1)}}^{\infty} \sim dX_{(n-j+1)} = n! \frac{1}{\sigma} \phi\left(\frac{X_{(1)}-\theta}{\sigma}\right) \cdots \frac{1}{\sigma j!} \phi\left(\frac{X_{(n-j)}-\theta}{\sigma}\right) \cdot \left(1 - \Phi\left(\frac{X_{(n-j)}-\theta}{\sigma}\right)\right)^j$$

Step n-2 : $(j = n-2)$

$$\int_{X_{(3)}}^{\infty} \sim dX_{(3)} = \frac{n!}{(n-2)!} \frac{1}{\sigma} \phi\left(\frac{X_{(1)}-\theta}{\sigma}\right) \cdot \frac{1}{\sigma} \phi\left(\frac{X_{(2)}-\theta}{\sigma}\right) \left(1 - \Phi\left(\frac{X_{(2)}-\theta}{\sigma}\right)\right)^{n-2}$$

$$= \frac{n!}{(n-2)!} \frac{1}{\sigma^2} \phi\left(\frac{X_{(1)}-\theta}{\sigma}\right) \cdot \phi\left(\frac{X_{(2)}-\theta}{\sigma}\right) \cdot \left(1 - \Phi\left(\frac{X_{(2)}-\theta}{\sigma}\right)\right)^{n-2}$$

$$\therefore L(\theta, \sigma^2 \mid X_{(1)}, X_{(2)}) = \frac{n!}{(n-2)!} \frac{1}{\sigma^2} \phi\left(\frac{X_{(1)}-\theta}{\sigma}\right) \phi\left(\frac{X_{(2)}-\theta}{\sigma}\right) \left(1 - \Phi\left(\frac{X_{(2)}-\theta}{\sigma}\right)\right)^{n-2}$$

$(n = 11)$

⑪

$\boxed{6}$ $Pr(X=x|\lambda) = e^{-\lambda} \cdot \frac{\lambda^{x}}{x!}$

$Pr(X_1=x_1, \cdots X_n=x_n | \lambda) = e^{-n\lambda} \cdot \frac{\lambda^{(x_1+ .. x_n)}}{x_1! \cdots x_n!}$

$= L(\lambda | X_1 \cdots X_n)$

$\therefore L(\lambda | x_1 \cdots x_n) = e^{-n\lambda} \cdot \frac{\lambda^{(x_1+ .. +x_n)}}{x_1! \cdots x_n!}$

$\log L(\lambda | x_1 .., x_n) = -n\lambda + (x_1 + .. x_n)\log\lambda - \log x_1! - \log x_2! \cdots - \log x_n!$

$\frac{\partial}{\partial\lambda}\log L = -n + \frac{1}{\lambda}(X_1 + \cdots + X_n) = 0$ $\quad \frac{\partial}{\partial\lambda^2}\log L = \frac{-1}{\lambda^2}(X_1 + .. X_n) < 0$

$\therefore \lambda = \frac{X_1 + .. X_n}{n} = \overline{X}$ 時 L 最大.

$\therefore \hat{\lambda}_{MLE} = \overline{X}$

$\therefore \frac{58 + 50 + \cdots + 59 + 57}{62} = 56.01613$

$\boxed{17}$

(1) $E[X] = \int_{x \geq \theta} x \exp(-x+\theta)\,dx \qquad (t := x-\theta,\ \frac{d\theta}{dx}=1)$

$= \int_{t \geq 0} (t+\theta) \exp(-t)\,dt$

$= \Gamma(2) + \theta \int_{t \geq 0}^{\infty} \exp(-t)\,dt =$

$= 1! + \theta = \theta+1$

$\therefore E\left[\frac{X_1+\cdots X_n}{n}\right] = \theta+1$

$\frac{X_1+\cdots X_n}{n} \xrightarrow{P} \theta+1$ （弱大数法則）

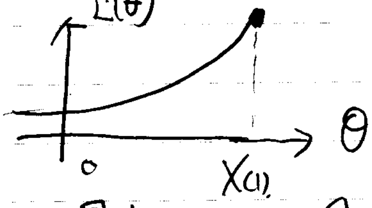
$\therefore \bar{X}-1 \xrightarrow{P} \theta$

$\therefore \hat{\theta}_{MME} = \bar{X}-1$

(2) $f(x_1 \cdots x_n \mid \theta) = \exp(-(x_1+\cdots x_n - n\theta)) \cdot I(X_1 \sim X_n \geq \theta)$

$= \exp(-(x_1+\cdots x_n) + n\theta) \cdot I(\theta \leq X_{(1)})$

$= L(\theta)$



$\therefore \theta = X_{(1)}$ 時, $L(\theta)$ 最大、 $\therefore \hat{\theta}_{MLE} = X_{(1)}$

$\boxed{8}$ 題意不太清楚，但猜想題意是「$\varepsilon_1 \sim \varepsilon_n$ 互相獨立，其pdf為 $f(\cdot)$，出題者希望利用條件分布來求解。」

(a) $p(e_1, e_2, \cdots e_n) = f(e_n | e_1 \cdots e_{n-1}) \cdot f(e_1 \cdots e_{n-1})$

$= f(e_n | e_1 \cdots e_n) \cdot f(e_{n-1} | e_1 \cdots e_{n-2}) \cdot f(e_1 \cdots e_{n-2})$

$\therefore p(e_1 \cdots e_n) = \prod_{j=1}^{n} f(e_j | e_1 \cdots e_{j-1})$

但 $e_i = \beta e_{i-1} + \varepsilon_i$ 可寫成 $p(e_1 \sim e_n) = \prod_{j=1}^{n} f(e_j | e_{j-1})$

（e_i 只有跟 e_{i-1} 有關）

$e_i | e_{i-1} \sim \varepsilon_i + \beta e_{i-1}$　　$\varepsilon_i = e_i - \beta e_{i-1}$

$\therefore f(\varepsilon_i) = f(e_i - \beta e_{i-1})$　　$\therefore p(e_1 \sim e_n) = \prod_{i=1}^{n} f(e_i - \beta e_{i-1})$

(b) $X_i = \mu + \varepsilon_i$ 應改為 $\underline{X_i = \mu + e_i}$.（題目有錯誤）

（PS 後來老師發了更新版）

$$\begin{cases} e_i = X_i - \mu \\ e_{i-1} = X_{i-1} - \mu \end{cases}$$

$\underline{e_i - \beta e_{i-1}} = (X_i - \mu) - \beta(X_{i-1} - \mu) = X_i - \beta X_{i-1} - \mu + \beta\mu$.

$= X_i - \beta X_{i-1} - (1-\beta)\mu$

跟(a)的情況一樣, x_i, e_i 之係數皆為 1. 所以, 直接將 $e_i-\beta e_{i-1}$ 改為 $x_i-\beta x_{i-1}-(1-\beta)\mu$ 即可.

$$\prod_{i=1}^{n} f(\underset{(e_0=0)}{e_i-\beta e_{i-1}}) = f(e_1)\cdot\prod_{j=2}^{n} f(e_j-\beta e_{j-1})$$

$$= \underbrace{f(e_1)}_{\substack{\| \\ f(x_1-\mu)}}\cdot\prod_{j=2}^{n} f(x_j-\beta x_{j-1}-(1-\beta)\mu)$$

∴ 證明完成

(c) $f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(\frac{-x^2}{2}\right)$

$$f(x_1-\mu)=\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(\frac{-1}{2}(x_1-\mu)^2\right)$$

$$f(x_j-\beta x_{j-1}-(1-\beta)\mu)=\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(\frac{-1}{2}\left(x_j-\beta x_{j-1}-(1-\beta)\mu\right)^2\right) \quad (j\geq 2)$$

$$\therefore p(x_1\ldots x_n)=\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n\exp\left(\frac{-(x_1-\mu)^2}{2}-\frac{1}{2}\sum_{j=2}^{n}\left(x_j-\beta x_{j-1}-(1-\beta)\mu\right)^2\right)$$

$\boxed{9}$

(a) $$\Pr\left(Y \leqq y \mid X \in \left[-\frac{1}{2}, \frac{1}{2}\right]\right) = \begin{cases} 1 & (y \geqq 0) \\ 0 & (y < 0) \end{cases}$$

$\parallel$

$$\frac{\Pr\left(Y \leqq y,\ X \in \left[-\frac{1}{2}, \frac{1}{2}\right]\right)}{\Pr\left(X \in \left[-\frac{1}{2}, \frac{1}{2}\right]\right)} = 2\Pr\left(Y \leqq y,\ X \in \left[-\frac{1}{2}, \frac{1}{2}\right]\right)$$

$$\therefore \Pr\left(Y \leqq y,\ X \in \left[-\frac{1}{2}, \frac{1}{2}\right]\right) = \begin{cases} \frac{1}{2} & (y \geqq 0) \\ 0 & (y < 0) \end{cases}$$

接下来求:

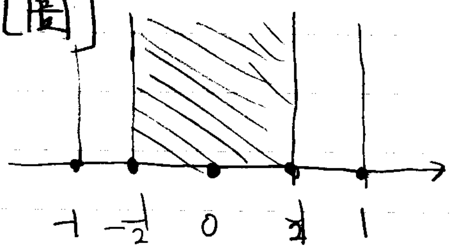
$$\Pr\left(Y \leqq y \mid X \notin \left[-\frac{1}{2}, \frac{1}{2}\right]\right) \quad \left(= \frac{\Pr\left(Y \leqq y,\ X \notin \left[-\frac{1}{2}, \frac{1}{2}\right]\right)}{\underbrace{\Pr\left(X \notin \left[-\frac{1}{2}, \frac{1}{2}\right]\right)}_{=\frac{1}{2}}} \right)$$

$$= \Pr\left(X \leqq y \mid X \notin \left[-\frac{1}{2}, \frac{1}{2}\right]\right)$$

$$= \frac{\Pr\left(X \notin \left[-\frac{1}{2}, \frac{1}{2}\right],\ X \leqq y\right)}{\Pr\left(X \notin \left[-\frac{1}{2}, \frac{1}{2}\right]\right)} = 2\Pr\left(X \notin \left[-\frac{1}{2}, \frac{1}{2}\right],\ X \leqq y\right)$$

$$= \begin{cases} 0 & (y < -1) \\ y+1 & \left(-1 \leqq y < -\frac{1}{2}\right) \\ \frac{1}{2} & \left(-\frac{1}{2} \leqq y < \frac{1}{2}\right) \\ y & \left(\frac{1}{2} \leqq y \leq 1\right) \\ 1 & (y > 1) \end{cases}$$

[图]



$$\therefore \Pr\left(Y \leq y,\ X \notin \left[-\tfrac{1}{2}, \tfrac{1}{2}\right]\right) = \begin{cases} 0 & (y<-1) \\ \frac{1}{2}(y+1) & (-1 \leq y < -\frac{1}{2}) \\ \frac{1}{4} & (-\frac{1}{2} \leq y < \frac{1}{2}) \\ \frac{y}{2} & (\frac{1}{2} \leq y \leq 1) \\ \frac{1}{2} & (y>1) \end{cases}$$

$$\therefore \Pr\left(Y \leq y,\ X \in \left[-\tfrac{1}{2}, \tfrac{1}{2}\right]\right)$$

$$+ \Pr\left(Y \leq y,\ X \notin \left[-\tfrac{1}{2}, \tfrac{1}{2}\right]\right) = \Pr(Y \leq y) = \text{cdf of } Y$$

$$= \begin{cases} \cdot\ 0 & (y<-1) \\ \cdot\ \frac{1}{2} & (-1 \leq y < -\frac{1}{2}) \\ \cdot\ \frac{1}{4} & (-\frac{1}{2} \leq y < 0) \\ \cdot\ \frac{3}{4} & (0 \leq y < \frac{1}{2}) \\ \cdot\ \frac{y+1}{2} & (\frac{1}{2} \leq y \leq 1) \\ \cdot\ 1 & (y>1) \end{cases}$$

(b) $Y: -\infty \to \infty$

$Z: 0 \to 1$

$\frac{dz}{dy} = f(y)$

$\int_{-\infty}^{\infty} f(y)\,dy = \int_0^1 dz$

$\therefore Z \sim U(0,1)$

(c) $E[Y|X] =$ ① $x \in [\frac{-1}{2}, \frac{1}{2}]$ … $P_r(Y=0\,|X=x) = 1$

$\therefore E[Y|X=x] = 0$ ② $x \notin [\frac{-1}{2}, \frac{1}{2}]$ … $P_r(Y=x\,|X=x) = 1$

$\therefore E[Y|X=x] = x$

$E[Y] = EE[Y|X] = \int_{-1}^{1} E[Y|X] \cdot \frac{1}{2} dx$

$= \int_{\frac{-1}{2}}^{\frac{1}{2}} 0 \cdot \frac{1}{2} dx + \int_{\frac{1}{2}}^{1} \frac{x}{2} dx + \int_{-1}^{\frac{-1}{2}} \frac{x}{2} dx$

$= \left[\frac{x^2}{4}\right]_{\frac{1}{2}}^{1} + \left[\frac{x^2}{4}\right]_{-1}^{\frac{-1}{2}} = \frac{3}{16} - \frac{3}{16} = 0 \quad \therefore E[Y] = 0$

同様道理 $E[Y^2|X=x] = \begin{cases} 0 & (x \in [\frac{-1}{2}, \frac{1}{2}]) \\ x^2 & (x \notin [\frac{-1}{2}, \frac{1}{2}]) \end{cases}$

$\therefore E[Y^2] = EE[Y^2|X=x]$

$= \int_{-1}^{1} E[Y^2|x] \frac{1}{2} dx = \int_{\frac{-1}{2}}^{\frac{1}{2}} 0 \cdot \frac{1}{2} dx + \int_{\frac{1}{2}}^{1} \frac{x^2}{2} dx$

$+ \int_{-1}^{\frac{-1}{2}} \frac{x^2}{2} dx = \left[\frac{x^3}{6}\right]_{\frac{1}{2}}^{1} + \left[\frac{x^3}{6}\right]_{-1}^{\frac{-1}{2}} = \frac{7}{48} \times 2 = \frac{7}{24}$

$\therefore E[Y^2] - E[Y]^2 = V[Y] = \frac{7}{24}$

$E[Z) = \int_0^1 z\,dz = \left[\frac{z^2}{2}\right]_0^1 = \frac{1}{2}$

$E[Z^2) = \int_0^1 z^2 dz = \left[\frac{z^3}{3}\right]_0^1 = \frac{1}{3}$

$$V[Z] = E[Z^2) - E[Z)^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

$$\therefore \begin{cases} E[Y) = 0 \\ V[Y) = \frac{7}{24} \\ E[Z] = \frac{1}{2} \\ V[Z) = \frac{1}{12} \end{cases}$$

$\boxed{10}$ $E[X] = \lim_{N\to\infty} \sum_{x=0}^{N} x \cdot Pr(X=x)$

$= \lim_{N\to\infty} \sum_{x=1}^{N} x \cdot Pr(X=x) = \lim_{N\to\infty} \sum_{x=1}^{N} \sum_{y=1}^{x} Pr(X=x)$

$\sum_{x=1}^{N} \sum_{y=1}^{x} Pr(X=x)$ 可以寫成 $\sum_{y=1}^{N} \sum_{x=y}^{N} Pr(X=x)$.

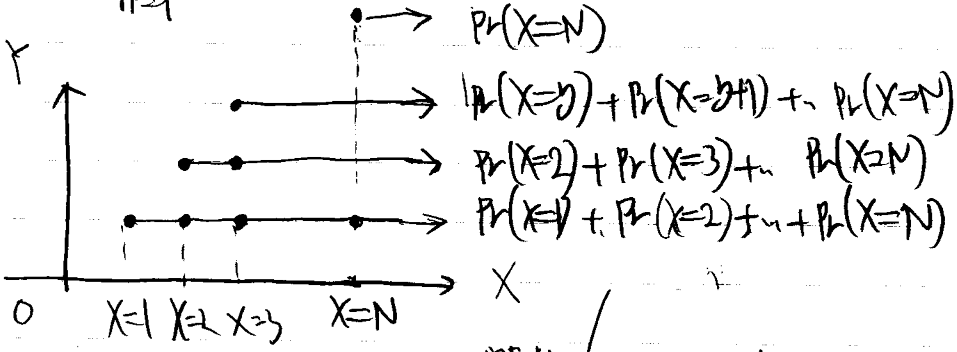
$\sum_{x=y}^{N} Pr(X=x) = Pr(X \geq y)$

$\therefore \sum_{x=1}^{N} \sum_{y=1}^{x} Pr(X=x) = \sum_{y=1}^{N} Pr(X \geq y)$

$\therefore \lim_{N\to\infty} \sum_{x=0}^{N} x \cdot Pr(X=x) = \lim_{N\to\infty} \sum_{y=1}^{N} Pr(X \geq y)$

$= \sum_{n=1}^{\infty} Pr(X \geq n)$ $(N\to\infty,\ y\to n)$ ∴ 證明完成

㊟

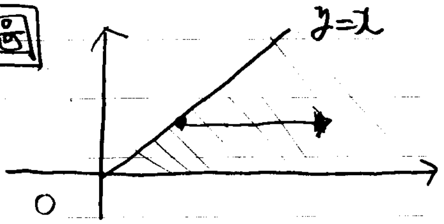


一般式 ↓ $\sum_{y=1}^{N} \sum_{x=y}^{N} Pr(X=x)$

$\boxed{1}$ $E[X] = \int_0^\infty x\, f_X(x)\,dx = \int_0^\infty \left(\int_0^x dy\right) f_X(x)\,dx$

$(X \geq 0)$

圖



$\int_0^\infty \int_0^x f_X(x)\,dy\,dx$

$= \int_0^\infty \int_y^\infty f_X(x)\,dx\,dy = \int_0^\infty \left[F_X(x)\right]_y^\infty dy$

$= \int_0^\infty (1-F_X(y))\,dy = \int_0^\infty (1-F_X(x))\,dx$

∴ 證明完成

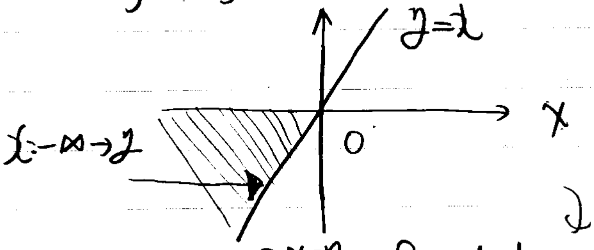
接著考慮一般的情形.

$$E[X] = \underbrace{\int_0^\infty x\, f_X(x)\,dx}_{\int_0^\infty (1-F_X(x))\,dx} + \underbrace{\int_{-\infty}^0 x\, f_X(x)\,dx}_{\int_{-\infty}^0 \int_0^x dy\, f_X(x)\,dx}$$

$$\int_{-\infty}^0 \int_0^x dy\, f_X(x)\,dx = \int_{-\infty}^0 \int_0^x f_X(x)\,dy\,dx$$

注 $x \leq 0$ ∴ $\int_0^x \rightarrow -\int_x^0$

$$= -\int_{-\infty}^0 \int_x^0 f_X(x)\,dy\,dx$$



替換積分順序

$$= -\int_{y=-\infty}^{y=0} \int_{x=-\infty}^{x=y} f_X(x)\,dx\,dy$$

$$= -\int_{y=-\infty}^{y=0} \left[F_X(x)\right]_{-\infty}^{y} dy = -\int_{y=-\infty}^{y=0} F_X(y)\,dy$$

∴ 證明完成

$\boxed{12}$

$$\begin{pmatrix} \frac{dy_1}{dx_1} = \frac{x_1}{\sqrt{x_1^2+x_2^2}} & \frac{dy_1}{dx_2} = \frac{x_2}{\sqrt{x_1^2+x_2^2}} \\ \frac{dy_2}{dx_1} = \frac{1}{x_2} & \frac{dy_2}{dx_2} = \frac{-x_1}{x_2^2} \end{pmatrix}$$

$$\det = \frac{-x_1^2}{\sqrt{x_1^2+x_2^2}\,x_2^2} - \frac{1}{\sqrt{x_1^2+x_2^2}} = \frac{-1}{\sqrt{x_1^2+x_2^2}}\left(\frac{x_1^2}{x_2^2}+1\right)$$

$$-\frac{\sqrt{x_1^2+x_2^2}}{x_2^2} = -y_1 \cdot \frac{1+y_2^2}{y_1^2} = -\frac{1+y_2^2}{y_1}$$

$\because$

$$\left(\begin{array}{l} y_1^2 = x_1^2+x_2^2, \quad x_1 = x_2 y_2 \\ y_1^2 = x_2^2 y_2^2 + x_2^2 = (1+y_2^2)x_2^2 \\ x_2^2 = \frac{y_1^2}{1+y_2^2} \end{array} \right.$$

$$\therefore \quad dy_1 dy_2 = \frac{1+y_2^2}{y_1} dx_1 dx_2$$

$$全機率 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \underbrace{\frac{1}{2\pi}\exp\left(\frac{-1}{2}(x_1^2+x_2^2)\right)}_{\frac{1}{2\pi}\exp\left(\frac{-1}{2}y_1^2\right)} \underbrace{dx_1 dx_2}_{\frac{y_1}{1+y_2^2}dy_1 dy_2}$$

接下來考慮 y_1, y_2 取的範圍

$$1=\int_{\underset{x_1}{-\infty}}^{\infty}\int_{\underset{x_2}{-\infty}}^{\infty}\sim dx_1dx_2=\overset{①}{\int_0^\infty\int_0^\infty\sim dx_1dx_2}+\overset{②}{\int_0^\infty\int_{-\infty}^0\sim dx_1dx_2}$$

$$+\underset{③}{\int_{-\infty}^0\int_0^\infty\sim dx_1dx_2}+\underset{④}{\int_{-\infty}^0\int_{-\infty}^0\sim dx_1dx_2}$$

$$\begin{cases} ①\ x_1: 0\to\infty \quad x_2: 0\to\infty \\ \Rightarrow y_1: 0\to\infty \quad y_2: 0\to\infty \end{cases}$$

$$\begin{cases} ②\ x_1: 0\to\infty \quad x_2: -\infty\to 0 \\ \Rightarrow y_1: 0\to\infty \quad y_2: -\infty\to 0 \end{cases}$$

$$\begin{cases} ③\ x_1: -\infty\to 0 \quad x_2: 0\to\infty \\ \Rightarrow y_1: 0\to\infty \quad y_2: -\infty\to 0 \end{cases}$$

$$\begin{cases} ④\ x_1: -\infty\to 0 \quad x_2: -\infty\to 0 \\ \Rightarrow y_1: 0\to\infty \quad y_2: 0\to\infty \end{cases}$$

$$\Rightarrow\begin{cases} ①,④ \Rightarrow \int_{y_1:0\to\infty}\int_{y_2:0\to\infty}\sim dy_1dy_2 \\ ②,③ \Rightarrow \int_{y_1:0\to\infty}\int_{y_2:-\infty\to 0} dy_1dy_2 \end{cases}$$

因為出現兩個相同的部份，所以 ×2

$\therefore$ ①+②+③+④

$$=\int_{y_1:0\to\infty}\int_{y_2:-\infty\to\infty}\frac{1}{2\pi}e^{\frac{-y_1^2}{2}}\cdot\frac{y_1}{1+y_2^2}\times\underset{\sim\sim}{2}\,dy_1dy_2$$

$$=\int_0^\infty\int_{-\infty}^\infty\underbrace{\left(y_1\exp\left(\frac{-y_1^2}{2}\right)\right)\left(\frac{1}{\pi\cdot 1+y_2^2}\right)}_{f_{Y_1,Y_2}(y_1,y_2)}dy_1dy_2$$

㉓

$$\therefore\ f_{Y_1Y_2}(y_1,y_2) = y_1 \exp\left(\frac{-y^2}{2}\right) \cdot \frac{1}{\pi} \cdot \frac{1}{1+y_2^2}$$

$$f_{Y_1}(y_1) = \int_{-\infty}^{\infty} f_{Y_1Y_2}(y_1,y_2)\,dy_2 = y_1 \exp\left(-\frac{1}{2}y^2\right)$$

$$f_{Y_2}(y_2) = \int_{0}^{\infty} f_{Y_1Y_2}(y_1,y_2)\,dy_1 = \frac{1}{\pi} \cdot \frac{1}{1+y_2^2}$$

$$\therefore\ f_{Y_1}(y_1) \cdot f_{Y_2}(y_2) = f_{Y_1,Y_2}(y_1,y_2)$$

∴ Y_1, Y_2 獨立

$\boxed{13}$ $E[e^{tX}] = \frac{1}{(1-\lambda t)^{\theta}} = M(t)$

$\log M(t) = -\theta \log(1-\lambda t) = K(t)$

$K'(t) = \frac{\lambda\theta}{1-\lambda t}$ $\quad K'(0) = \lambda\theta$

$K''(t) = \frac{\lambda^2\theta}{(1-\lambda t)^2}$ $\quad K''(0) = \lambda^2\theta$

∴ $E[X] = \lambda\theta \quad V[X] = \lambda^2\theta$

$E[X^2] = \lambda^2\theta + \lambda^2\theta^2$

根據弱大數法則 $\frac{X_1+\cdots X_n}{n} \xrightarrow{P} \lambda\theta$，$\frac{X_1^2+\cdots X_n^2}{n} \xrightarrow{P} \lambda^2\theta+\lambda^2\theta^2$.

$\overline{X}$, $\frac{X_1^2+\cdots X_n^2}{n}$ 分別為 $\lambda\theta$, $\lambda^2\theta+\lambda^2\theta^2$ 之一致估計量.

先解以下方程式:

$$\begin{cases} \lambda\theta = \overline{X} & \cdots ① \\ \lambda^2\theta^2+\lambda^2\theta = \frac{X_1^2+\cdots X_n^2}{n} & \cdots ② \end{cases}$$

$$∴ \lambda^2\theta = \frac{X_1^2+\cdots X_n^2}{n} - \overline{X}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i-\overline{X})^2 \cdots ③$$

$$\begin{cases} ∴ \hat{\lambda}_{(MME)} = \frac{1}{n\overline{X}}\sum_{i=1}^{n}(X_i-\overline{X})^2 & (③/①) \Rightarrow ④ \\ \hat{\theta}_{(MME)} = \frac{n\overline{X}^2}{\sum_{i=1}^{n}(X_i-\overline{X})^2} & (①/④) \end{cases}$$

（根據Slutzky定理，$\hat{\lambda}$, $\hat{\theta}$ 為 λ, θ 之一致估計量）

$\boxed{14}$ 機率函數可寫為：

$$\Pr(X=x)=\left\{\theta^{x-1}(1-\theta)\right\}^{1-I(x=r+1)}\cdot\left\{\theta^{r}\right\}^{I(x=r+1)}$$

$$=\theta^{(x-1)(1-I(x=r+1))+rI(x=r+1)}\cdot(1-\theta)^{(1-I(x=r+1))}$$

$$=\theta^{(x-1)+(r-x+1)I(x=r+1)}(1-\theta)^{1-I(x=r+1)}$$

∴ 聯合機率函數為 $\Pr(X_1=x_1, X_2=x_2 \ldots X_n=x_n)$

$$=\theta^{\sum_{i=1}^{n}(x_i-1)+\sum_{i=1}^{n}(r-x_i+1)I(x_i=r+1)}\cdot(1-\theta)^{\sum_{i=1}^{n}(1-I(x_i=r+1))}$$

$$=L(\theta|x_1 \ldots x_n)$$

$$\log L(\theta|x_1 \ldots x_n)=\left\{\sum_{i=1}^{n}(x_i-1)+\sum_{i=1}^{n}(r-x_i+1)I(x_i=r+1)\right\}\log\theta$$
$$+\left\{\sum_{i=1}^{n}(1-I(x_i=r+1))\right\}\log(1-\theta)$$

$$\therefore \frac{\partial}{\partial\theta}\log L(\theta)=\frac{1}{\theta}\left\{\sum_{i=1}^{n}(x_i-1)+\sum_{i=1}^{n}(r-x_i+1)I(x_i=r+1)\right\}$$
$$-\frac{1}{1-\theta}\left(\sum_{i=1}^{n}(1-I(x_i=r+1))\right)=0$$

$$\Leftrightarrow (1-\theta)\left\{\sum_{i=1}^{n}(x_i-1)+\sum_{i=1}^{n}(r-x_i+1)I(x_i=r+1)\right\}$$
$$-\theta\left(\sum_{i=1}^{n}(1-I(x_i=r+1))\right)=0$$

$$\Leftrightarrow \sum_{i=1}^{n}(x_i-1)+\sum_{i=1}^{n}(r-X_i+1)I(X_i=r+1)=\theta\cdot\left\{\sum_{i=1}^{n}(x_i-1)+\sum_{i=1}^{n}(r-X_i+1)I(X_i=r+1)+\sum_{i=1}^{n}\left(1-I(X_i=r+1)\right)\right\}$$

$$\therefore\ \hat{\theta}_{MLE}=\frac{\sum_{i=1}^{n}(x_i-1)+\sum_{i=1}^{n}(r-X_i+1)I(X_i=r+1)}{\sum_{i=1}^{n}(x_i-1)+\sum_{i=1}^{n}(r-X_i+1)I(X_i=r+1)+\sum_{i=1}^{n}\left(1-I(X_i=r+1)\right)}$$

$$M\overset{\mathrm{def}}{:=}\sum_{i=1}^{n}I(X_i=r+1)$$

$$\hat{\theta}_{MLE}=\frac{\sum_{i=1}^{n}X_i-n+M(r+1)-\sum_{i=1}^{n}x_i I(X_i=r+1)}{\sum_{i=1}^{n}X_i-\cancel{n}+M(r\cancel{+1})-\sum_{i=1}^{n}x_i I(X_i=r+1)+\cancel{n}-\cancel{M}}$$

$$=\frac{\sum_{i=1}^{n}X_i-n+M(r+1)-\sum_{i=1}^{n}\overset{r+1}{\boxed{X_i}}I(X_i=r+1)}{\sum_{i=1}^{n}x_i+Mr-\sum_{i=1}^{n}\underset{\downarrow\ r+1}{\boxed{X_i}}I(X_i=r+1)}$$

$$=\frac{\sum_{i=1}^{n}x_i-n+M(r+1)-M(r+1)}{\sum_{i=1}^{n}x_i+Mr-(r+1)M}$$

$$=\underbrace{\frac{\sum_{i=1}^{n}x_i-n}{\sum_{i=1}^{n}x_i-M}}$$

∴ 證明完成

$\boxed{15}$ 在固定N, n之情況下，求b之MLE.

（在此假設觀察到一個樣本X）

$$Pr(X=x|b) = \frac{{}_bC_x \cdot {}_{N-b}C_{n-x}}{{}_NC_x} = L(b|X)$$

b為離散數值，因此觀察 L(b|X) vs L(b-1|X) 之比例

$$\frac{L(b|X)}{L(b-1|x)} = \frac{{}_bC_x \cdot {}_{N-b}C_{n-x}}{{}_{b-1}C_x \cdot {}_{N-b+1}C_{n-x}} \cdot \frac{\cancel{{}_NC_x}}{\cancel{{}_NC_x}}$$

$$= \frac{\overset{b}{\cancel{b!}}}{\cancel{x!}\underset{(b-x)}{\cancel{(b-x)!}}} \cdot \frac{\cancel{x!}\cancel{(b-1-x)!}}{\cancel{(b-1)!}} \cdot \frac{\cancel{(N-b)!}}{\cancel{(n-x)!}\cancel{(N-b-n+x)!}}$$

$$\cdot \frac{\cancel{(n-x)!}\cancel{(N-b+1-n+x)!}}{\cancel{(N-b+1)!}} \quad \rightarrow (N-b+1-n+x) \quad \rightarrow (N-b+1)$$

$$= \frac{b}{b-x} \cdot \frac{(N-b+1-n+x)}{(N-b+1)}$$

分母－分子 ≥ 0

$$\Leftrightarrow b(N-b+1-n+x) - (b-x)(N-b+1) \geq 0$$

$$\Leftrightarrow (\cancel{bN}-\cancel{b^2}+\cancel{b}-nb+\cancel{bx}) - (\cancel{bN}-\cancel{b^2}+\cancel{b}-Nx+\cancel{bx}-x)$$

$$= Nx+x-nb \geq 0$$

$x(N+1) \geq nb$

$b \leq x\left(\frac{N+1}{n}\right) \Leftrightarrow \frac{L(b|X)}{L(b-1|X)} \geq 1$

∴

$1 < \frac{L(2|X)}{L(1|X)}, \quad 1 < \frac{L(3|X)}{L(2|X)}, \cdots 1 \underset{(<)}{\leq} \frac{L([\frac{N+1}{n}x]|X)}{L([\frac{N+1}{n}x]-1|X)}, \quad \frac{L([\frac{N+1}{n}x]+1|X)}{L([\frac{N+1}{n}x]|X)} < 1$

∴ $L(1|X) < L(2|X) < \cdots < L([\frac{N+1}{n}x]-1|X) \underset{(<)}{\leq} L([\frac{N+1}{n}x]|X) > \cdots$

若 $\frac{N+1}{n}x$ 為整數，且 $b=\frac{N+1}{n}x$，

則 $\frac{L(b|X)}{L(b-1|X)}=1$ ∴ $b=\frac{N+1}{n}x$ 和 $\frac{N+1}{n}x-1$ 時，L為最大。

總而言之，① $\frac{N+1}{n}x$ 非整數 ∴ $b=[\frac{N+1}{n}x]$ 時 L為最大。

$\hat{b}_{MLE} = [\frac{N+1}{n}x)$

② $\frac{N+1}{n}x$ 為整數 ∴ $\frac{N+1}{n}x, \frac{N+1}{n}x-1$ 皆令L為最大。

$\hat{b}_{MLE} = \frac{N+1}{n}x, \frac{N+1}{n}x-1$

$\boxed{16}$ $f_\theta(x_1\cdots x_n) = \theta^n (x_1\cdots x_n)^{\theta-1} \exp(-x_1^\theta - x_2^\theta \cdots - x_n^\theta)$

$\log f_\theta(x_1\cdots x_n) = \log\theta^n + (\theta-1)\log(x_1x_2\cdots x_n) - x_1^\theta - x_2^\theta \cdots - x_n^\theta$

$L'(\theta) = \frac{\partial \log f_\theta(x_1\cdots x_n)}{\partial\theta} = \frac{n}{\theta} + \log(x_1x_2\cdots x_n) - x_1^\theta\cdot\log x_1 \cdots - x_n^\theta\log x_n$

$= \frac{n}{\theta} - \sum_{j=1}^{n}(x_j^\theta - 1)\log x_j$

$L'(\theta)=0 \Leftrightarrow \underbrace{\frac{n}{\theta} = \sum_{j=1}^{n}(x_j^\theta - 1)\log x_j}$ ㊧ $= h(\theta)$ ㊨ $= g(\theta)$

$g(\theta|\vec{x}) \overset{def}{:=} \sum_{j=1}^{n}(x_j^\theta - 1)\log x_j$ (右)

$\frac{\partial g}{\partial\theta} = \sum_{j=1}^{n} x_j^\theta (\log x_j)^2 \; (= g'(\theta))$

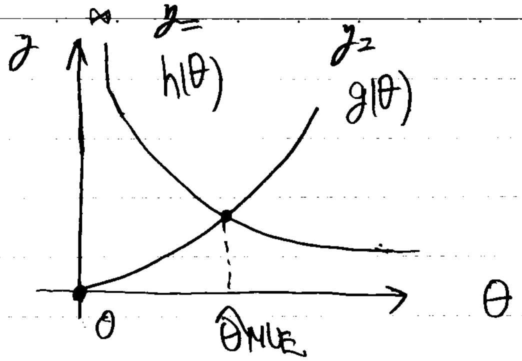
$g'(\theta) \geqq 0$ ($\because$ $(\log x_j)^2 \geqq 0$. 方$\geqq 0$)

$\therefore$ $g(\theta|x_1 \sim x_n)$ 為遞增函數. 且 $g(1)=0$.

$h(\theta) \overset{def}{:=} \frac{n}{\theta}$

$\therefore$ $\hat{\theta}_{MLE}$ 為 $h(\theta) = g(\theta|\vec{x})$ 之實數解。

$\lim_{\theta\to+0} h(\theta) = +\infty$, $\lim_{\theta\to\infty} h(\theta) = +0$.



$x_j > 0$ $\quad {}^{\exists}\theta > 0$ st $g(\theta|x) > 0$

($h(\theta)$ 為遞減函數, 且 收斂至 0 (+0)
$g(\theta)$ 為遞增函數 且 ${}^{\exists}\theta$. st $g(\theta|X) > 0$

∴ 一定會存在唯一的 θ 使得 $h(\theta) = g(\theta)$. (如圖)

總而言之, 存在唯一的最大概似估計量 $\hat{\theta}_{MLE}$

(註 $L''(\theta) = \frac{-n}{\theta^2} - \sum_{j=1}^{n} x_j^{\theta} (\log x_j)^2 < 0$)

$\boxed{17}$ $f(x_1 \cdots x_n \mid \theta) = \left(\frac{1}{\sqrt{2\pi}}\right)^n \exp\left(-\frac{1}{2}\sum_{i=1}^{n}(x_i-\theta)^2\right)$

$L(\theta) = \log f(x_1 \cdots x_n \mid \theta) = -\frac{1}{2}\log(2\pi) - \frac{1}{2}\sum_{i=1}^{n}(x_i-\theta)^2$

$L'(\theta) = \frac{\partial}{\partial\theta}\log f(x_1 \cdots x_n \mid \theta) = \sum_{i=1}^{n}(x_i-\theta) = n(\overline{X}-\theta)$

case: $\overline{X}>0$

θ	0		$\overline{X}$	
$L'(\theta)$		+	0	−
$L(\theta)$		↗	max	↘

∴ $\theta = \overline{X}$ 時 $L(\theta)$ 為最大

case. $\overline{X} \leq 0$

θ	0	
$L'(\theta)$		−
$L(\theta)$		↘

$\theta>0$ 時, $L(\theta)$ 為遞減函數.

θ 不得取 0, 因此 MLE 不存在

(若允許 θ 取 0, 則 θ 之 MLE 為 0)

$(\theta \geq 0)$

(弟hu)

Advanced Statistical Inference I

Homework 6: Estimation, Likelihood, and Kernel smoother

Due Date: January 3rd 2017

1. Let X be a random variable with $EX^2 < \infty$, and $Y = |X|$. Assume that X has a density symmetric about 0. Show that random variables X and Y are uncorrelated, but they are not independent.

2. Suppose U and V are independent with exponential distribution with parameter λ. (A random variable T is exponentially distributed with parameter λ if its density is given by $f(t) = \lambda \exp(-\lambda t)$ with support $T > 0$.) Define $X = U + V$ and $Y = UV$.

   (a) Derive the joint density of (X, Y).

   (b) Find the best linear predictor of Y given X.

   (c) Find the best predictor of Y given X.

3. Let $X_1, \cdots, X_n$ be IID Cauchy random variables. What is the distribution of $\bar{X}_n$? Does this result make sense?
   Hint: look up the characteristic function of the Cauchy RV.

4. Let X_i be i.i.d. exponential random variables with rate one, $i \geq 1$. Let N be a geometric random variable with success probability p, $0 < p < 1$, i.e. $P(N = k) = (1-p)^{k-1}p$, $k = 1, 2, \cdots$, and independent of all X_i, $i \geq 1$. Find the distribution of $\sum_{i=1}^{N} X_i$.

5. Let X_i be independent Gamma(a_i, b) random variables, $i = 1, \cdots, n$.

   (a) Use the characteristic or moment generating function to show that $\sum_{i=1}^{n} X_i$ is Gamma$(\sum_{i=1}^{n} a_i, b)$.

   (b) For a positive constant C, what is the distrition of CX_i?

   (c) Show that $Y_1 = X_1/(X_1+X_2)$ and $Y_2 = X_1+X_2$ are independent. Derive the distribution of Y_1.

~~6.~~ 4 (2.) Suppose U and V are independent with exponential distribution with parameter λ. (A random variable T is exponentially distributed with parameter λ if its density is given by $f(t) = \lambda \exp(-\lambda t)$ with support $T > 0$.) Define $X = U + V$ and $Y = UV$.

   (a) Derive the joint density of (X, Y).

   (b) Find the best linear predictor of Y given X.

   (c) Find the best predictor of Y given X.

7. Suppose that X and Y have a joint pdf given by

$$f_{X,Y}(x, y) = \begin{cases} 2 & 0 < x < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Find $E[Y|X = x_0]$.

   (b) Plot the CEF (conditional expectation function) $E[Y|X = x]$. Can you explain heuristically why the function has this particular form?

   (c) Find $E[YX^3 + 1|X = x_0]$.

(d) Find $V(Y|X = x_0)$. Does its dependence on x_0 make sense to you?

8. The Weibull cumulative distribution function is

$$F(x) = 1 - \exp\left[-\left(\frac{x}{\alpha}\right)^{\beta}\right], \quad x \geq 0, \alpha > 0, \beta > 0.$$

   (a) Find the density function.

   (b) Show that if W follows a Weibull distribution, then $X = (W/\alpha)^{\beta}$ follows an exponential distribution.

   (c) How could Weibull random variables be generated from a uniform random number generator?

9. Let $X_1,$ ~~$X+2$~~ X_2 $\sim Uniform(0, \theta)$ where $\theta > 0$.

   (a) Find the distribution of (X_1, X_2) given T where $T = \max\{X_1, X_2\}$.

   (b) Show that $X_1 + X_2$ is not sufficient.

10. Let $X_1, \cdots, X_n \sim Uniform(-\theta, 2\theta)$ where $\theta > 0$. Find the likelihood function.

11. Consider four observations -1, 0, 0.5, and 3 and evaluation at $x = 0$, $x = 0.5$, and $x = 1$. Using a bandwidth of 1, determine Gaussian kernel density estimate at $x = 0$, $x = 0.5$, and $x = 1$. Note that the resulting estimate at $x = 0$ should be 0.249.

12. ✓ You are given a kernel $K(\cdot)$ which satisfies $K(u) \geq 0$, $\int K(u)du = 1$, $\int uK(u)du = 0$, $\int u^2K(u)du = \sigma_K^2 < \infty$. You are also given a bandwidth $h > 0$, and a collection of n univariate observations $x_1, x_2, \cdots, x_n$. Assume that the data are independent samples from some unknown density f.

   (a) Give the formula for $\hat{f}_h$, the kernel density estimate corresponding to these data, this bandwidth, and this kernel.

   (b) Find the expectation of a random variable whose density is $\hat{f}$, in terms of the sample moments, h, and the properties of the kernel function.

   (c) Find the variance of a random variable whose density is $\hat{f}_h$, in terms of the sample moments, h, and the properties of h the kernel function.

   (d) How must h change as n grows to ensure that the expectation and variance of $\hat{f}_h$ will converge on the expectation and variance of f?

13. ✓ Let $X_1, \cdots, X_n$ be a random sample from the density

$$f(x|\theta) = \theta x^{\theta-1} I_{(0,1)}(x).$$

   The parameter space is $\Theta = (0, \infty)$.

   (a) Verify that $-\log X_1 = Y$ has an exponential distribution.

   (b) Find the Cramer Rao lower bound for unbiased estimators of $\tau(\theta) = 1/\theta$.

   (c) Show that $-\sum_{i=1}^{n} \log X_i/n$ is an UMVUE of $1/\theta$.

✓14. Let $X_1, \cdots, X_n$ be iid $U[0, \theta]$, and suppose that we want to estimate θ.

(a) Show that $X_{(n)} = \max_{1 \le i \le n} X_i$ is sufficient for θ.

(b) Let $\tilde{\theta} = 2X_1$, show that $\tilde{\theta}$ is an unbiased estimator for θ.

(c) Find $E(\tilde{\theta}|X_{(n)})$ and show that it is a UNVUE of θ

✓15. Suppose $X_1, \cdots, X_n$ are iid Poisson(λ), and let $\theta = \exp(-\lambda)$ which is $P(X_1 = 0)$.

(a) Show that $T = \sum_{i=1}^{n} X_i$ is sufficient for θ.

(b) Consider an estimator $\tilde{\theta} = 1_{X_1=0}$. Show that $\tilde{\theta}$ is an unbiased estimator of θ.

(c) Show that $E(\tilde{\theta}|T = t) = (1 - 1/n)^{\sum_i X_i}$.

✓16. Let $\mathbf{X} = (X_1, \ldots, X_n)$ be a sample from an exponential distribution with individual densities

$$f(x; \theta) = \frac{1}{\theta} \exp\left(-\frac{x}{\theta}\right), \quad x > 0,$$

where $\theta > 0$ is unknown.

θ^2

(a) Show that $\tilde{\theta} = X_1^2/2$ is an unbiased estimator of $g(\theta) = \boxed{\theta;}$

(b) Show that $t(\mathbf{X}) = \sum_{i=1}^{n} X_i$ is sufficient for θ.

(c) Rao-Blackwellize $\tilde{\theta}$ to find an improved unbiased estimator of $g(\theta)$ which is denoted by $\hat{\theta}_u$;
Hint: You may use without proof that the distribution of $U = X_1/(\sum_{i=1}^{n} X_i)$ follows a Beta-distribution $\mathcal{B}(1, n-1)$, where $\mathcal{B}(\alpha, \beta)$ is the distribution with density

$$f(t; \alpha, \beta) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} t^{\alpha - 1}(1 - t)^{\beta - 1}, \quad 0 < t < 1.$$

(d) Find the MLE of θ^2 and denote it by $\hat{\theta}_{mle}$. Compare the mean square error of the two estimates.

$\frac{T^{n-1}}{\theta^n \Gamma(n)} \exp\left(\frac{-t}{\theta}\right)$

$X_1 + X_n \sim \Gamma(n, \theta)$

$\frac{t^{n-1}}{\theta^n \Gamma(n)} \exp\left(-\frac{t}{\theta}\right)$ $\quad \frac{t^{n+1}}{\theta} \exp\left(\frac{-t}{\theta}\right)$ $\quad \frac{t}{\theta} = \lambda$ $\quad \frac{dx}{dt} = \theta$

$\frac{(\theta x)^{n+1}}{\theta^n} \exp(-x)\, \theta\, dx = n\Gamma(n+1)\theta^2$

$n\Gamma$

$\theta^2 \cdot \Gamma(n+2)$

$\frac{t^{n+1}}{\theta^n \Gamma(n)} \exp\left(\frac{t}{\theta}\right)$ $\quad \frac{\theta^2 \cdot \Gamma(n+2)}{\Gamma(n)}$ $\quad (n+1)n\theta^2$

①

高等統計推論 (I) 作業6. R05246013 森元俊成

$\boxed{1}$ $E[XY] = E[X|X|] = \int_{x\geq 0} x^2 f(x)\,dx + \int_{x<0} (-x^2) f(x)\,dx$

$= \int_0^\infty x^2 f(x)\,dx + \int_{-\infty}^0 (-x^2) f(x)\,dx$

$-x = y \quad \frac{dx}{dy} = -1$

$= \int_0^\infty x^2 f(x)\,dx + \int_\infty^0 (-y^2) f(-y)\cdot(-1)\,dy$

$= \int_0^\infty x^2 f(x)\,dx - \int_0^\infty y^2 \underset{\underset{f(y)}{\|}}{f(-y)}\,dy$

$= 0$

$E[X] = \int_{-\infty}^\infty x f(x)\,dx = 0 \quad (\because f(x) = f(-x))$

$\therefore E[XY] - E[X]E[Y] = 0 \ (= cov[X,Y])$

$\therefore$ X與Y為 uncorrelated. 」

「 $\underset{\underset{Pr(X\leq x,\ |X|\leq y)}{\|}}{Pr(X\leq x,\ Y\leq y)}$ vs $\underset{\underset{Pr(X\leq x)\,Pr(-y\leq X\leq y)}{\|}}{Pr(X\leq x)\,Pr(Y\leq y)}$ ⊛ 考慮 $y\leq x$ 之情況

$\|$

$Pr(-y \leq X \leq \min\{x,y\})$

$\|$ $(\because y\leq x)$

$Pr(-y \leq X \leq y)$

假設 $\Pr(X \leqq x, Y \leqq y) = \Pr(X \leqq x) \Pr(Y \leqq y)$ $(0<y<x)$

$\Rightarrow$. $\Pr(X \leqq x) = 1$. or $\Pr(-y \leqq X \leqq y) = 0$ (for all $0<y<x$)

矛盾) $\therefore$ $\Pr(X \leqq X, Y \leqq y) \neq \Pr(X \leqq X) \Pr(Y \leqq y)$

$\therefore$ 並非獨立.

$\boxed{2}$ $U, V \sim e(\lambda)$ (iid) (mean $\frac{1}{\lambda}$)

(a) $J = \begin{pmatrix} \frac{\partial X}{\partial u} & \frac{\partial X}{\partial v} \\ \frac{\partial Y}{\partial u} & \frac{\partial Y}{\partial v} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ v & u \end{pmatrix}$ $\therefore \det J = u - v$

$\therefore |\det J| = |u-v|$

$dxdy = |u-v| \, dudt$

考慮t之2次方程式… $(t-U)(t-V)=0$

$t^2-(U+V)t+UV=0$ $g(t) \overset{def}{:=} t^2-(U+V)t+UV = t^2-Xt+Y$

$g(t)=0$ 有兩個正實數解（含重解）

$\therefore \frac{X}{2} \geq 0$, $X^2-4Y \geq 0$, $g(0) = Y \geq 0$

$g(t)=0$ 之解為 U, V $\left(\frac{X \pm \sqrt{X^2-4Y}}{2}\right)$

case① $(U,V) = \left(\frac{X+\sqrt{X^2-4Y}}{2}, \frac{X-\sqrt{X^2-4Y}}{2}\right)$ $|U-V| = \sqrt{X^2-4Y}$

case② $(U,V) = \left(\frac{X-\sqrt{X^2-4Y}}{2}, \frac{X+\sqrt{X^2-4Y}}{2}\right)$ $|U-V| = \sqrt{X^2-4Y}$

$$\int_{U,V \geq 0} f(u)f(t)\,dudt = \int_{u,v \geq 0} \lambda^2 \exp(-\lambda(t+u))\,dudt$$

$$= \int_{①} f\left(\frac{X+\sqrt{X^2-4Y}}{2}\right) f\left(\frac{X-\sqrt{X^2-4Y}}{2}\right) \frac{1}{\sqrt{X^2-4Y}}\,dxdy$$

$$+ \int_{②} f\left(\frac{X-\sqrt{X^2-4Y}}{2}\right) f\left(\frac{X+\sqrt{X^2-4Y}}{2}\right) \frac{1}{\sqrt{X^2-4Y}}\,dxdy$$

$$= \int_{\substack{X \geq 0, Y \geq 0 \\ \sqrt{X^2-4Y} \geq 0}} 2\lambda^2 \exp(-\lambda X) \cdot \frac{1}{\sqrt{X^2-4Y}}\,dxdy$$

∴ $f_{XY}(x,y) = 2\lambda^2\exp(-\lambda x)\cdot\frac{1}{\sqrt{x^2-4y}}$ $(x\geq0, y\geq0, x^2-4y\geq0)$

(b) 求 α, β 使 $E[(Y-\alpha X-\beta)^2] = S(\alpha,\beta)$ 為最小.

$\frac{\partial S}{\partial\alpha}=0,\ \frac{\partial S}{\partial\beta}=0\ldots$ 得 $\hat\beta = \frac{cov[X,Y]}{V[X]}$ $\hat\alpha = E[Y] - \frac{cov[X,Y]}{V[X]}E[X]$

$\hat\alpha = -\frac{1}{\lambda^2},\ \hat\beta = \frac{1}{\lambda}$ ∴ $-\frac{1}{\lambda^2}X + \frac{1}{\lambda}$ 由

$\alpha X+\beta$ (best linear predictor)

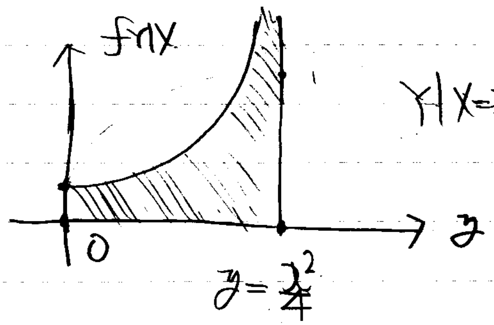
詳細的計算過程, 請參閱 Note.

(c) 求 Y|X=x 之分布.

$$\int_{y=0\to\frac{x^2}{4}} 2\lambda^2\exp(-\lambda x)\cdot\frac{1}{\sqrt{x^2-4y}}dy = 2\lambda^2\exp(-\lambda x)\left[-\frac{1}{2}(x^2-4y)^{\frac{1}{2}}\right]_{y=0}^{y=\frac{x^2}{4}}$$

$= \lambda^2 x\exp(-\lambda x)$ ∴ $f_X(x) = \lambda^2 x\exp(-\lambda x)$

∴ $f_{Y|X}(y|x) = \frac{1}{x}\frac{2}{\sqrt{x^2-4y}}$ $(0\leq y\leq\frac{x^2}{4})$ $(F_{Y|X}(y|x) = 1-\frac{1}{x}\sqrt{x^2-4y})$



Y|X=x 之 pdf

$y=\frac{x^2}{4}$ 時, 其機率密度為最高.

∴ $\frac{X^2}{4}$

$\boxed{3}$ $X_1 \cdots X_n \sim \text{Cauchy}(0,1)$

利用 characteristic function. $\phi(t) \overset{\text{def.}}{=}$

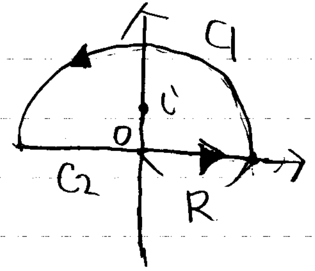
$$E[e^{itX}] = \int_{x=-\infty}^{x=\infty} \frac{1}{\pi}\cdot\frac{1}{1+x^2}\, e^{itx}\, dx$$

$(t>0)$

· 考慮以下複函數之積分

$$\oint_C \frac{1}{\pi}\cdot\frac{e^{itz}}{1+z^2}\,dz \quad (t>0)$$

$(C = C_1 + C_2)$



特異點: $z=i$ $\quad \operatorname*{Res}_{z=i} f(z) = \lim_{z\to i}(z-i)f(z)$

$$\lim_{z\to i}\frac{1}{\pi}\frac{e^{itz}}{z+i} = \frac{e^{-t}}{2\pi i}$$

$$\therefore \oint_C \frac{1}{\pi}\frac{1}{1+z^2}e^{itz}\,dz\ (t>0) = (2\pi i)\operatorname*{Res}_{z=i} f(z) = e^{-t}\ (t>0);$$

接著考慮 $\int_{C_1}\frac{1}{\pi}\frac{e^{itz}}{1+z^2}\,dz$ 之積分

$$\left|\int_{C_1}\frac{1}{\pi}\frac{e^{itz}}{1+z^2}\,dz\right| \le \int_{C_1}\frac{1}{\pi}\frac{|e^{itz}|}{|1+z^2|}\,dz \le \int_{C_1}\frac{1}{\pi}\frac{1}{|z|^2-1}\,dz$$

$$= \frac{1}{\pi}\cdot\frac{2\pi R}{R^2-1} = \frac{2R}{R^2-1} \qquad \lim_{R\to\infty}\frac{2R}{R^2-1} = 0$$

∵ $R\to\infty$ ∴ $\oint_C \frac{1}{\pi}\cdot\frac{e^{itz}}{1+z^2}dz = \lim_{R\to\infty}\int_{C_1}\frac{1}{\pi}\cdot\frac{e^{itz}}{1+z^2}dz$

$= \int_{-\infty}^{\infty}\frac{1}{\pi}\cdot\frac{e^{itx}}{1+x^2}dx = e^{-t}\ (t>0)$

∴ $\phi(t)=e^{-t}\ (t>0)$ $\phi(t)$ 應滿足 $\phi(t)=\overline{\phi(t)}$

∴ $\underset{\sim\sim\sim\sim\sim\sim}{\phi(t) = e^{-|t|}}$

$E\left[e^{it\left(\frac{X_1+\cdots+X_n}{n}\right)}\right] = E\left[e^{i\left(\frac{t}{n}\right)(X_1)}\right]^n = \left(e^{-\left|\frac{t}{n}\right|}\right)^n = e^{-|t|}$

∴ $\frac{X_1+\cdots+X_n}{n} \sim \text{Cauchy}(0,1)$

Cauchy 分佈不存在 $E[X]$, $E[X^2]$, 因此 ~~中央極限定理~~ (or 大數法則) 不成立. ∵ $\bar{X}$ ~~$\xrightarrow{d}$ Normal~~.

⑦

$\boxed{4}$. $T \overset{\text{def}}{:=} \sum_{i=1}^{N} X_i$

$T \mid N=n \sim \Gamma(n, 1)$

$N \sim G(p)$ （N表示試驗次數，而並非失敗次數）

$Pr(N=n) = (1-p)^{n-1} \cdot p$

$f(T=t, N=n) = f_{T|N}(T=t \mid N=n) \cdot Pr(N=n)$

$= \frac{t^{n-1}}{\Gamma(n)} \exp(-t) \cdot (1-p)^{n-1} \cdot p = \frac{1}{\Gamma(n)} \exp(-t) \cdot p \, (t(1-p))^{n-1}$

$\sum_{N=1}^{\infty} f(T=t, N=n) = f_T(t) = \sum_{n=1}^{\infty} p\exp(-t) \cdot \frac{(t(1-p))^{n-1}}{\Gamma(n)}$

$= \sum_{n=0}^{\infty} p\exp(-t) \cdot \frac{(t(1-p))^{n}}{\Gamma(n+1)} = p\exp(-t) \cdot \exp t(1-p) = p\exp(-pt)$

㊟ $e^{\lambda} = 1+\lambda+\frac{\lambda^2}{2!}+\dots\frac{\lambda^n}{n!}+\dots$

由此可知 $T\left(=\sum_{i=1}^{N} X_i\right)$ 之周邊分布為 $\exp(p)$

$\boxed{5}$ $E[e^{tX_i}] = \int_0^\infty \frac{x^{a_i-1}}{\Gamma(a_i)\cdot b^{a_i}} e^{tx} \cdot \exp\left(-\frac{x}{b}\right) dx$

(a)

$$= \int_0^\infty \frac{x^{a_i-1}}{\Gamma(a_i)\cdot b^{a_i}} \exp\left(-\left(\underbrace{\frac{1}{b}-t}_{\lambda>0}\right)x\right) dx \qquad y := \lambda x \qquad \frac{dy}{dx} = \lambda$$

$$= \int_0^\infty \frac{\left(\frac{y}{\lambda}\right)^{a_i-1}}{\Gamma(a_i)\, b^{a_i}} \exp(y) \cdot \frac{dy}{\lambda} = \int_0^\infty \frac{y^{a_i-1}}{\Gamma(a_i)} \frac{1}{(\lambda b)^{a_i}} dy = \frac{1}{(\lambda b)^{a_i}}$$

$$= \frac{1}{(1-bt)^{a_i}} \quad \therefore M_{X_i}(t) = (1-bt)^{-a_i}$$

$$E\left[e^{t(X_1+\cdots X_n)}\right] = E\left[e^{tX_1}\right] E\left[e^{tX_2}\right] \cdot E\left[e^{tX_n}\right]$$

$= (1-bt)^{-(a_1+\cdots a_n)}$ ∴ 由此可知 $X_1+\cdots X_n \sim \Gamma(a_1+\cdots a_n, b)$

(b) $E\left[e^{t(cX_i)}\right] = E\left[e^{(tc)X_i}\right] = M_{X_i}(ct) = (1-bc)^{-a_i}$

$\therefore cX_i \sim \Gamma(a_i, bc)$

(c) $\begin{cases} X_1 = Y_1Y_2 & (\geqq 0) \\ X_2 = Y_2 - Y_1Y_2 = Y_2(1-Y_1) & (\geqq 0) \end{cases}$

Jacobian : $\begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{pmatrix} = \begin{pmatrix} y_2 & y_1 \\ -y_2 & 1-y_1 \end{pmatrix}$ $\qquad \det J = y_2$

$X_1 \geqq 0,\ X_2 \geqq 0 \quad \therefore Y_1 Y_2 \geqq 0$ and $Y_2(1-Y_1) \geqq 0$

$\Rightarrow 0 \leqq Y_1 \leqq 1,\ Y_2 \geqq 0$

⑨

$\boxed{5}$ (c)

$$1=\iint \frac{x_1^{a_1-1}}{\Gamma(a_1)\,b^{a_1}}\exp\left(-\frac{x_1}{b}\right)\frac{x_2^{a_2-1}}{\Gamma(a_2)\,b^{a_2}}\exp\left(\frac{-x_2}{b}\right)dx_1dx_2$$

$$=\iint_{\substack{0\le y_1\le 1\\ 0\le y_2<\infty}} \frac{(y_1y_2)^{a_1-1}\,(y_2(1-y_1))^{a_2-1}}{\Gamma(a_1)\Gamma(a_2)\,b^{a_1+a_2}}\exp\left(\frac{-1}{b}y_2\right)y_2\,dy_1dy_2$$

$$=\iint_{\substack{0\le y_1\le 1\\ 0\le y_2<\infty}} \frac{y_1^{a_1-1}(1-y_1)^{a_2-1}\,y_2^{a_1+a_2-1}}{\Gamma(a_1)\Gamma(a_2)\,b^{a_1+a_2}}\exp\left(\frac{-y_2}{b}\right)dy_1dy_2$$

$$=\iint_{\substack{0\le y_1\le 1\\ 0\le y_2<\infty}} \frac{y_1^{a_1-1}(1-y_1)^{a_2-1}}{Be(a_1,a_2)}\,\frac{y_2^{a_1+a_2-1}}{\Gamma(a_1+a_2)}\,\frac{\exp\left(\frac{-1}{b}(y_1+y_2)\right)}{b^{a_1+a_2}}dy_1dy_2$$

$$f_{Y_1}(y_1)=\frac{y_1^{a_1-1}(1-y_1)^{a_2-1}}{Be(a_1,a_2)}\ ,\quad f_{Y_2}(y_2)=\frac{y_2^{a_1+a_2-1}}{\Gamma(a_1+a_2)}\,\frac{\exp\left(\frac{-1}{b}(y_1+y_2)\right)}{b^{a_1+a_2}}$$

$$f_{Y_1}(y_1)\,f_{Y_2}(y_2)=f_{Y_1Y_2}(y_1,y_2)$$

∴ Y_1 與 Y_2 獨立,

$Y_1\sim Be(a_1,a_2)\quad Y_2\sim\Gamma(a_1+a_2,\,b)$

($\boxed{6}$ 與 $\boxed{2}$ 題目完全一樣，請參閱 $\boxed{2}$)

$\boxed{7}$

(a) $f_X(x) = \int_x^1 2\,dy = [2y]_x^1 = 2(1-x)$

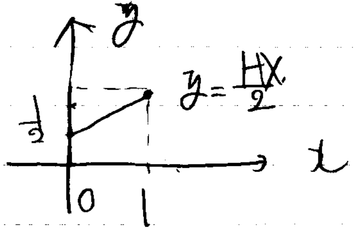
$\frac{f_{X,Y}(x,y)}{f_X(x)} = f_{Y|X}(y|X=x) = \frac{1}{1-x}$ $(x<y<1)$

$X=x_0$ 時 $Y \sim \mathrm{Uni}(x_0,1)$

$\therefore E[Y|X=x_0] = \frac{1+x_0}{2}$ $\left(\int_{y=x_0}^{y=1} \frac{1}{1-x_0}\cdot y\,dy = \frac{1+x_0}{2}\right)$

(b) 由(a)可知，$Y|X=x \sim U(x,1)$

$E[Y|X=x] = \frac{1+x}{2}$



(c) $X=x_0$ 時 $YX^3+1 = x_0^3Y+1$ (X可以當成常數)

$\therefore E[YX^3+1\,|X=x_0] = x_0^3\underbrace{E[Y|X=x_0]}_{\frac{1+x_0}{2}}+1$

$\therefore \underbrace{\frac{x_0^4+x_0^3}{2}+1}$

⑪

$\boxed{7}$

$(1-y)(1\cdots)$

(d) $V[Y|X=x_0] = \underset{\sim\sim\sim}{E[Y^2|X=x_0]} - E[Y|X=x_0]^2$

$$\int_{x_0}^{1} \frac{y^2\,dy}{1-x_0} - \left(\frac{1+x_0}{2}\right)^2$$

$$= \frac{1-x_0^3}{3(1-x_0)} - \frac{1}{4}(1+x_0)^2$$

$$= \frac{1}{3}(1+x_0+x_0^2) - \frac{1}{4}(1+x_0)^2$$

$$= \frac{1}{12}(4x_0^2+4x_0+4-3x_0^2-6x_0-3)$$

$$= \frac{1}{12}(x_0^2-2x_0+1) = \frac{1}{12}(1-x_0)^2$$

$\therefore V[Y|X=x_0] = \frac{1}{12}(1-x_0)^2$

$X=x_0$ 時. $Y \sim Uni(x_0,1)$. (跟 $Uni(\theta,1)$ 一樣)

所以變異數包含 x_0 (θ) 是一件很正常的事.

$\boxed{8}$ (a) $\frac{dF(x)}{dx} = \frac{d}{dx}\left\{\left(\frac{x}{\alpha}\right)^{\beta}\right\} \exp\left(-\left(\frac{x}{\alpha}\right)^{\beta}\right) = \underset{\sim\sim\sim\sim}{\frac{\beta x^{\beta-1}}{\alpha^{\beta}} \exp\left(-\left(\frac{x}{\alpha}\right)^{\beta}\right)}$

(b) $\Pr(W \leq w) = 1 - \exp\left(-\left(\frac{w}{\alpha}\right)^{\beta}\right)$ （W服從Weibull分佈）

$\quad\, \|$

$\Pr(X^{\frac{1}{\beta}} \cdot \alpha \leq w) = \Pr\left(X^{\frac{1}{\beta}} \leq \frac{w}{\alpha}\right) = \Pr\left(X \leq \underbrace{\left(\frac{w}{\alpha}\right)^{\beta}}_{\overset{\|}{x}}\right)$

$\therefore \Pr(X \leq \lambda) = 1 - \exp(-\lambda)$ $\left(\because 1 - \exp\left(-\left(\frac{w}{\alpha}\right)^{\beta}\right) \therefore \left(\frac{w}{\alpha}\right)^{\beta} \to x\right)$

$\therefore X \sim \exp(1)$

(c) 若 $X \sim \exp(1)$ $\quad W := \alpha X^{\frac{1}{\beta}} \sim \text{Weibull}(\alpha, \beta)$

故此. 考慮由均勻分佈產生指數分佈的方法.

$X_1 \cdots X_n \sim \text{Uni}(0,1)$ $\quad \Pr(nX_{(1)} \geq \lambda) = \Pr\left(X_{(1)} \geq \frac{\lambda}{n}\right)$

$= \left(1 - \frac{\lambda}{n}\right)^{n}$ $\quad \therefore \Pr(nX_{(1)} \leq \lambda) = 1 - \left(1 - \frac{\lambda}{n}\right)^{n} = 1 - \left\{\left(1 - \frac{\lambda}{n}\right)^{-\frac{n}{\lambda}}\right\}^{(-\lambda)}$

$n \to \infty \cdots$ $\quad \Pr(nX_{(1)} \leq \lambda) \to 1 - e^{-\lambda}$

$\therefore nX_{(1)} \xrightarrow{d} \exp(1)$ (as $n \to \infty$)

$\therefore \alpha (nX_{(1)})^{\frac{1}{\beta}} \xrightarrow{d} \text{Weibull}(\alpha, \beta)$

$\boxed{9}$ $X_1, X_2 \sim Uni(0,\theta)$

(1) $Pr(T \leqq t) = Pr(X_1, X_2 \leqq t) = \left(\frac{t}{\theta}\right)^2 \cdot I_{(0,\theta)}(t)$

$\frac{d}{dt} Pr(T \leqq t) = \frac{2t}{\theta^2} \cdot I_{(0,\theta)}(t)$ $\quad \therefore f_T(t) = \frac{2t}{\theta^2} \cdot I_{(0,\theta)}(t)$

$$f_{X_1,X_2,T}(x_1,x_2,t) = \frac{1}{\theta^2} \begin{cases} X_1 < X_2 = t & \text{or} \\ X_2 < X_1 = t & \text{or.} \quad (0<t<\theta) \\ X_1 = X_2 = t \end{cases}$$

$$= \frac{1}{\theta^2} \cdot I_{(0,\theta)}(t) \cdot I_{(0,t]}(x_1) \, I_{(0,t]}(x_2)$$

$$\therefore \frac{f_{X_1,X_2,T}(x_1,x_2,t)}{f_T(t)} = \frac{I_{(0,t]}(x_1) \cdot I_{(0,t]}(x_2)}{2t}$$

∵ $X_1, X_2 | T=t$ 之分布跟 θ 無關,

∴ T 為 θ 之充分統計量.

(2) 求 θ 之最小充分統計量…

$$\because \frac{f(Y_1,Y_2|\theta)}{f(X_1,X_2|\theta)} = \frac{\frac{1}{\theta^2} \cdot I(Y_{(2)} < \theta)}{\frac{1}{\theta^2} I(X_{(2)} < \theta)} = \frac{I(Y_{(2)} < \theta)}{I(X_{(2)} < \theta)} \;\cdots \text{與 } \theta \text{ 無關}$$

$\Rightarrow X_{(2)} = Y_{(2)}$.

$X_{(2)} = Y_{(2)} \Rightarrow \frac{f(Y_1, Y_2|\theta)}{f(X_1, X_2|\theta)}$ 與 θ 無關

∴ $\max\{X_1, X_2\}$ 為 θ 之最小充分統計量

由 X_1+X_2 無法得知 $\max\{X_1, X_2\}$ 之值

∴ X_1+X_2 並非充分統計量

⑮

$\boxed{10}$ $f(x_1 .. x_n \mid \theta) = \left(\frac{1}{3\theta}\right)^n I_{(-\theta, 2\theta)}(x_1) \cdots I_{(-\theta, 2\theta)}(x_n)$

$$= \left(\frac{1}{3\theta}\right)^n I(-\theta < X_{(1)}, X_{(n)} < 2\theta)$$

$$= \left(\frac{1}{3\theta}\right)^n I(-X_{(1)} < \theta) \cdot I\left(\frac{X_{(n)}}{2} < \theta\right)$$

$$= \left(\frac{1}{3\theta}\right)^n \cdot I\left(\max\left\{-X_{(1)}, \frac{X_{(n)}}{2}\right\} < \theta\right)$$

$$L(\theta) = \left(\frac{1}{3\theta}\right)^n \cdot I\left(\max\left\{-X_{(1)}, \frac{X_{(n)}}{2}\right\} < \theta\right)$$

(由此可知 θ之MLE為 $\max\left\{-X_{(1)}, \frac{X_{(n)}}{2}\right\}$)

$\boxed{11}$ 核密度估計…

$$\hat{f_h}(x) = \frac{1}{nh} \sum_{i=1}^{n} K\left(\frac{x - x_i}{h}\right)$$

$n=4$ $x_1 = -1$ $x_2 = 0$ $x_3 = 0.5$ $x_4 = 3$

$$\hat{f_h}(x) = \frac{1}{4} \sum_{i=1}^{4} K(x - x_i) \quad (K = \phi)$$

① $\hat{f_h}(0) = \frac{1}{4}\left\{ \phi(1) + \phi(0) + \phi(-0.5) + \phi(-3) \right\}$

$= 0.249353.$

② $\hat{f}_h(0.5) = \frac{1}{4}\sum_{i=1}^{4} K(0.5-x_i) = \frac{1}{4}\{\phi(1.5)+\phi(0.5)+\phi(0)+\phi(-2.5)\}$

$= 0.224513$

③ $\hat{f}_h(1) = \frac{1}{4}\sum_{i=1}^{4} K(1-x_i) = \frac{1}{4}\{\phi(2)+\phi(1)+\phi(0.5)+\phi(-2)\}$

$= 0.175504$

$\boxed{12}$

(a) $\hat{f}_h(x) = \frac{1}{nh}\sum_{i=1}^{n} K\left(\frac{x-x_i}{h}\right)$ → (X為隨機變數、其pdf為 $\hat{f}$)

(b) $X \sim \hat{f}$ ($\hat{f}(X=x) = \frac{1}{nh}\sum_{i=1}^{n} K\left(\frac{x-x_i}{h}\right)$)

$$E[X \mid X_1=x_1 \cdots X_n=x_n] = \int_{-\infty}^{\infty} \frac{1}{nh}\sum_{i=1}^{n} K\left(\frac{x-x_i}{h}\right) dx$$

$$= \frac{1}{nh}\sum_{i=1}^{n}\int_{-\infty}^{\infty} x K\left(\frac{x-x_i}{h}\right) dx \qquad u=\frac{x-x_i}{h} \qquad \frac{du}{dx}=\frac{1}{h}$$

$$= \frac{1}{nh}\sum_{i=1}^{n}\int_{-\infty}^{\infty} (hu+x_i) K(u)\, h\, du$$

$$= \frac{1}{n}\sum_{i=1}^{n}\int_{-\infty}^{\infty} \left(hu K(u) + x_i K(u)\right) du$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left\{ h\underbrace{\int_{-\infty}^{\infty} u K(u)\, du}_{0} + x_i \underbrace{\int_{-\infty}^{\infty} K(u)\, du}_{1} \right\}$$

$$= \frac{1}{n}\sum_{i=1}^{n} x_i = \bar{x}$$ → ㊟ x_i $(i=1\sim n)$ 為隨機變數 其pdf為 f

$\boxed{12}$

(c) $\underset{X^2|X_1=x_1\cdots X_n=x_n}{E[X^2]} = \int_{-\infty}^{\infty} \frac{x^2}{nh}\sum_{i=1}^{n} K\left(\frac{x-x_i}{h}\right) dx$

$$= \sum_{i=1}^{n}\int_{-\infty}^{\infty} \frac{x^2}{nh} K\left(\frac{x-x_i}{h}\right) dx \qquad \frac{x-x_i}{h}=u$$

$$= \sum_{i=1}^{n}\int_{u=-\infty}^{u=\infty} \frac{(hu+x_i)^2}{nh} K(u)\cdot h\,du$$

$$= \sum_{i=1}^{n}\int_{u=-\infty}^{u=\infty} \frac{1}{n}(h^2u^2+2hux_i+x_i^2) K(u)du$$

$$= \sum_{i=1}^{n}\left\{\frac{h^2}{n}\int_{-\infty}^{\infty} u^2K(u)du + \frac{2hx_i}{n}\int_{-\infty}^{\infty} uK(u)du + \frac{x_i^2}{n}\int_{-\infty}^{\infty} K(u)du\right\}$$

$$= \sum_{i=1}^{n}\left(\frac{h^2}{n}\sigma_k^2 + \frac{x_i^2}{n}\right)$$

$$= h^2\sigma_k^2 + \frac{1}{n}\sum_{i=1}^{n} x_i^2$$

$\therefore \underset{|X_1=x_1\cdots X_n=x_n}{V[X]} = E[X^2]-E[X]^2 = h^2\sigma_k^2 + \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \bar{x}^2$

$$= h^2\sigma_k^2 + \sum_{i=1}^{n}\frac{(X_i-\bar{X})^2}{n}$$

(d) 由(b),(c)可知… X的 機率密度函數为 $\hat{f}_n$ 時

$$E[X] = \frac{1}{n}(x_1+\cdots x_n)$$

$$V[X] = h^2\sigma_k^2 + \sum_{i=1}^{n}\frac{(X_i-\bar{X})^2}{n}$$

（在此假設 $\int_{-\infty}^{\infty} x f(x)\,dx < \infty$，$\int_{-\infty}^{\infty} x^2 f(x)\,dx < \infty$）

$n \to \infty$ 時　$E[X] = \frac{X_1+\cdots+X_n}{n} \xrightarrow{P} \mu$（$\mu$⋯f之期望值）

（$E[X \mid X_1=x_1 \cdots X_n=x_n]$）

（若h固定）　$V[X] = h^2\sigma_K^2 + \sum_{i=1}^{n} \frac{(X_i-\bar{X})^2}{n} \xrightarrow{P} h^2\sigma_K^2 + \sigma^2$（$\sigma^2$⋯f之變異數）

（$V[X \mid X_1 \cdots X_n]$）

若希望 $V[X] \xrightarrow{P} \sigma^2$，需要 $h^2 \to 0$ (as $n \to \infty$)

㊟ $E[X] = \frac{X_1+X_2+\cdots+X_n}{n} \xrightarrow{P} \mu$

根據 Khinchin 弱大數法則，

$X_1, X_2 \cdots X_n \sim$ iid　期望值為 $\mu < \infty$

$\frac{X_1+\cdots+X_n}{n} \xrightarrow{P} \mu$

同樣道理，$\sum_{i=1}^{n} \frac{1}{n}(X_i-\bar{X})^2 \xrightarrow{P} \sigma^2$（f為變異數）

⑲

$\boxed{13}$

(a) $x = e^{-y}$ $\left(\begin{array}{l} x:\ 0 \to 1 \\ y:\ \infty \to 0 \end{array} \right.$

$$\frac{dx}{dy} = -e^{-y} \qquad dx = -e^{-y}dy$$

$$1 = \int_0^1 \theta \cdot x^{\theta-1} dx = \int_\infty^0 \theta \cdot e^{-y(\theta-1)} \cdot (-e^{-y}) dy$$

$$= \int_0^\infty \theta e^{-\theta y} dy \quad \therefore \ Y \sim \exp(\theta) \ (\text{mean}: \tfrac{1}{\theta})$$

(b) T…$\tau(\theta)$之不偏估計量。在此只考慮$\frac{\partial}{\partial\theta}$、$\int$可替換之情形。

$$E\left[(T-\tau(\theta))\left(\frac{\partial}{\partial\theta}\log f(x_1..x_n|\theta)\right)\right]^2 \leqq E\left[(T-\tau(\theta))^2\right] \cdot$$

$$E\left[\left(\frac{\partial}{\partial\theta}\log f(x_1..x_n|\theta)\right)^2\right] = V[T] \cdot \underset{\text{Fisher 資訊量}}{\underbrace{I_n(\theta)}}$$

($\because$ Cauchy-Schwartz 不等式)

$$E\left[(T-\tau(\theta))\frac{\partial}{\partial\theta}\log f(X_1..X_n|\theta)\right] = \frac{\partial}{\partial\theta}\int T(X_1..X_n) \cdot f(x_1..x_n|\theta)\,dx$$

$$-\frac{\partial}{\partial\theta}\int f(x_1..x_n|\theta)\,dx = \tau'(\theta)$$

$$V[T] \cdot I_n(\theta) \geqq (\tau'(\theta))^2 \quad \therefore \ V[T] \geqq \frac{(\tau'(\theta))^2}{I_n(\theta)}$$

在適當的条件下「$V[T] \geqq \frac{(\tau'(\theta))^2}{I_n(\theta)}$」成立

$\frac{\tau'(\theta)^2}{I_n(\theta)}$ 為 Cramer Rao's Lower Bound.

$I_n(\theta) = nI_1(\theta)$. $I_1(\theta) = E\left[\frac{\partial}{\partial\theta}\log f(x|\theta)\right] = E\left[\left(\frac{1}{\theta}+\log x\right)^2\right]$

$E\left[\left(\frac{1}{\theta}-Y\right)^2\right] = V[Y] = \frac{1}{\theta^2}$ （ $\exp(\theta)$ 之變異數）

∴ $I_n(\theta) = \frac{n}{\theta^2}$ $\tau'(\theta) = \frac{-1}{\theta^2}$ $(\tau'(\theta))^2 = \frac{1}{\theta^4}$

∴ $\frac{(\tau'(\theta))^2}{I_n(\theta)} = \frac{\theta^2}{n}\cdot\frac{1}{\theta^4} = \frac{1}{n\theta^2}$ …… CRLB of $\tau(\theta)$ 之不偏估計量.

(C) $-\sum_{i=1}^{n}\log X_i \sim \Gamma(n,\theta)$

$V\left[-\sum_{i=1}^{n}\log X_i\right] = \frac{n}{\theta^2}$, $E\left[-\sum_{i=1}^{n}\log X_i\right] = \frac{n}{\theta}$

∴ $V\left[\frac{-1}{n}\sum_{i=1}^{n}\log X_i\right] = \frac{1}{n\theta^2}$ $E\left[\frac{-1}{n}\sum_{i=1}^{n}\log X_i\right] = \frac{1}{\theta}$

$-\sum_{i=1}^{n}\frac{1}{n}\log X_i$ …為 $\frac{1}{\theta}$ 之不偏估計量，且達到 CRLB，因此它必定為 $\frac{1}{\theta}$ 之 UMVUE.

$\boxed{14}$ $X_1 \cdots X_n \sim U(0,\theta)$

(a) $f(x|\theta) = \frac{1}{\theta} I_{(0,\theta)}(x)$

$$f(x_1 \cdots x_n|\theta) = \left(\frac{1}{\theta}\right)^n I_{(0\theta)}(x_1) \cdot I_{(0,\theta)}(x_2) \cdots I_{(0\theta)}(x_n)$$

$$= \underbrace{I_{(0,\infty)}(X_{(1)})}_{h(x)} \cdot \underbrace{I_{(0,\theta)}(X_{(n)}) \cdot \frac{1}{\theta^n}}_{g(X_{(n)}|\theta)}$$

Neyman Fisher 分解定理 ∴ $X_{(n)}$ 為 θ 之充分統計量

(b) $E[2X_1] = 2E[X_1] = 2\int_0^\theta \frac{1}{\theta} x\,dx = \frac{2}{\theta}\cdot\left[\frac{X^2}{2}\right]_0^\theta = \theta$

∴ $2X_1$ 為 θ 之不偏估計量

(c) 證明 $X_{(n)}$ 為 θ 之完備充分統計量

$T \overset{def}{=} X_{(n)}$ $\quad Pr(T\le t) = \left(\frac{t}{\theta}\right)^n = \frac{t^n}{\theta^n}$ $\quad \frac{dPr(T\le t)}{dt} = \frac{nt^{n-1}}{\theta^n}$

∴ $f_T(t) = \frac{nt^{n-1}}{\theta^n}$

若 $E[g(T)]=0 \Rightarrow \int_0^\theta \frac{nt^{n-1}}{\theta^n} g(t)\,dt = 0 \overset{(\text{乘以}\theta^n)}{\Longrightarrow}$

∴ $\int_0^\theta nt^{n-1}g(t)\,dt = 0 \overset{(\theta\text{微分})}{\Longrightarrow} n\theta^{n-1}g(\theta)=0 \overset{(\text{除以}n\theta^{n-1})}{\Longrightarrow}$

得 $g(\theta)=0$ ∴ $g=0$

∴ $E[g(T)]=0 \Rightarrow Pr[g(T)=0]=1$ ∴ T 為 θ 之完備充分統計量

$X_{(n)}$ 為 θ 之充分統計量，$\hat{\theta}(X) | X_{(n)}$ 的分佈與 θ 無關.

$E[\hat{\theta} | X_{(n)}]$ 為 $X_{(n)}$ 之函數. $EE[\hat{\theta} | X_{(n)}] = E[\hat{\theta}] = \theta$

∴ $E[\hat{\theta} | X_{(n)}]$ 為完備統計量且亦為 θ 之不偏估計量

∴ $E[\hat{\theta} | X_{(n)}]$ 為 θ 之 UMVUE. (Lehmann-scheffe 定理) $= \frac{n+1}{n}X_{(n)}$ $\left(E(T) = \frac{n}{n+1}\theta \therefore \frac{n+1}{n}T \text{ 為 UMVUE}\right)$

$\boxed{15}$ $Pr(X=\lambda) = e^{-\lambda}\cdot\frac{\lambda^{x}}{x!}$

(a) $Pr(X_1=\lambda_1,\dots,X_n=\lambda_n) = e^{-n\lambda}\cdot\frac{\lambda^{(x_1+\dots+x_n)}}{x_1!\cdots x_n!} = \underbrace{\frac{1}{x_1!\cdots x_n!}}_{h(x)}\cdot\underbrace{e^{-n\lambda}\cdot\lambda^{(x_1+\dots+x_n)}}_{g(T|\lambda)}$

註 $= e^{-n\lambda}\cdot\exp((x_1+\dots+x_n)\log\lambda)$ … 指數族

$\{\log\lambda \mid \lambda>0\}$ 之維度 $=1$ ∴ $X_1+\dots+X_n$ 為 λ 之完備充分統計量

$(T = X_1+\dots+X_n \sim Po(n\lambda))$

根據 Neyman-Fisher 分解定理，T 為 λ 之充分統計量

(b) $\tilde{\theta} = \begin{cases} 1 & \text{if } X_1=0 \\ 0 & \text{else} \end{cases}$

$E[\hat{\theta}] = Pr(\hat{\theta}=1)\cdot 1 = Pr(X_1=0) = e^{-\lambda}\cdot\frac{\lambda^0}{0!} = e^{-\lambda}$

∴ $\hat{\theta}$ 為 $e^{-\lambda}$ 之不偏估計量

註 $X_2+X_3+\dots X_n \sim Po((n-1)\lambda)$

(c) $E[\hat{\theta} | T=t] = \sum_{\hat{\theta}=0,1}\tilde{\theta}\cdot Pr(\tilde{\theta} | T=t) = Pr(\hat{\theta}=1 | T=t)$

$= Pr(X_1=0 | X_1+\dots+X_n=t) = \frac{Pr(X_1=0, X_1+\dots+X_n=t)}{Pr(T=t)} = \frac{Pr(X_1=0)Pr(X_2+X_3+\dots+X_n=t)}{Pr(T=t)}$

$= \frac{(\cancel{e^{-\lambda}})\left(\cancel{e^{-(n-1)\lambda}}\cdot\frac{((n-1)\lambda)^t}{\cancel{t!}}\right)}{\left(\cancel{e^{-n\lambda}}\cdot\frac{(n\lambda)^t}{\cancel{t!}}\right)} = \left(1-\frac{1}{n}\right)^t = \left(1-\frac{1}{n}\right)^{X_1+\dots+X_n}$ ∴ 證明完成

$(= e^{-\lambda}$ 之 UMVUE ∵ Lehmann-scheffe)

$\boxed{16}$ $E[e^{tX}] = \frac{1}{(1-\theta t)} = M_X(t)$

(a) $\frac{d}{dt^2}M_X(t) = \frac{d}{dt}\frac{\theta}{(1-\theta t)^2} = \frac{2\theta^2}{(1-\theta t)^3}$

$\therefore M_X''(0) = 2\theta^2 \quad \therefore E[X_1^2] = 2\theta^2 \quad \therefore \underline{E[\frac{X_1^2}{2}] = \theta^2}$

(b) $f(x_1 \cdots x_n|\theta) = \frac{1}{\theta^n}\exp(\frac{-1}{\theta}(x_1+\cdots x_n)) = \underbrace{\frac{1}{\theta^n}\exp(\frac{-1}{\theta}T)}_{g(T|\theta)} \cdot \underbrace{1}_{h(x)}$

∴ T為θ之充份統計量（∵ Neyman-Fisher 分解定理）

(c) $f(x_1 \cdots x_n|\theta) = \frac{1}{\theta^n}\exp(\frac{-1}{\theta}T)$ ：指數族

∴ 參數之維度 $\dim\Theta = 1$

觀察 exp(~)內，T之係數 $(\frac{-1}{\theta})$

$\theta>0$ 時，$(\frac{-1}{\theta})$ 呈現 維度1的矩形
（不會停留在固定的點）

由此可知，T為θ之完備充分統計量

$E[\frac{X_1^2}{2}|X_1+\cdots+X_n]$ 為θ之UMVUE 」

先求 $X_1|X_1+\cdots X_n = t$ 之分布

$X \overset{def}{=} X_1$，$Y \overset{def}{=} X_2+\cdots X_n$ （X與Y為獨立）

$W \overset{def}{=} X$，$T = X+Y \ (=X_1+\cdots X_n)$

$$\Rightarrow \left(\begin{array}{l} X=W \\ Y=T-W \end{array}\right) \quad J=\begin{pmatrix} \frac{\partial X}{\partial W} & \frac{\partial X}{\partial T} \\ \frac{\partial Y}{\partial W} & \frac{\partial Y}{\partial T} \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

$\therefore dxdy = dwdt$

$$\int f_{XY}(x,y)\,dxdy = \int \frac{1}{\theta}\exp\left(\frac{-x}{\theta}\right)\cdot \frac{y^{n-2}}{\Gamma(n-1)\cdot\theta^{n-1}}\exp\left(\frac{-y}{\theta}\right)dxdy$$

$$= \int \frac{(t-w)^{n-2}}{\theta^n}\cdot\frac{1}{\Gamma(n-1)}\cdot\exp\left(\frac{-1}{\theta}t\right)dwdt$$

$-\left(1-\frac{w}{t}\right)^{n-1}$

[illegible]

$$\therefore f_{W,T}(w,t) = \frac{(t-w)^{n-2}}{\theta^n}\cdot\frac{1}{\Gamma(n-1)}\cdot\exp\left(\frac{-t}{\theta}\right)$$

$$f_{W|T}(w|t) = \frac{f_{W,t}(w,t)}{f_T(t)} = \frac{(t-w)^{n-2}}{\theta^n}\cdot\frac{\exp\left(\frac{-t}{\theta}\right)}{\Gamma(n-1)} \Big/ \frac{1}{\theta^n}\cdot\frac{t^{n-1}}{\Gamma(n)}\exp\left(\frac{-t}{\theta}\right)$$

$$= (n-1)\left(1-\frac{w}{t}\right)^{n-2}\cdot\frac{1}{t} \quad (\text{注. } W=X_1 \quad T=X_1+X_2+\dots X_n)$$

$$\therefore X_1 \mid X_1+\dots X_n = t \overset{pdf}{\sim} (n-1)\left(1-\frac{x_1}{t}\right)^{n-2}\cdot\frac{1}{t} \quad (0\le x_1\le t)$$

$$\therefore E[X_1^2 \mid X_1+\dots X_n=t] = \int_0^t (n-1)\left(1-\frac{x}{t}\right)^{n-2}\frac{1}{t}\cdot x^2\,dx$$

$$\left(\frac{x}{t}=u \quad \frac{dy}{dx}=\frac{1}{t}\ ; \quad x:0\to t, \quad u:0\to 1\right)$$

$$= \int_0^1 (ut)^2\cdot(n-1)\cdot(1-u)^{n-2}\cdot\frac{1}{t}\cdot t\,dy$$

$$= (n-1)t^2\int_0^1 u^2(1-u)^{n-2}du = (n-1)t^2\,Be(3,n-1)$$

$$= t^2\cdot(n-1)\cdot\frac{\Gamma(3)\Gamma(n-1)}{\Gamma(n+2)} = t^2\cdot(n-1)\cdot\frac{2!(n-2)!}{(n+1)!} = t^2\cdot\frac{2!(n-1)!}{(n+1)!}$$

$$= \frac{2t^2}{n(n+1)} \Rightarrow E\left[\frac{X_1^2}{2}\,\Big|\, X_1+\dots X_n=t\right] = \frac{t^2}{n(n+1)} \quad \therefore \frac{(X_1+\dots X_n)^2}{n(n+1)} \dots \theta^2\text{之UMVUE}$$

㉕

$\boxed{16}$

(d) $f(x_1\cdots x_n|\theta)=\frac{1}{\theta^n}\exp\left(\frac{-1}{\theta}(x_1+\cdots+x_n)\right)$

$\log f(x_1\cdots x_n|\theta)=-n\log\theta-\frac{1}{\theta}(x_1+\cdots x_n)$

$\frac{\partial}{\partial\theta}\log f(x_1\cdots x_n|\theta)=\frac{-n}{\theta}+\frac{1}{\theta^2}(x_1+\cdots x_n)=\frac{(x_1+\cdots x_n)-n\theta}{\theta^2}$

$\theta=\bar{X}$ 時… MAX ∴ $\hat{\theta}_{MLE}=\bar{X}$

根據 MLE 之不變性，$\hat{\theta}^2$ 之 MLE $=\bar{X}^2=\frac{T^2}{n^2}$

① $MSE[\hat{\theta}^2_{UMVUE}]=E\left[\left(\frac{T^2}{n(n+1)}-\theta^2\right)^2\right]=V\left[\frac{T^2}{n(n+1)}\right]$

② $MSE[\hat{\theta}^2_{MLE}]=E\left[\left(\frac{T^2}{n^2}-\theta^2\right)^2\right]=E\left[\left(\left(\frac{T^2}{n^2}-\frac{n+1}{n}\theta^2\right)+\left(\frac{n+1}{n}\theta^2-\theta^2\right)\right)^2\right]$

$=V\left[\frac{T^2}{n^2}\right]+\frac{\theta^4}{n^2}$ （由 $\hat{\theta}^2_{UMVUE}=\frac{(x_1+\cdots x_n)^2}{n(n+1)}=\frac{T^2}{n(n+1)}$）

顯然 $V\left[\frac{T^2}{n(n+1)}\right]<V\left[\frac{T^2}{n^2}\right]$

∴ ① < ②

∴ $\hat{\theta}^2_{UMVUE}$ 之 MSE 較小。

(NOTE)

作業6 [2] (b) 計算過程 　　　　[2] (b)-note p1

$$f_{XY}(x,y) = 2\lambda^2 \exp(-\lambda x)\cdot \frac{1}{\sqrt{x^2-4y}} \quad (x \geqq 0,\ y \geqq 0,\ x^2-4y \geqq 0)$$

$$\begin{cases} \alpha = E[Y] - \dfrac{\mathrm{cov}[X,Y]}{V[X]} E[X] \\ \beta = \dfrac{\mathrm{cov}[X,Y]}{V[X]} \end{cases}$$

$$f_X(x) = \int_{y=0\to\frac{x^2}{4}} f_{XY}(x,y)\,dy = \lambda^2 x \exp(-\lambda x)$$

- $E[X] = \int_{x=0}^{x=\infty} \lambda^2 x^2 \exp(-\lambda x)\,dx$

$\lambda x = t \quad \frac{dt}{dx} = \lambda \qquad \int_{t=0}^{t=\infty} t^2 \exp(-t)\frac{dt}{\lambda} = \Gamma(3)\cdot\frac{1}{\lambda}$

$= \underline{\frac{2}{\lambda}} \qquad \therefore E[X] = \frac{2}{\lambda}$

- $E[X^2] = \int_{x=0}^{x=\infty} \lambda^2 x^3 \exp(-\lambda x)\,dx$

$$= \int_{t=0}^{t=\infty} \frac{t^3}{\lambda} \exp(-t)\frac{dt}{\lambda} = \frac{1}{\lambda^2}\Gamma(4) = \frac{6}{\lambda^2}$$

$\therefore V[X] = \frac{6}{\lambda^2} - \left(\frac{2}{\lambda}\right)^2 = \underline{\frac{2}{\lambda^2}}$

- $E[Y] = \int_{x=0}^{x=\infty}\int_{y=0}^{y=\frac{x^2}{4}} 2\lambda^2 \exp(-\lambda x)\cdot\frac{y}{\sqrt{x^2-4y}}\,dy\,dx$

$$= \int_0^\infty \int_0^{\frac{x^2}{4}} \lambda^2 \exp(-\lambda x) \cdot \frac{y}{\sqrt{\frac{x^2}{4} - y}} \, dy \, dx$$

$$= \int_0^\infty \int_0^{\frac{\lambda^2}{4}} \lambda^2 \exp(-\lambda x) \cdot \frac{\left(y - \frac{x^2}{4}\right) + \frac{\lambda^2}{4}}{\sqrt{\frac{x^2}{4} - y}} \, dy \, dx$$

$$= \int_0^\infty \int_0^{\frac{x^2}{4}} \lambda^2 \exp(-\lambda x) \cdot \left\{ \frac{\frac{x^2}{4}}{\sqrt{\frac{x^2}{4} - y}} - \sqrt{\frac{x^2}{4} - y} \right\} dy \, dx$$

$$= E\left[\frac{X^2}{4}\right] - \underbrace{\int_0^\infty \int_0^{\frac{x^2}{4}} \lambda^2 \exp(-\lambda x) \cdot \sqrt{\frac{x^2}{4} - y} \, dy \, dx}_{\parallel}$$

$$= \int_0^\infty \lambda^2 \exp(-\lambda x) \left[\frac{-2}{3} \left(\frac{x^2}{4} - y \right)^{\frac{3}{2}} \right]_{y=0}^{y=\frac{x^2}{4}} dx$$

$$= \int_0^\infty \lambda^2 \exp(-\lambda x) \cdot \frac{2}{3} \cdot \left(\frac{x}{2}\right)^3 dx$$

$$= \int_0^\infty \frac{1}{12} \lambda^2 x^3 \exp(-\lambda x) \, dx \qquad \left(\lambda x = t, \ \frac{dt}{dx} = \lambda \right)$$

$$= \int_0^\infty \frac{1}{12} \cdot \lambda^2 \cdot \left(\frac{t}{\lambda}\right)^3 \exp(-t) \cdot \frac{dt}{\lambda}$$

$$= \int_0^\infty \frac{1}{12} \cdot \frac{1}{\lambda^2} t^3 \exp(-t) \, dt$$

$$= \frac{1}{12\lambda^2} \cdot \Gamma(4) = \underline{\frac{1}{2\lambda^2}}$$

$$\therefore E(Y) = E\left[\frac{X^2}{4}\right] - \frac{1}{2\lambda^2} = \frac{3}{2\lambda^2} - \frac{1}{2\lambda^2} = \underline{\frac{1}{\lambda^2}}$$

$\boxed{2}$-(b) note p.3

- $E[XY]$

$$\int_{x=0}^{x=\infty}\int_{y=0}^{y=\frac{x^2}{4}} 2\lambda^2\cdot\exp(-\lambda x)\cdot\frac{xy}{\sqrt{x^2-4y}}\,dy\,dx$$

$$=\int_{x=0}^{x=\infty}\int_{y=0}^{y=\frac{x^2}{4}} \lambda^2\exp(-\lambda x)\cdot\frac{xy}{\sqrt{\frac{x^2}{4}-y}}\,dy\,dx$$

$$=\int_{x=0}^{x=\infty}\int_{y=0}^{y=\frac{x^2}{4}} \lambda^2\exp(-\lambda x)\cdot x\cdot\left\{\frac{\left(y-\frac{x^2}{4}\right)+\frac{x^2}{4}}{\sqrt{\frac{x^2}{4}-y}}\right\}dy\,dx$$

$$=\int_{x=0}^{x=\infty}\int_{y=0}^{y=\frac{x^2}{4}} \lambda^2\exp(-\lambda x)\cdot x\cdot\left\{\frac{\frac{x^2}{4}}{\sqrt{\frac{x^2}{4}-y}}-\sqrt{\frac{x^2}{4}-y}\right\}dy\,dx$$

$$=\int_0^\infty\int_0^{\frac{x^2}{4}} \lambda^2\exp(-\lambda x)\cdot\left(\frac{\frac{x^3}{4}}{\sqrt{\frac{x^2}{4}-y}}-x\sqrt{\frac{x^2}{4}-y}\right)dy\,dx$$

$$=E\left[\frac{X^3}{4}\right]-\underbrace{\int_0^\infty\int_0^{\frac{x^2}{4}} \lambda^2\exp(-\lambda x)\cdot x\sqrt{\frac{x^2}{4}-y}\,dy\,dx}$$

$$=\int_0^\infty \lambda^2\exp(-\lambda x)\cdot x\cdot\left[-\frac{2}{3}\left(\frac{x^2}{4}-y\right)^{\frac{3}{2}}\right]_0^{\frac{x^2}{4}}dx$$

$$=\int_0^\infty \lambda^2\exp(-\lambda x)\cdot\frac{2x}{3}\cdot\left(\frac{x}{2}\right)^3dx$$

$$=\int_0^\infty \lambda^2\exp(-\lambda x)\cdot\frac{x^4}{12}\,dx\qquad\left(\lambda x=t\quad\frac{dt}{dx}=\lambda\right)$$

$$=\int_0^\infty \lambda^2\exp(-t)\frac{1}{12}\left(\frac{t}{\lambda}\right)^4\cdot\frac{dt}{\lambda}$$

$$=\int_0^\infty\frac{1}{\lambda^3}\cdot\frac{1}{12}t^4\exp(-t)\,dt=\frac{1}{12\lambda^3}\cdot\Gamma(5).$$

$$= \frac{2}{\lambda^3}$$

$$\therefore E[XY] = \underbrace{E\left[\frac{\lambda^3}{4}\right]} - \frac{2}{\lambda^3}$$

$$\parallel$$

$$\int_0^\infty \frac{\lambda^3}{4} \cdot f_X(x)\,dx$$

$$= \int_0^\infty \frac{x^3}{4} \cdot \lambda^2 \cdot x \exp(-\lambda x)\,dx \quad \left(\lambda x = t \quad \frac{dt}{dx} = \lambda\right)$$

$$= \int_0^\infty \frac{x^4}{4} \lambda^2 \exp(-\lambda x)\,dx$$

$$= \int_0^\infty \frac{1}{4}\left(\frac{t}{\lambda}\right)^4 \cdot \lambda^2 \exp(-t) \cdot \frac{dt}{\lambda}$$

$$= \int_0^\infty \frac{1}{4} \cdot \frac{t^4}{\lambda^3} \exp(t)\,dt$$

$$= \frac{1}{4\lambda^3}\Gamma(5) = \frac{4!}{4\lambda^3} = \frac{6}{\lambda^3}$$

$$\therefore E[XY] = \frac{6}{\lambda^3} - \frac{2}{\lambda^3} = \frac{4}{\lambda^3}$$

$$\mathrm{cov}[X,Y] = \underbrace{E[XY]}_{\frac{4}{\lambda^3}} - \underbrace{E[X]}_{\frac{2}{\lambda}} \cdot \underbrace{E[Y]}_{\frac{1}{\lambda^2}} = \frac{4}{\lambda^3} - \frac{2}{\lambda^3} = \underline{\frac{2}{\lambda^3}}$$

$$\therefore \alpha = E[Y] - \frac{\mathrm{cov}[X,Y]}{V[X]} \cdot E[X] = \frac{1}{\lambda^2} - \frac{\left(\frac{2}{\lambda^3}\right)}{\left(\frac{2}{\lambda^2}\right)} \cdot \frac{2}{\lambda} = \cdot$$

$$= \frac{1}{\lambda^2} - \frac{1}{\lambda} \cdot \frac{2}{\lambda} = \underline{\frac{-1}{\lambda^2}}$$

$\boxed{2}$-(b) note P5

$$\hat{\beta} = \frac{\mathrm{cov}[X,Y]}{V[X]} = \left(\frac{2}{\lambda^2}\right)^{-1} \cdot \frac{2}{\lambda^3} = \frac{\lambda^2}{2} \cdot \frac{2}{\lambda^3} = \frac{1}{\lambda}$$

$$\therefore \hat{\alpha}X + \hat{\beta} = \underline{\left(\frac{-1}{\lambda^2}\right)X + \frac{1}{\lambda}} \quad \cdots \boxed{2}\,(b)$$

Best Linear Predictor.

(Note) HW6. [14] (c) 別の解き方（別的解法）

[14] (c) では $2X_1 | X_{(n)} = t$ が完備十分統計量 T の関数になることが分かっており $E[T] = \int_0^\theta t \cdot \frac{nt^{n-1}}{\theta^n} dt$ $= \frac{n\theta}{n+1}$ であることから $E\left[\frac{n+1}{n}T\right] = \theta$ なので $\frac{n+1}{n}T$ が θ の UMVUE であるという解き方を用いたが、

真面目に $X_1 | X_{(n)} = t$ の分布から求める方法を以下に示しておく。 $(X_{(n)} = T)$

① $x_1 < t$ のとき…

$$Pr(X_1 \leq x_1, X_{(n)} \leq t) = Pr(X_1 \leq x_1, X_1 \sim X_n \leq t)$$

$$= Pr(X_1 \leq \min\{X_1, t\}, X_2 \sim X_n \leq t)$$

$$= Pr(X_1 \leq x_1, X_2 \sim x_n \leq t) = \frac{x_1 \cdot t^{n-1}}{\theta^n} \quad (x_1 < t)$$

$$= F_{X_1,T}(x_1,t) \qquad \nearrow f_{X_1,T}$$

$$\frac{\partial^2 F_{X_1,t}}{\partial x_1 \partial t} = \frac{(n-1)t^{n-2}}{\theta^n}$$ である

$$\therefore f_{X_1|T}(x_1|T) = \frac{f_{X_1,T}(x_1,t)}{f_T(t)} = \frac{\frac{(n-1)t^{n-2}}{\theta^n}}{\left(\frac{nt^{n-1}}{\theta^n}\right)} = \left(1-\frac{1}{n}\right) \cdot \frac{1}{t}$$ である

また $\Pr(X_1 = t \mid T = t) = \frac{1}{n}$ である

（$\because \Pr(X_1 = \max\{X_1, \cdots X_n\})$ となるため）

以上の事から $X_1 \mid T = t$ の分布は

$$\begin{cases} 0 \leq x_1 < t \cdots & (1-\frac{1}{n})\cdot\frac{1}{t} \text{ の密度} \\ x_1 = t \cdots & \frac{1}{n} \text{ （確率）} \end{cases}$$

$$E[2X_1 \mid T] = \int_0^t (1-\tfrac{1}{n})\cdot\frac{2x_1}{t}\,dx_1 + \underbrace{\frac{1}{n}}_{\Pr(X_1=t \mid T=t)}\cdot(2t)$$

$$= (1-\tfrac{1}{n})\left[\frac{x_1^2}{t}\right]_0^t + \frac{2t}{n}$$

$$= (1-\tfrac{1}{n})t + \frac{2t}{n} = (1+\tfrac{1}{n})t = \frac{n+1}{n}t$$

より求めた結果はたしかに一致する。

(メモ) HW2 [3] sub-Gaussianの振り返り.

[3] 今さらながら sub-Gaussian に関する問題を振り返っておく

(問) $E[X]=0$ $\exists\sigma>0$ s.t. $M_X(t) \leq \exp\left(\frac{t^2}{2\sigma^2}\right)$ (for all t)

⇒ Xは sub-Gaussian であると言う.

(1) Xが sub-Gaussian の性質を満足すれば

$$E[e^{tX}] = E[e^{(-t)(-X)}] = M_{(-X)}(-t) \leq \exp\left(\frac{t^2}{2\sigma^2}\right) = \exp\left(\frac{(-t)^2}{2\sigma^2}\right)$$

(for all t)

であるから $M_{(-X)}(t) \leq \exp\left(\frac{t^2}{2\sigma^2}\right)$ となり −Xも sub-Gaussian の性質を満足する

(2) 次に $\theta>0$ のとき $Pr(|X-\mu| \geq \theta) \leq 2\exp\left(\frac{-\theta^2}{2\sigma^2}\right)$ を示す.

ここで Xの平均をμとする 1は Xの平均を0と見るに

$Pr(|X| \geq \theta) \leq 2\exp\left(\frac{-\theta^2}{2\sigma^2}\right)$ を示せば十分

(左) $\underbrace{Pr(X \geq \theta)}_{①} + \underbrace{Pr(-X \geq \theta)}_{②}$ $(\theta>0)$

Xも−Xも sub-Gaussian なので ①について

$① \leq \exp\left(\frac{-\theta^2}{2\sigma^2}\right)$ を言えば ②についても同様に示せる

① $Pr(X \geqq \theta)$ $(\theta > 0)$

ここで $t>0$ として $= Pr(tX \geqq t\theta) = Pr(e^{tX} \geqq e^{t\theta})$

Markov不等式より $\leqq e^{-t\theta} \cdot E[e^{tX}] = e^{-t\theta} \cdot M_X(t)$

$\leqq e^{-t\theta} \cdot \exp\left(\frac{\sigma^2 t^2}{2}\right)$

ここで $t = \frac{\theta}{\sigma^2}$ を代入することで $\exp\left(\frac{-\theta^2}{\sigma^2}\right) \exp\left(\frac{\theta^2}{2\sigma^2}\right)$

$\leqq \exp\left(\frac{-\theta^2}{2\sigma^2}\right)$ が得られる

②も同様、う証明が完成した。

(3)に関しては $X \geqq 0$ (a.s) であるとき

$E[X] = \int_0^\infty (1-F(x))\, dx = \int_0^\infty Pr(X>t)\, dt$

である事を用いれば良い。(注 $X \to |X|^p$ として)

2016 中間テスト

Midterm $\boxed{4}$

本問も sub Gaussian に関する問題であるので復習しておく

$\vec{x} = \begin{pmatrix} X_1 \\ \vdots \\ X_p \end{pmatrix} \sim N(0, \sigma^2 I_p)$ であり

$\vec{x}^t\vec{x} \sim \chi_p$ である (平均 p 分散 2p)

このとき $Pr(|\vec{x}^t\vec{x} - p| \geq p\varepsilon) \leq 2\exp\left(\frac{-p\varepsilon^2}{8}\right)$ を示す

これは $\vec{x}^t\vec{x} - p$ は sub Gaussian であることを利用する

$Y \overset{def}{=} \vec{x}^t\vec{x} - p$ とおく $Pr(|Y| \geq p\varepsilon) \leq 2\exp\left(\frac{-p\varepsilon^2}{8}\right)$ を示す

$$\overset{||}{\underbrace{Pr(Y \geq p\varepsilon)}_{①}} + \underbrace{Pr(-Y \geq p\varepsilon)}_{②}$$

①に関して $Pr(e^{\theta Y} \geq e^{p\varepsilon\theta})$

$$\leq e^{-p\varepsilon\theta} \cdot E[e^{\theta Y}] = e^{-p\varepsilon\theta} \cdot M_Y(\theta)$$

与えられた条件 $M_Y(\theta) \leq \exp(2p\theta^2)$

より $\leq e^{-p\varepsilon\theta}\exp(2p\theta^2) = \exp(2p\theta^2 - p\varepsilon\theta)$

ここで $\theta = \frac{\varepsilon}{4}$ $\exp\left(\frac{p\varepsilon^2}{8} - \frac{p\varepsilon^2}{4}\right) = \exp\left(\frac{-p\varepsilon^2}{8}\right)$

②も同様となり $\leq 2\exp\left(\frac{-p\varepsilon^2}{8}\right)$ である